A FINE SEARCH METHOD FOR THE CUBIC-PHASE FUNCTION-BASED ESTIMATOR

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ABSTRACT

In this paper, we propose a refinement procedure for the estimation of polynomial-phase signal (PPS) parameters. The PPS parameters are estimated using a method based on the cubic-phase function (CPF), known to outperform the standard high-order ambiguity function (HAF) in terms of both the estimation accuracy and performance threshold. The method is referred to as the hybrid CPF-HAF (HCH). Since in the calculation of the HCH we cannot use fast Fourier transform algorithms, it is very important to decrease the number of HCH samples required for precise parameter estimation. To that end, we propose an approach based on the dichotomous search of the HCH peak. With the calculation of only several additional HCH samples, we are able to estimate parameters accurately without using time consuming oversampling or maximum-likelihood approach.

Index Terms— Dichotomous search, cubic-phase function, high-order ambiguity function, polynomial-phase signal.

1. INTRODUCTION

Polynomial-phase signals (PPSs) arise in a number of applications. For example, in passive intelligent radar surveillance, we try to determine what type of radar pulse is being transmitted, a linear FM, quadratic FM, or some other type. Another application includes signals produced by brown bats [1]. The parameters of these signals can serve to indicate the activity of the bat.

One of the most popular approaches for parameter estimation of PPSs is based on the high-order ambiguity function (HAF) [2] and its derivatives [3, 4]. In the HAF-based approach, the original signal is transformed to a complex sinusoid whose frequency is related to the highest order phase coefficient. The signal is transformed using the auto-correlation procedure and in order to estimate the Pth order phase coefficient, we have to perform $P-1$ auto-correlations. However, each auto-correlation increases the number of interference terms and increases the performance threshold by approximately 6 dB [5].

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The cubic-phase function (CPF) is introduced in order to enable more accurate estimation of parameters of a cubic-phase signal with lower signal-to-noise ratio (SNR) threshold than with the HAF [1]. This is due to the fact that only one auto-correlation of the original signal is performed instead of two required in the HAF. The CPF-based method requires one-dimensional maximizations, whereas the maximum likelihood (ML) method requires three-dimensional maximizations. Moreover, the CPF parameter estimates are asymptotically optimal or near optimal at high SNRs.

Recently, an extension of the CPF to higher order PPSs, referred to as the hybrid CPF-HAF (HCH), has been proposed [6]. In comparison to the HAF, the HCH performs one auto-correlation less and the error-propagation effect is reduced two times. Specifically, the SNR threshold is decreased by about 9 dB, while the estimation mean-squared error (MSE) is around 2 dB lower [6].

The parameter estimation is performed using the coarse and fine searches strategy. The coarse search is performed by locating the peak of the considered function (HAF, CPF or HCH), where the function is calculated over the predefined grid of values. The fine search performs the peak maximization through oversampling or some iterative procedure. Unlike the HAF, the calculation of the CPF/HCH cannot be facilitated by the fast Fourier transform (FFT) algorithms. Therefore, it is of high importance to reduce the number of calculated CPF/HCH samples.

In this paper, we propose an efficient algorithm for the PPS parameter estimation from the CPF and HCH. With the calculation of only a few additional samples, we can estimate the peak position very accurately, thus preventing the need for oversampling. The method is based on the dichotomous search of the spectral peak [7, 8].

Paper is organized as follows. In Section 2, a summary of the HAF, CPF and HCH is presented. Section 3 describes the proposed fine search algorithm, while the simulation results are given in Section 4. Conclusions are drawn in Section 5.

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2. HAF, CPF AND HCH

2.1. PPS and HAF

The Pth order PPS is defined as

$$x(n) = Ae^{j\sum_{i=0}^{P} a_i n^i}, \quad n \in \left[-\frac{N-1}{2}, \frac{N-1}{2}\right], \quad (1)$$

where A is the amplitude, $a_i, i = 0, ..., P$, are the phase coefficients and N is the signal length. Without the loss of generality, we will assume that N is odd and that the considered time interval is symmetric around $n = 0$. Our goal is to estimate A and $a_i, i = 0, ..., P$, by observing $x(n)$ embedded in noise $v(n)$, i.e., from the signal

$$y(n) = x(n) + v(n), \quad (2)$$

where $v(n)$ is zero mean white Gaussian noise with variance σ^2. The SNR is defined as $\text{SNR} = 20\log_{10}(A/\sigma)$.

The auto-correlation or the phase-differentiation (PD) is defined recursively as

$$\begin{aligned} \text{PD}_0[n,\tau] &= y(n), \\ \text{PD}_1[n,\tau] &= y(n+\tau)y^*(n-\tau), \\ &\vdots \\ \text{PD}_Q[n,\tau] &= \text{PD}_{Q-1}[n+\tau,\tau]\text{PD}^*_{Q-1}[n-\tau,\tau], \end{aligned} \quad (3)$$

where τ is the lag parameter. A single stage of the PD operator reduces the phase order of $x(n)$ by one. Therefore, for $x(n)$ given by (1), $\text{PD}_{P-1}[n,\tau]$ is a complex sinusoid whose frequency ω_0 is related to the highest order phase parameter a_P as follows [2]:

$$a_P = \frac{\omega_0}{2^{P-1}P!\tau^{P-1}}. \quad (4)$$

The coefficient a_P can be estimated very efficiently using the HAF which is calculated as the discrete Fourier transform (DFT) of $\text{PD}_{P-1}[n,\tau]$, i.e.

$$\text{HAF}(\omega) = \sum_{n=-\frac{N-1}{2}+(P-1)\tau}^{\frac{N-1}{2}-(P-1)\tau} \text{PD}_{P-1}[n,\tau]e^{-j\omega n}. \quad (5)$$

Hence, the coefficient estimation is reduced to the HAF maximization:

$$\hat{a}_P = \frac{1}{2^{P-1}P!\tau^{P-1}} \arg\max_{\omega} |\text{HAF}(\omega)|, \quad (6)$$

where $\hat{a}_P$ represents the estimate of a_P.

Lower order coefficients can be estimated by repeating the procedure on the signal $y_d(n) = y(n)e^{-j\hat{a}_P n^P}$. Ideally, $y_d(n)$ is a $(P-1)$-order PPS. However, in reality, a_P and $\hat{a}_P$ are not the same and the estimation error $\delta a_P = \hat{a}_P - a_P$ influences the estimation of lower order coefficients. This is known as the error-propagation.

2.2. CPF and HCH

In the special case that $x(n)$ is a third-order PPS, we can use the CPF defined as [1]

$$\text{CPF}(n,\Omega) = \sum_{m=-\frac{N-1}{2}+|n|}^{\frac{N-1}{2}-|n|} y(n+m)y(n-m)e^{-j\Omega m^2}. \quad (7)$$

In the noise-free case, the CPF peaks along the second-order phase derivative of $x(n)$, i.e., along $2(a_2 + 3a_3 n)$. Coefficients a_2 and a_3 are estimated by locating maxima of the CPF calculated at two time instants n and solving a set of two linear equations [1]. Therefore, to estimate a_3 using the CPF, we have to perform one auto-correlation only (7), instead of the two employed in (5). Due to the lower number of auto-correlations, the CPF results in a more accurate estimation and lower performance threshold compared to the HAF [1]. In addition, the CPF parameter estimates are asymptotically optimal or near optimal at high SNR values.

For PPSs of order higher than three, the HCH is proposed as follows [6]:

$$\begin{aligned} \text{HCH}(n,\Omega) = \sum_{k=-N_1}^{N_1} &\text{PD}_{P-3}[n+k,\tau] \\ &\times \text{PD}_{P-3}[n-k,\tau]e^{-j\Omega k^2}, \end{aligned} \quad (8)$$

where $N_1 = (N-1)/2 - |n| - (P-3)\tau$. The HCH reaches maxima at [6]

$$\Omega(n) = 2^{P-3}\tau^{P-3}P!a_P n + 2^{P-3}\tau^{P-3}(P-1)!a_{P-1}. \quad (9)$$

Therefore, coefficients a_P and a_{P-1} can be estimated by locating maxima of the HCH evaluated at two instants $n = 0$ and $n = n_1$, i.e., by using

$$\hat{a}_P = \frac{\Omega(n_1) - \Omega(0)}{2^{P-3}n_1\tau^{P-3}P!}, \quad (10)$$

$$\hat{a}_{P-1} = \frac{\Omega(0)}{2^{P-3}\tau^{P-3}(P-1)!}. \quad (11)$$

The lower order phase coefficients are calculated from the de-chirped signal $y_d(n) = y(n)e^{-j\hat{a}_P n^P - j\hat{a}_{P-1}n^{P-1}}$.

Note that the CPF represents the special case of the HCH obtained for $P = 3$ in (8).

From (8), we see that in the HCH calculation we cannot use the FFT algorithms, but we have to calculate it using definition. The peak maximization in the HCH domain can be performed using the coarse and fine search approach. The coarse search constitutes the evaluation of the HCH on the grid of equally spaced points in interval $[\Omega_{\min}, \Omega_{\max}]$ and locating the HCH peak. The fine search consists in refining the coarse search, either by oversampling the Ω domain around the coarse estimate or by using some maximization procedure. Since our aim is to reduce the complexity of the HCH procedure, it is highly desirable to use

some approach other than oversampling. To this end, we propose to apply the dichotomous search on the HCH peak. It will be shown that the HCH peak can be maximized by using only a few HCH samples in addition to the coarse search.

3. PROPOSED REFINEMENT APPROACH

Let us denote the coarse search estimate as Ω_c. Instead of oversampling the HCH spectrum around Ω_c, we propose to perform a dichotomous (binary) search around Ω_c. In a dichotomous search, we always consider three HCH samples, one in the middle of interval and two at interval limits. At each iteration, we narrow the considered interval to the one limited by the middle sample and the larger of limit samples. We calculate new middle sample and repeat the procedure. The procedure is iterated L times.

The proposed method can be summarized as follows:

Step 1. Calculate $\text{HCH}(n,\Omega)$ using (8) along the Ω grid with the resolution of $\Delta\Omega$ and find the maximum position $\Omega_m = \Omega_c$. The HCH value at Ω_m will be denoted as HCH_0, i.e., $\text{HCH}_0 = \text{HCH}(n,\Omega_m)$. Next, calculate two HCH samples at the distance of $\pm\Delta\Omega/2$ from Ω_m, i.e.

$$\text{HCH}_{\pm 1} = \text{HCH}(n, \Omega_m \pm \Delta\Omega/2),$$

Step 2. Iterate L times the following steps

$\Delta\Omega = \Delta\Omega/2$
if $\text{HCH}_1 > \text{HCH}_{-1}$ then
  $\text{HCH}_{-1} = \text{HCH}_0$ and $\Omega_m = \Omega_m + \Delta\Omega$
else
  $\text{HCH}_1 = \text{HCH}_0$ and $\Omega_m = \Omega_m - \Delta\Omega$
Set $\text{HCH}_0 = \text{HCH}(n,\Omega_m)$.

Step 3. The final estimation is obtained as

$$\hat{\Omega} = \Omega_m.$$

3.1. Computational complexity analysis

Performing L iterations of the proposed algorithm is equivalent to interpolating the Ω grid around Ω_c with 2^L samples [8]. The complexity of the proposed refinement method equals the complexity of calculating $(L+2)$ HCH samples. We can neglect division by 2 and comparison operations. Therefore, by implementing the proposed algorithm in the HCH maximization procedure instead of the oversampling, the complexity will be reduced by $(2^L - (L+2))U$ operations per maximization, where U is the number of operations required for the calculation of one HCH sample.

Table 1. Phase coefficients of three considered PPSs.

PPS 1	PPS 2	PPS 3
$a_1 = (37+r)\Delta$	$a_1 = (27+r)\Delta$	$a_1 = (31+r)\Delta$
$a_2 = (71+r)\Delta^2$	$a_2 = (65+r)\Delta^2$	$a_2 = (57+r)\Delta^2$
$a_3 = (41+r)\Delta^3$	$a_3 = (39+r)\Delta^3$	$a_3 = (41+r)\Delta^3$
—	$a_4 = (27+r)\Delta^4$	$a_4 = (19+r)\Delta^4$
—	$a_5 = (19+r)\Delta^5$	$a_5 = (23+r)\Delta^5$
—	—	$a_6 = (11+r)\Delta^6$

The computational complexity of one HCH sample is determined as follows. The calculation of $\text{PD}_{P-3}[n,\tau]$ for all values of n requires $(P-3)(N-2(P-2)\tau)$ complex multiplications. Since the calculation of each term in sum (8) requires two complex multiplications and one complex exponential (the argument of complex exponential is not included due to its low complexity), the overall complexity of one HCH sample is $(P-3)(N-2(P-2)\tau) + 2(N-2|n_1|-2(P-3)\tau)$ complex multiplications, $N-2|n_1|-2(P-3)\tau$ complex additions and $N-2|n_1|-2(P-3)\tau$ complex exponentials. Taking into account that one complex multiplication requires four real multiplications and two real additions, and that one complex addition requires two real additions, one HCH sample requires $4PN - 8\tau P^2 + 24P\tau - 4N - 16n_1$ real multiplications, $2PN - 4\tau P^2 + 8P\tau + 12\tau - 12n_1$ real additions and $2N - 4|n_1| - 4(P-3)\tau$ sines/cosines. In terms of the big O notation, the complexity of one HCH samples requires $O(PN)$ real multiplications, $O(PN)$ real additions and $O(N)$ sines/cosines.

4. SIMULATIONS

In this section, we study the performance of the proposed algorithm on three different PPSs, one third-order, one fifth-order and one sixth-order PPS. The phase coefficients are given in Table 1. In Table 1, Δ represents the sampling period and equals $\Delta = 2/(N-1)$. The number r represents a random number with uniform distribution in interval $[0,1]$. In each simulation trial, r takes different value for each considered coefficient.

Signals PPS 1, PPS 2 and PPS 3 are embedded in zero-mean white Gaussian noise with variance σ^2. The total number of samples of each signal is $N = 1001$. The parameters are estimated by the HAF, the HCH with oversampling and the HCH with the dichotomous search. A coarse search in the HCH is performed over the interval $[-300, 300]$ with the resolution of $\Delta\Omega_c = 1$. The oversampling is done with 1 (no oversampling - coarse search), 10 and 30 samples around the coarse estimate. The dichotomous search refinement is performed in $L = 5$ iterations. In the calculation of the HAF and HCH, lags are chosen following instructions from [2] and [6], respectively. The CPF

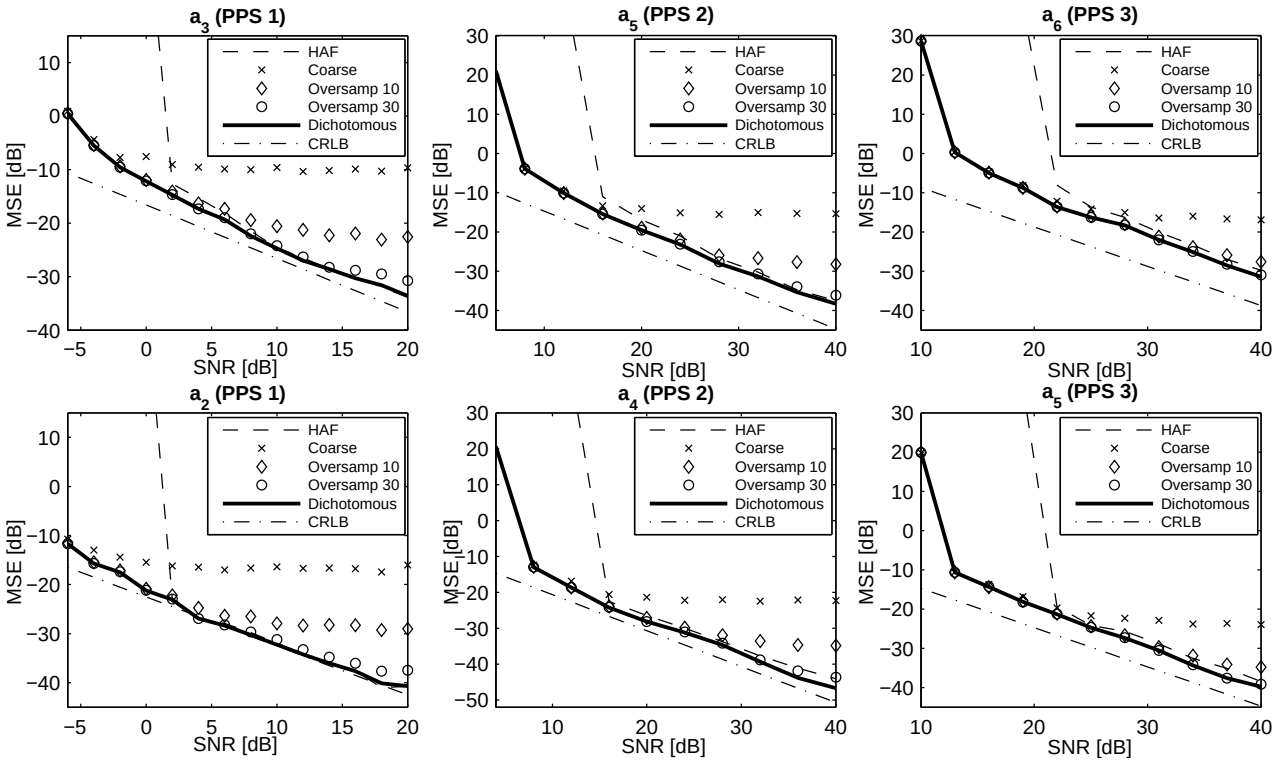

Fig. 1. MSE and Cramer-Rao lower bound (CRLB) versus noise variance σ^2. *Left column:* MSE in the estimation of a_3 and a_2 of PPS 1; *Middle column:* MSE in the estimation of a_5 and a_4 of PPS 2; *Right column:* MSE in the estimation of a_6 and a_5 of PPS 3.

is considered as a special case of the HCH according to (7) and (8) and is used in the estimation of PPS 1. In the HAF-based method, estimates are refined using the oversampling approach with 256 samples around the coarse estimate.

The obtained MSE curves are shown in Fig. 1. The MSE values are obtained over 200 Monte Carlo simulations. The left two subplots correspond to the estimation of coefficients a_3 and a_2 of PPS 1, the middle two subplots to coefficients a_5 and a_4 of PPS 2, and the right two subplots correspond to the estimation of coefficients a_6 and a_5 of PPS 3. In other words, we presented results obtained for the highest two coefficients of the considered signals.

For all the considered coefficients, the least accurate results are obtained using the coarse search. On the other hand, the oversampling with 30 samples and dichotomous search with $L = 5$ iterations perform approximately the same and give the best results. However, the dichotomous search is characterized by more than four times lower complexity. In addition, the oversampling method is characterized by fixed error at high SNRs, which is not the case for the dichotomous search. The oversampling with 10 samples has approximately the same complexity as the dichotomous search with $L = 5$ iterations, but also has worse performance.

Note that the proposed refinement achieves the accuracy bounds in the estimation of coefficients a_3 and a_2 of PPS 1 as given in Table I in [1].

Finally, the proposed method outperforms the HAF-based method in terms of both the accuracy and threshold. The difference in accuracy is more pronounced with higher PPS orders.

5. CONCLUSIONS

The CPF and its extension the HCH outperform the HAF in accuracy and SNR threshold. However, as opposed to the HAF, the calculation of the CPF/HCH cannot be facilitated by the FFT routine. Therefore, it is highly desirable to reduce the number of CPF/HCH samples required for the precise estimation of the PPS parameters. We proposed to perform the fine parameter estimation through the maximization of the CPF/HCH peak using the dichotomous search. We showed that only several additional CPF/HCH samples are needed to obtain a desirable accuracy. This way, we do not have to use time consuming techniques such as oversampling or the ML approach.

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