HYPERCOMPLEX BARK-SCALE FILTER BANK DESIGN BASED ON ALLPASS-PHASE SPECIFICATIONS

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ABSTRACT

With telephony and hearing aids in mind, we present in this research paper an I-channel Bark-scale filter bank approach ($I = 2^D$, $D \in \mathbb{N}$) that employs a hypercomplex (i.e. multi-hyperbolic) allpass filter as a filter bank. Efficient computation is assured by maximal orthogonal decomposition of the selected multi-hyperbolic algebra. The main advantages of this approach are: i) parallel processing of the subband signals, ii) designability of arbitrarily *non-uniform filter banks, and iii) robustness (= losslessness) as a result of the double complementarity of (hypercomplex) allpass functions.*

1. INTRODUCTION

For a long time, one approach to the design and implementation of two-channel IIR (analysis) filter banks has been based on the concepts of *i)* two coupled real allpass functions [1, 2], or *ii)* one complex allpass function by splitting its complex output signal into its real and imaginary parts [3].

By using instead a hypercomplex 4-dimensional allpass function, the original approach has been extended [4]: The subband filters $H_\nu(z)$, $\nu = 0, \ldots, 3$, of the filter bank are represented by the four real subsystems of a quaternion-valued allpass function. This approach can be based on any hypercomplex algebra of suitable dimension being related to the number of filter bank channels [5].

Subsequently, we present a filter bank approach based on d-dimensional multi-hyperbolic algebras, where the number of channels of *arbitrary* bandwidth and allocation is given by: $I = d = 2^D$, $D \in \mathbb{N}$. To this end, we recall the properties of these algebras, introduce multi-hyperbolic allpass functions, specify the desired phase functions to perform the allpass function design, and give a design example.

2. MULTI-HYPERBOLIC NUMBERS AND SYSTEMS

A 2-dimensional hyperbolic number $\boldsymbol{a} = a_\mathrm{R} + \boldsymbol{i}_1 a_\mathrm{I} \in \mathbb{D}$ is composed of a real part $a_\mathrm{R} \in \mathbb{R}$ and an imaginary part $a_\mathrm{I} \in \mathbb{R}$ where, in contrast to complex numbers $\mathbb{C}$, the square of the imaginary unit $\boldsymbol{i}_1$ is defined by [6]: $\boldsymbol{i}_1^2 = 1$. In general, a multi-hyperbolic number $\mathbb{D}_D$, $D \in \mathbb{N}$, of dimension $d = 2^D > 2$ is derived from $\mathbb{D}$ by consecutive doubling (similar to that described in [7]) subject to the squaring rule and commutativity [6, 8, 9]: $\boldsymbol{i}_\nu^2 = 1, \boldsymbol{i}_\mu \boldsymbol{i}_\nu = \boldsymbol{i}_\nu \boldsymbol{i}_\mu, \forall \mu, \nu = 1, \ldots, 2^D - 1$.

Hence, each multi-hyperbolic algebra $\mathbb{D}_D$ is both commutative and associative, but possesses particular zero divisors $\boldsymbol{a}_0 \neq 0$ *and* $\boldsymbol{b}_0 \neq 0$ with zero product [7]: $\boldsymbol{a}_0 \boldsymbol{b}_0 = 0$. As a result of the latter property, each d-dimensional regular multi-hyperbolic number $\boldsymbol{a} \in \mathbb{D}_D$ can be decomposed into d orthogonal real-valued components $\tilde{\boldsymbol{a}} = [\tilde{a}_0, \ldots, \tilde{a}_{d-1}]^\mathrm{T}$. While a regular multiplication in $\mathbb{D}_D$ obviously requires d^2 real multiplications, the multiplication count is reduced to just d when using the orthogonal representation $\tilde{\boldsymbol{a}}$ [8].

Regular and maximally decomposed orthogonal representations of $\boldsymbol{a} \in \mathbb{D}_D$ are related by

$$\boldsymbol{a} = \mathbf{E}\tilde{\boldsymbol{a}} \in \mathbb{R}^d \quad \text{and} \quad \tilde{\boldsymbol{a}} = \mathbf{F}\boldsymbol{a} = \mathbf{E}^{-1}\boldsymbol{a} \in \mathbb{R}^d, \tag{1}$$

where $\boldsymbol{a} = [a_0, \ldots, a_{d-1}]^\mathrm{T}$. The quadratic real, symmetric and orthogonal $\mathbb{D}_D$ basis-matrix is given by [6, 10]:

$$\mathbf{E} = \frac{1}{2^D} \bigotimes_{\delta=1}^{D} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \mathbf{E}^\mathrm{T} = \frac{1}{2^D}\mathbf{E}^{-1} = \frac{1}{2^D}\mathbf{F}. \tag{2}$$

Here, $\bigotimes_{\delta=1}^{D}$ represents a D-fold KRONECKER-Product [11] of the basis-matrix of algebra $\mathbb{D}$. As shown in [10], the basis-matrix $\mathbf{E}$ and its inverse $\mathbf{F}$ constitute a WALSH-HADAMARD transform pair (WHT).

A d-dimensional multi-hyperbolic system is most efficiently implemented by using the orthogonal system representation [5, 8]

$$\mathbf{Y}(z) = \underbrace{\mathbf{E}\tilde{\mathbf{H}}(z)\mathbf{F}}_{\mathbf{H}(z)} \cdot \mathbf{X}(z) = \underbrace{\mathbf{E}\tilde{\mathbf{H}}(z)\mathbf{E}^{-1}}_{\mathbf{H}(z)} \cdot \mathbf{X}(z), \tag{3}$$

where the d-dimensional input and output vectors, $\mathbf{X}(z)$ and $\mathbf{Y}(z)$, comprise the z-transforms of the d real input and output signals assigned to the d components of the multi-hyperbolic input and output signals $\boldsymbol{x}(n)$, $\boldsymbol{y}(n) \in \mathbb{D}_D$, and the diagonal $d \times d$ transfer matrix $\tilde{\mathbf{H}}(z)$ is composed of the real orthogonal subsystems $\tilde{H}_\mu(z)$, $\mu = 0, \ldots, d-1$, derived from the regular representation according to (1) and (2):

$$\tilde{\mathbf{h}}(z) = \begin{bmatrix} \tilde{H}_0(z) \\ \vdots \\ \tilde{H}_{d-1}(z) \end{bmatrix} = \mathbf{F} \cdot \begin{bmatrix} H_0(z) \\ \vdots \\ H_{d-1}(z) \end{bmatrix}. \quad (4)$$

A block diagram of the system implementation (3) is shown in Fig. 1. Obviously, signal processing by the real orthogonal subsystems $\tilde{h}_\mu(n) \longleftrightarrow \tilde{H}_\mu(z)$, $\mu = 0, \ldots, d-1$, is performed separately in a parallel manner. As anticipated, a d-dimensional multi-hyperbolic convolution requires just d real convolutions as a result of d-fold (maximum) orthogonal decomposition [8].

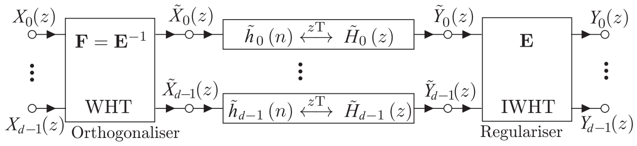


Fig. 1: System implementation based on orthogonal representation (3); $\tilde{h}_\mu(n) \in \mathbb{R}$, $\mu = 0, \ldots, d-1$,

3. MULTI-HYPERBOLIC ALLPASS FUNCTION

We devise a multi-hyperbolic allpass system (MHB-AP) as an analysis filter bank for real input and real output signals. To this end, we assign the real input signal $x(n) \longleftrightarrow X(z)$ to the real part $x_0(n)$ of the hyperbolic MHB-AP input signal $\boldsymbol{x}(n) \in \mathbb{D}_D$, while all $d-1$ imaginary parts are set to zero: Fig. 2. The real components $y_\nu(n) \longleftrightarrow Y_\nu(z)$, $\nu = 0, \ldots, d-1$, of the MHB-AP output signal $\boldsymbol{y}(n) \in \mathbb{D}_D$ represent the $I = d$ real subband signals of the I-channel analysis filter bank.

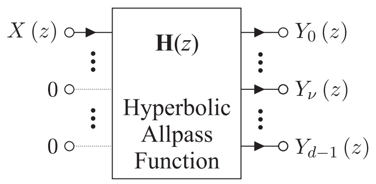


Fig. 2: Hyperbolic allpass system based analysis filter bank

The transfer function of a $\mathbb{D}_D$ MHB-AP function is given by

$$\boldsymbol{H}(z) = \mathcal{Z}\{\boldsymbol{h}(n)\} = \underbrace{\mathcal{Z}\{h_0\}}_{H_0(z)} + \sum_{\nu=1}^{d-1} \underbrace{\mathcal{Z}\{h_\nu\}}_{H_\nu(z)} \boldsymbol{i}_\nu \quad (5)$$

with the allpass properties of power complementarity (losslessness) and allpass-complementarity [2, 3]:

$$\left|\boldsymbol{H}\left(\mathrm{e}^{\mathrm{j}\Omega}\right)\right| = \sum_{\nu=0}^{d-1} \left|H_\nu\left(\mathrm{e}^{\mathrm{j}\Omega}\right)\right|^2 = \left|\sum_{\nu=0}^{d-1} H_\nu\left(\mathrm{e}^{\mathrm{j}\Omega}\right)\right| = 1, \ \forall\Omega. \quad (6)$$

The latter property follows from the orthogonality of the $\mathbb{D}_D$ basis-matrix $\mathbf{E}$ as a result of the relation (7).

For efficient implementation, we again decompose the MHB-AP system into the maximum number d of orthogonal components in compliance with (3) and (4). Since the MHB-AP input signal is real, relation (3) simplifies to:

$$\mathbf{h}(z) = \begin{bmatrix} H_0(z) \\ \vdots \\ H_{d-1}(z) \end{bmatrix} = \mathbf{E} \cdot \tilde{\mathbf{h}}(z) = \mathbf{E} \cdot \begin{bmatrix} \tilde{H}_0(z) \\ \vdots \\ \tilde{H}_{d-1}(z) \end{bmatrix} \quad (7)$$

where the filter vector $\mathbf{h}(z)$ (1st column of transfer matrix $\mathbf{H}(z)$, Fig. 2) comprises the real-coefficient filter bank transfer functions, and $\tilde{\mathbf{h}}(z)$ the respective real allpass functions.

Since $\boldsymbol{x}(n) = x_0(n) = x(n) \in \mathbb{R}$ according to Fig. 2, the block diagram Fig. 1 likewise simplifies: The orthogonaliser block is discarded, and all real allpass functions $\tilde{H}_\mu(z) \longleftrightarrow \tilde{h}_\mu(n)$, $\mu = 0, \ldots, d-1$, are concurrently excited by the real filter bank input signal $x(n)$.

The transfer functions of the d real orthogonal allpass subsystems are given by [2]:

$$\tilde{H}_\mu(z) = z^r \frac{\tilde{N}_\mu(z^{-1})}{\tilde{N}_\mu(z)}, \quad \tilde{N}_\mu(z) = \sum_{\varrho=0}^{r} \tilde{a}_{\mu,\varrho} z^\varrho, \quad (8)$$

where $\mu = 0, \ldots, d-1$ and r represents the common order of the d allpass functions, with the resulting magnitude and phase responses:

$$\left|\tilde{H}_\mu\left(\mathrm{e}^{\mathrm{j}\Omega}\right)\right| \equiv 1 \, \forall\Omega, \ \tilde{H}_\mu\left(\mathrm{e}^{\mathrm{j}\Omega}\right) = \mathrm{e}^{-\mathrm{j}\tilde{b}_\mu(\Omega)}, \ \tilde{b}_\mu(\Omega) \in \mathbb{R}. \quad (9)$$

The coefficients $\tilde{a}_{\mu,\varrho} \in \mathbb{R}$ of the orthogonal allpass subsystems (8) are related to the regular multi-hyperbolic allpass coefficients $\boldsymbol{a}_\varrho \in \mathbb{D}_D$, $\varrho = 0, \ldots, r$, by (7). Hence, the frequency responses $H_\nu\left(\mathrm{e}^{\mathrm{j}\Omega}\right)$, $\nu = 0, \ldots, d-1$, of the regular MHB-AP subsystems are derived from the phase responses $\tilde{b}_\mu(\Omega)$, $\mu = 0, \ldots, d-1$, of the real orthogonal allpass systems $\tilde{H}_\mu(z)$ as follows:

$$\mathbf{h}\left(\mathrm{e}^{\mathrm{j}\Omega}\right) = \begin{bmatrix} H_0\left(\mathrm{e}^{\mathrm{j}\Omega}\right) \\ \vdots \\ H_{d-1}\left(\mathrm{e}^{\mathrm{j}\Omega}\right) \end{bmatrix} = \mathbf{E} \cdot \begin{bmatrix} \mathrm{e}^{-\mathrm{j}\tilde{b}_0(\Omega)} \\ \vdots \\ \mathrm{e}^{-\mathrm{j}\tilde{b}_{d-1}(\Omega)} \end{bmatrix} \quad (10)$$

$$= \begin{bmatrix} \sum_{\mu=0}^{d-1} E_{0,\mu} \mathrm{e}^{-\mathrm{j}\tilde{b}_\mu(\Omega)} \\ \vdots \\ \sum_{\mu=0}^{d-1} E_{d-1,\mu} \mathrm{e}^{-\mathrm{j}\tilde{b}_\mu(\Omega)} \end{bmatrix}, \quad (11)$$

where use is made of (7), (9) and (2).

4. ALLPASS SPECIFICATION AND DESIGN

For each magnitude response $\left|H_\nu\left(\mathrm{e}^{\mathrm{j}\Omega}\right)\right|$ of the d real filter bank transfer functions (7) we define a passband region $\mathcal{D}_\nu$,

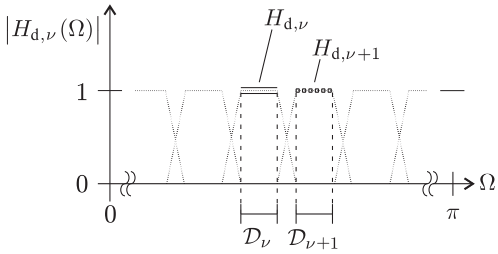


Fig. 3: Illustration of filter bank magnitude specification

$\mathcal{D}_\nu \subset \{\Omega : 0 \leq \Omega \leq \pi\}$ being concurrently a stopband region of all other subsystems with the desired functions:

$$|H_{\mathrm{d},\nu}(\Omega)| = 1 \ \cap \ |H_{\mathrm{d},\xi}(\Omega)| = 0 \ \forall \xi \neq \nu, \ \Omega \in \mathcal{D}_\nu, \quad (12)$$

implying disjoint passbands $\mathcal{D}_\nu \cap \mathcal{D}_\xi = \emptyset \ \forall \nu \neq \xi$, and assuming ordered indexing: $\max \mathcal{D}_\nu < \min \mathcal{D}_{\nu+1}$. Let $\nu = 0, \ldots, d$, since $\left|H_0\left(\mathrm{e}^{\mathrm{j}\Omega}\right)\right|$ always represents a bandstop filter transfer function (passbands about $\Omega = 0, \pi$) as a result of allpass properties investigated in the appendix. Hence, we split the associated zeroth passband into two regions: $\mathcal{D}_0 := \mathcal{D}_0 \cap \mathcal{D}_d$. Finally, all remaining transition bands $\mathcal{T}$ are given by (cf. example depicted in Fig. 3):

$$\mathcal{T} = [0, \pi] \setminus \mathcal{D}, \quad \mathcal{D} = \bigcup_{\nu=0}^{d} \mathcal{D}_\nu. \quad (13)$$

Using (10) and (11) in conjunction with (9), the design problem of the d real regular filter bank transfer functions $\mathbf{h}(z)$ is mapped to the design of the real orthogonal allpass functions $\tilde{\mathbf{h}}(z)$, as defined in (7), by specifying the desired orthogonal allpass function phase responses relative to that of index $\mu = 0$ (cf. Fig. 1):

$$\tilde{b}_{\mathrm{d},\mu}(\Omega) = \tilde{b}_{\mathrm{d},0}(\Omega) + \Delta\tilde{b}_{\mathrm{d},\mu}(\Omega), \quad \mu = 1, \ldots, d-1, \quad (14)$$

where $\tilde{H}_0(z) = z^{-r}$ is assumed linear phase: $\tilde{b}_{\mathrm{d},0}(\Omega) = r\Omega$. As a result, all $d-1$ real allpass functions can be devised separately.

Next, we introduce the desired (regular) filter bank magnitude responses (12) and the desired (orthogonal) phase responses (14) component-wise into (11)

$$|H_{\mathrm{d},\nu}(\Omega)| = \left|\sum_{\mu=0}^{d-1} E_{\nu \bmod d,\mu} \mathrm{e}^{-\mathrm{j}\tilde{b}_{\mathrm{d},\mu}(\Omega)}\right| =$$
$$\left|\mathrm{e}^{-\mathrm{j}\tilde{b}_{\mathrm{d},0}(\Omega)}\right| \cdot \left|E_{\nu,0} + \sum_{\mu=1}^{d-1} E_{\nu \bmod d,\mu} \mathrm{e}^{-\mathrm{j}\Delta\tilde{b}_{\mathrm{d},\mu}(\Omega)}\right| =$$
$$\left|\frac{1}{d} + \sum_{\mu=1}^{d-1} E_{\nu \bmod d,\mu} \mathrm{e}^{-\mathrm{j}\Delta\tilde{b}_{\mathrm{d},\mu}(\Omega)}\right| = 1, \quad \Omega \in \mathcal{D}_\nu. \quad (15)$$

Note that the (regular) filter index is extended according to $\nu := \nu \bmod d$ to cope with the passband splitting $\mathcal{D}_0 := \mathcal{D}_0 \cap \mathcal{D}_d$ of $H_0(z)$. Furthermore, we take into account that all elements of the $\mathbb{D}_D$ basis matrix $\mathbf{E}$ defined by (2) are represented by $E_{\nu,\mu} = \pm 1/d, \ \ \nu, \mu = 0, \ldots, d-1$. Hence, for each passband region $\mathcal{D}_\nu$, $\nu = 0, \ldots, d$, the desired phase differences (14) must meet the (sufficient) condition:

$$\Delta\tilde{b}_{\mathrm{d},\mu}(\Omega) = \begin{cases} 0, & E_{\nu \bmod d,\mu} = \frac{1}{d} \\ \pm\pi, & E_{\nu,\mu} = -\frac{1}{d} \end{cases}, \quad \Omega \in \mathcal{D}_\nu, \quad (16)$$

in order to comply with (15) and, as a consequence, to meet the requirements (12) for the (regular) transfer functions $H_\nu(z)$, $\nu = 1, \ldots, d-1$. Note that with the minimum phase difference definition of (16), the required common order r of the allpass functions is minimal (cf. appendix).

Finally, for (14) we have to define suitable desired phase differences for the d transition regions

$$\mathcal{T}_\nu = \{\Omega : \max \mathcal{D}_\nu < \Omega < \min \mathcal{D}_{\nu+1}\}, \quad \nu = 0, \ldots, d-1, \quad (17)$$

between any pair of contiguous passband regions. To this end, we introduce a continuous yet arbitrary desired phase difference function in each transition region $\mathcal{T}_\nu$:

$$\Delta\tilde{b}_{\mathrm{d},\mu}(\Omega) = \Delta\tilde{b}_{\mathrm{d},\mu}(\mathcal{D}_\nu) + \Delta^2\tilde{b}_{\mathrm{d},\mu}(\mathcal{D}_\nu) \cdot f_{\mathrm{U}}\left(\frac{\Omega - \Omega_{\mathrm{u}}}{\Omega_{\mathrm{o}} - \Omega_{\mathrm{u}}}\right), \quad (18)$$

where $\Omega \in \mathcal{T}_\nu$, $f_{\mathrm{U}}(W) \in \mathbb{R}$ with non-negative slope and $f_{\mathrm{U}}(0) = 0$, $f_{\mathrm{U}}(1) = 1$, $\Omega_{\mathrm{u}} = \min \mathcal{T}_\nu$, $\Omega_{\mathrm{o}} = \max \mathcal{T}_\nu$, $\mu, \nu = 0, \ldots, d-1$, and

$$\Delta^2\tilde{b}_{\mathrm{d},\mu}(\mathcal{D}_\nu) = \Delta\tilde{b}_{\mathrm{d},\mu}(\mathcal{D}_{\nu+1}) - \Delta\tilde{b}_{\mathrm{d},\mu}(\mathcal{D}_\nu). \quad (19)$$

To obtain uniform slopes of the desired phase difference functions (18) in the transition regions (17), we modify the desired phase differences in the passband regions $\mathcal{D}_\nu$. To this end, we develop a recursive algorithm to replace (16).

The required minimum phase slopes (19) between any two passband regions to comply with (16) and (2) are given by:

$$\Delta^2\tilde{b}_{\mathrm{d},\mu}(\mathcal{D}_\nu) = \begin{cases} 0, & \left|E_{(\nu+1) \bmod d,\mu} - E_{\nu,\mu}\right| = 0 \\ \pm\pi, & \left|E_{(\nu+1) \bmod d,\mu} - E_{\nu,\mu}\right| = \frac{2}{d} \end{cases}, \quad (20)$$

where $\mu, \nu = 0, \ldots, d-1$. To unify the phase slopes (20) of all orthogonal allpass functions $\tilde{H}_\mu(z)$, $\mu = 1, \ldots, d-1$, we assign the two cases of (20) for each $\tilde{H}_\mu(z)$ to two different index sets according to:

$$\mathcal{I}_\nu = \left\{\mu : \left|\Delta^2\tilde{b}_{\mathrm{d},\mu}(\mathcal{D}_\nu)\right| > 0\right\}, \quad \nu = 0, \ldots, d-1 \quad (21)$$

and

$$\mathcal{J}_\nu = \left\{\mu : \Delta^2\tilde{b}_{\mathrm{d},\mu}(\mathcal{D}_\nu) = 0\right\} = \{1, \ldots, d-1\} \setminus \mathcal{I}_\nu. \quad (22)$$

Hence, for the real orthogonal allpass functions $\tilde{H}_\mu(z)$ of index set (22) we have in compliance with (20):

$$\Delta\tilde{b}_{\mathrm{d},\mu}(\mathcal{D}_{\nu+1}) = \Delta\tilde{b}_{\mathrm{d},\mu}(\mathcal{D}_\nu), \quad \mu \in \mathcal{J}_\nu, \quad \nu = 0, \ldots, d-1, \quad (23)$$

where, due to restriction (26), all desired phase differences of $\mathcal{D}_0$ of *all* orthogonal allpass functions are given by: $\Delta\tilde{b}_{\mathrm{d},\mu}(\mathcal{D}_0) \equiv 0, \quad \mu = 1, \ldots, d-1$. In contrast, compliant with (20), the desired phase differences of the orthogonal allpass functions $\tilde{H}_\mu(z)$, $\mu \in \mathcal{I}_\nu$, must change phase difference versus the transition bands (17), $\mathcal{T}_\nu$, by $\pm\pi$ as follows:

$$\Delta\tilde{b}_{\mathrm{d},\mu}(\mathcal{D}_{\nu+1}) = \begin{cases} \Delta\tilde{b}_{\mathrm{d},\mu}(\mathcal{D}_\nu) + \pi, & \Delta\tilde{b}_{\min} = 0 \\ \Delta\tilde{b}_{\mathrm{d},\mu}(\mathcal{D}_\nu) - \pi, & \Delta\tilde{b}_{\min} > 0 \end{cases}, \quad (24)$$

where $\Delta\tilde{b}_{\min} = \min_{\forall\kappa\in\mathcal{I}_\nu} \left\{\Delta\tilde{b}_{\mathrm{d},\kappa}(\mathcal{D}_\nu)\right\}$. As a result, all associated desired phase differences have either a common positive or a common negative slope. According to (24) the slope depends on the minimum value of all desired phase differences $\Delta\tilde{b}_{\mathrm{d},\mu}(\mathcal{D}_\nu)$, $\mu \in \mathcal{I}_\nu$, of the preceding passband region $\mathcal{D}_\nu$. Due to the allpass restriction (26), negative slopes of the desired phase differences are always required, as soon as all associated orthogonal allpass functions $\tilde{H}_\mu(z)$, $\mu \in \mathcal{I}_\nu$, have accumulated positive desired phase differences up to the respective preceding passband region $\mathcal{D}_\nu$.

5. DESIGN EXAMPLE

We present the design of a 16-channel filter bank (sampling rate: $f_\mathrm{s} = 8\mathrm{kHz}$) based on the quadrihyperbolic algebra $\mathbb{D}_4$ of dimension $d = 2^4 = 16$, where the filter centre frequencies are allocated compliant to the Bark-scale [12]: Fig. 4. Channels $\nu = 1, \ldots, 15$ encompass the usable overall input bandwidth ranging from 300Hz to 3.4kHz, while the bandstop channel $H_0(z)$ is unused. The passbands widths of all passband regions are set to zero, hence $\mathcal{D}_\nu = \Omega_{\mathrm{c},\nu}, \nu = 1, \ldots, 15$.

Following the procedure presented in section 4, each of the 15 desired phase functions (14) is separately approximated by an allpass function of order $r = 80$ using a least-squares design routine [13]. The resulting phase differences of the orthogonal allpass functions (being approx. linear phase, since the reference branch is fixed to $\tilde{H}_0(z) = z^{-r}$) along with the associated regular filter bank transfer functions are depicted in Fig. 5. From Fig. 5(a) it is readily recognised that in all domains with changing phase differences, the slopes of all changes are always either positive or negative, as required by the algorithm. In Fig. 5(b) the unused bandstop transfer function $H_0(z)$ is indicated by a dashed line.

ν	$\Omega_{\mathrm{c},\nu}$	$f_{\mathrm{c},\nu}/\mathrm{Hz}$	ν	$\Omega_{\mathrm{c},\nu}$	$f_{\mathrm{c},\nu}/\mathrm{Hz}$
1	0,0625	250	9	0,3425	1370
2	0,0875	350	10	0,4	1600
3	0,1125	450	11	0,4625	1850
4	0,1425	570	12	0,5375	2150
5	0,175	700	13	0,625	2500
6	0,21	840	14	0,725	2900
7	0,25	1000	15	0,85	3400
8	0,2925	1170			

Fig. 4: Bark-scale centre frequencies Ω_c, f_c of passband regions $\mathcal{D}_\nu$ of regular filter bank transfer functions $H_\nu(z)$, $\nu = 1, \ldots, 15$, for quadrihyperbolic filter bank; $f_\mathrm{s} = 8\mathrm{kHz}$

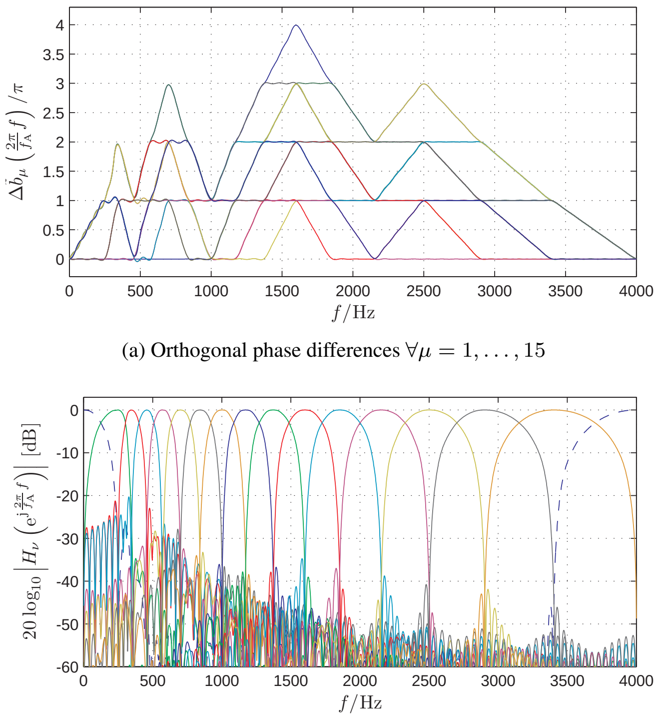
6. CONCLUSION

For the first time, we have presented a general approach to the design of one-stage I-channel hypercomplex analysis filter banks (AFB) for real input and output signals that are based on multi-hyperbolic (MHB) algebras $\mathbb{D}_D$ of dimension $d = 2^D$, $D \in \mathbb{N}$, where $I \equiv d$. Conceptually, the I-channel filter bank is implemented as a d-dimensional MHB allpass system, where the real input signal is assigned to the real part of the MHB input signal, while all its imaginary parts are set to zero, and the $d = I$ components of the MHB output signal are considered as the I real filter bank subband signals. As a result of maximal orthogonal decomposition into d components possible for *all* MHB algebras $\mathbb{D}_D$, the design and implementation of an MHB allpass is most efficiently reduced to the separate approximation and parallel realisation of just d real allpass functions of the same order.

Further desirable features of the allpass based MHB filter bank are: *i*) The set of filter bank transfer functions is allpass and power complementary (robustness/losslessness), and *ii*) the potential of arbitrary choice of centre frequencies and bandwidth allocation of all $d-1$ channels (with the exception of one bandstop channel with split passbands' centre frequencies at $\Omega = 0, \pi$) overcoming the restrictions of allpass-transformed uniform filter banks. As a consequence of allpass complementarity (6), the non-decimated AFB subband signals are perfectly reconstructed to the original AFB input signal by mere addition of all AFB subband signals. Finally, the dual synthesis filter bank is readily derived from the AFB by applying the transposition rules of real systems.

In our MHB filter bank approach the orders of all d parallel allpass functions are *identical* in contrast to well-known filter bank designs based on coupled allpass functions [2, 3, 13] that require different orders. It should also be noted that filter banks for complex input and output signals can likewise efficiently be based on tessarine algebras $\mathbb{CD}_D$ of dimension $d = 2^{D+1}$ [8, 9]. Here, in contrast to the reported MHB approach with real allpass functions, the (maximally possible) $d/2$ orthogonal allpass functions are *complex*-valued.

More detailed investigations of the presented filter bank approach to be reported in forthcoming papers will be related to the following topics: *i*) Avoid a *fixed linear phase* reference allpass branch $\tilde{H}_0(z)$, *ii*) design all real allpass functions commonly or iteratively, *iii*) use and compare different allpass function design algorithms, e.g. an equiripple or weighted LMS approach for uniform stopband attenuation, *iv*) define less order-consuming desired phase differences in the transition regions (17), and *v*) compare major properties (e.g. com-



(a) Orthogonal phase differences $\forall \mu = 1, \ldots, 15$

(b) Regular magnitude responses $\forall \nu = 0, \ldots, 15$

Fig. 5: Bark-scale quadrihyperbolic filter bank with uniform slopes of phase differences; $f_\mathrm{s} = f_\mathrm{A} = 8\mathrm{kHz}$, $I = d = 16$, order of allpass function $r = 80$

plexity, signal delay and distortion) of the reported design approach with those of allpass-transformed complex-modulated filter banks.

Appendix

The phase response of a real allpass function $\tilde{H}(z)$ (8), (9) of order r has fixed values at $\Omega = 0, \pi$ [2]:

$$\tilde{b}(0) = 0, \quad \tilde{b}(\pi) = r\pi. \tag{25}$$

Hence, the desired phase differences of all orthogonal allpass functions $\tilde{H}_\mu(z)$ of the common order r are restricted to:

$$\Delta\tilde{b}_{\mathrm{d},\mu}(0) = \Delta\tilde{b}_{\mathrm{d},\mu}(\pi) = 0, \quad \mu = 0, \ldots, d-1. \tag{26}$$

For the regular subsystem $H_0(z)$ of the hyperbolic allpass function (5) it follows from (10) in conjunction with (25):

$$\left|H_0\left(\mathrm{e}^{\mathrm{j}0}\right)\right| = \left|d \cdot \frac{1}{d} \cdot \mathrm{e}^{\mathrm{j}0}\right| = \left|H_0\left(\mathrm{e}^{\mathrm{j}\pi}\right)\right| = \left|d \cdot \frac{1}{d} \cdot \mathrm{e}^{-\mathrm{j}r\pi}\right| = 1, \tag{27}$$

and for the remaining magnitude responses:

$$\left|H_\nu\left(\mathrm{e}^{\mathrm{j}0}\right)\right| = \left|H_\nu\left(\mathrm{e}^{\mathrm{j}\pi}\right)\right| = 0, \quad \nu = 1, \ldots, d-1. \tag{28}$$

As a result, $H_0(z)$ is constrained to a bandstop filter transfer function, while all $H_\nu(z)$, $\nu = 1, \ldots, d-1$, can be specified as bandpass filters.

An allpass phase response $\tilde{b}(\Omega)$ is monotonically increasing [2]. To meet this condition, the slope of $\tilde{b}_{\mathrm{d},0}(\Omega)$ in (14) must be sufficiently great. This is controlled by the common allpass order r, since $\tilde{H}_0(z) = z^{-r}: \quad \tilde{b}_{\mathrm{d},0}(\Omega) = r\Omega$.

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