# RATIONAL-ORDERED DISCRETE FRACTIONAL FOURIER TRANSFORM

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## ABSTRACT

The discrete fractional Fourier transform (DFRFT) whose order parameter is a rational number has special interesting properties that ordinary DFRFT does not possess. In this paper, periodicity and eigendecomposition properties of the rational-ordered DFRFT (RODFRFT) are investigated. We find that RODFRFT must be periodic and periods of RODFRFT are derived for all possible orders. As to the eigendecomposition of RODFRFT, we first derive eigenvalue multiplicities of the RODFRFT of order $4/p$, where $p$ is its period. The results are then generalized to RODFRFT of any rational orders.

***Index Terms*—**Fractional Fourier transform, discrete fractional Fourier transform, DFT, Hermite-Gaussian function, eigenvalue multiplicity

## 1. INTRODUCTION

The discrete fractional Fourier transform (DFRFT) is the generalization of the DFT with one additional real parameter [1]-[2]. An eigendecomposition-based DFRFT whose output can approximate samples of the continuous fractional Fourier transform (FRT) has been defined [1]-[2]. The continuous FRT is useful in signal processing and optics [3]. In [1], Pei and Yeh compute eigenvectors of the DFRFT by those of the Dickinson-Steiglitz DFT-commuting matrix [4]. Eigenvectors of the Dickinson-Steiglitz DFT-commuting matrix are discrete Hermite-Gaussian functions (HGF) [1]-[2]. This is one of the main reasons that the DFRFT output is similar to the FRT [1]. In order to define DFRFTs closer to the FRT, several DFT-commuting matrices with even finer discrete HGF DFT eigenvectors have been proposed [5]-[8].

Discrete transforms frequently used in signal processing have multiple eigenvalues, e.g. DFT, DCT, DST, etc. Therefore, it is important to understand eigenvalue multiplicities of discrete transforms. Eigenvalue multiplicities are useful to fractionalize discrete transforms. To define DFRFT, eigenvalue multiplicities of the DFT, discovered by McClellan and Parks [9], are used in [1]. Eigenvalue multiplicities for DCT and DST are derived for defining their corresponding fractional transforms [10].

An $N\times N$ discrete transform $\mathbf{T}$ is defined to be periodic with period $p$ (a positive integer) if

$$\mathbf{T}^p = \mathbf{I}\,, \tag{1}$$

where $\mathbf{I}$ is the identity matrix. Periodic transform has special properties [6]. For example,

$$\mathbf{A} = \mathbf{M} + \mathbf{TMT}^{-1} + \mathbf{T}^2\mathbf{MT}^{-2} + \cdots + \mathbf{T}^{p-1}\mathbf{MT}^{-(p-1)} \tag{2}$$

is a commuting matrix of $\mathbf{T}$ (i.e., $\mathbf{TA}=\mathbf{AT}$), where $\mathbf{M}$ is an arbitrary $N\times N$ matrix [6]. Eigenvectors of $\mathbf{T}$ can then be computed from those of its commuting matrix $\mathbf{A}$ [6].

The $N\times N$ DFT matrix $\mathbf{F}$ is defined as

$$[\mathbf{F}]_{k,n} = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}kn}, \quad 0 \le k, n \le N-1\,. \tag{3}$$

Let the eigendecomposition of the DFT $\mathbf{F}$ be

$$\mathbf{F} = \sum_{k=0}^{N-1} \lambda_k \mathbf{e}_k \mathbf{e}_k^T\,, \tag{4}$$

where $\mathbf{e}_k$, $k$=0, 1, …, ($N$-1), are an orthonormal eigenvector basis and $\lambda_k$ are eigenvalues of $\mathbf{F}$. It is known that $\mathbf{F}$ has only 4 distinct eigenvalues {1, -$j$, -1, $j$} and their multiplicities are given in TABLE I [9]. The $N\times N$ DFRFT $\mathbf{F}^a$ with one order parameter $a$ is defined by [1]-[2]:

$$\mathbf{F}^a = \sum_{k=0}^{N-1} \lambda_k^a \mathbf{e}_k \mathbf{e}_k^T\,. \tag{5}$$

TABLE I EIGENVALUE MULTIPLICITIES OF THE $N\times N$ DFT $\mathbf{F}$

| $\lambda$ | $e^{-j\frac{2\pi}{4}\cdot 0}$ | $e^{-j\frac{2\pi}{4}\cdot 1}$ | $e^{-j\frac{2\pi}{4}\cdot 2}$ | $e^{-j\frac{2\pi}{4}\cdot 3}$ |
|---|---|---|---|---|
| $N$=4$m$ | $m$+1 | $m$ | $m$ | $m$-1 |
| $N$=4$m$+1 | $m$+1 | $m$ | $m$ | $m$ |
| $N$=4$m$+2 | $m$+1 | $m$ | $m$+1 | $m$ |
| $N$=4$m$+3 | $m$+1 | $m$+1 | $m$+1 | $m$ |

## 2. PERIODICITY OF THE RODFRFT

Generally, the DFRFT $\mathbf{F}^a$ is not periodic. Let $r$ and $s$ be two coprime integers, i.e., ($r$, $s$)=1. The $N\times N$ RODFRFT is the DFRFT whose order parameter is a rational number, i.e., $a = s/r$ in (5).The resulting RODFRFT $\mathbf{F}^{s/r}$ has a special

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property that it is periodic. Peridicity of $\mathbf{F}^{s/r}$ can be derived from the fact that $\mathbf{F}^4=\mathbf{I}$ as follows:

1) When $s$ is odd: $\mathbf{F}^{s/r}$ is periodic with period $p=4r$, since $4r$ the smallest positive integer power of $\mathbf{F}^{s/r}$ that is equal to $\mathbf{I}$. This can be seen by $(\mathbf{F}^{s/r})^{4r}=\mathbf{F}^{4s}=\mathbf{I}$.

2) When $s=2b$ with $b$ being an odd integer: $\mathbf{F}^{s/r}$ is periodic with period $p=2r$, since $(\mathbf{F}^{s/r})^{2r}=\mathbf{F}^{4b}=\mathbf{I}$.

3) When $s=4b$ with $b$ being any integer: $\mathbf{F}^{s/r}$ is periodic with period $p=r$, since $(\mathbf{F}^{s/r})^{r}=\mathbf{F}^{4b}=\mathbf{I}$.

We conclude that the period $p$ of the RODFRFT $\mathbf{F}^{s/r}$ is:

$$p=\begin{cases}4r, \text{if } s \text{ is odd,}\\ 2r, \text{if } s=2b \text{ with } b \text{ being odd,}\\ r, \text{if } s=4b.\end{cases} \tag{6}$$

Let the period of $\mathbf{F}^{s/r}$ be $p$ as given by (6). Then the eigenvalues of $\mathbf{F}^{s/r}$ must be $p$th roots of unity.

*Property 1:* The eigenvalues $\lambda$ of $\mathbf{F}^{s/r}$ must satisfy

$$\lambda^p=1. \tag{7}$$

## 3. EIGENDECOMPOSITION OF THE RODFRFT

In this section, we will discuss the eigendecomposition of the RODFRFT which can approximate samples of its corresponding rational-ordered FRT. We will first derive eigenvalue multiplicities of $\mathbf{F}^{4/p}$. The result will then be generalized to DFRFT of arbitrary rational orders $\mathbf{F}^{s/r}$.

### 3.1. DFRFT approximating the continuous FRT

It is known that the following $N\times N$ DFRFT definition with fractional order $a$ can approximate its corresponding continuous FRT [1]-[2]:

$$\mathbf{F}^a=\begin{cases}\sum_{k=0}^{N-1} e^{-j\frac{\pi}{2}ka}\mathbf{h}_k\mathbf{h}_k^T, \text{ if } N \text{ is odd}\\ \sum_{k=0}^{N-2} e^{-j\frac{\pi}{2}ka}\mathbf{h}_k\mathbf{h}_k^T+e^{-j\frac{\pi}{2}N\cdot a}\mathbf{h}_N\mathbf{h}_N^T, \text{ if } N \text{ is even}\end{cases} \tag{8}$$

where $\mathbf{h}_k$, $k=0,1,2,\ldots$, are orthonormal and are $k$th-order discrete Hermite-Gaussian function (HGF) DFT eigenvectors with $k$ zero-crossings. There are several methods that can be used to compute $\mathbf{h}_k$ from various DFT commuting matrices in the literature [4]-[8]. Note that the definition of $\mathbf{F}^a$ in (8) has different forms for odd $N$ and even $N$ because [1]-[2]:

1) When $N$ is odd: The orders (zero-crossings) of the discrete HGF DFT eigenvectors of $\mathbf{F}^a$ are 0, 1, 2, …, ($N$-1).

2) When $N$ is even: The largest order of the discrete HGF DFT eigenvectors of $\mathbf{F}^a$ is $N$, instead of ($N$-1) for the above DFRFT with odd $N$. Consequently, the orders of the discrete HGF DFT eigenvectors of $N\times N$ $\mathbf{F}^a$ with even $N$ are 0, 1, 2, …, ($N$-2), $N$.

TABLE II EIGENVALUES AND CORRESPONDING DISCRETE HGF DFT EIGENVECTORS OF $\mathbf{F}^{4/p}$

| | $\lambda$ | $e^{-j\frac{2\pi}{p}\cdot 0}$ | $e^{-j\frac{2\pi}{p}\cdot 1}$ | $\cdots$ | $e^{-j\frac{2\pi}{p}\cdot (p-1)}$ |
|---|---|---|---|---|---|
| eigenvectors | | $\mathbf{h}_0$ | $\mathbf{h}_1$ | $\cdots$ | $\mathbf{h}_{p-1}$ |
| | | $\mathbf{h}_p$ | $\mathbf{h}_{p+1}$ | $\cdots$ | $\mathbf{h}_{2p-1}$ |
| | | $\mathbf{h}_{2p}$ | $\mathbf{h}_{2p+1}$ | $\cdots$ | $\mathbf{h}_{3p-1}$ |
| | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

TABLE III EIGENVALUES AND CORRESPONDING DISCRETE HGF DFT EIGENVECTORS OF 16×16 RODFRFT $\mathbf{F}^{4/6}$

| | $\lambda$ | $e^{-j\frac{2\pi}{6}\cdot 0}$ | $e^{-j\frac{2\pi}{6}\cdot 1}$ | $e^{-j\frac{2\pi}{6}\cdot 2}$ | $e^{-j\frac{2\pi}{6}\cdot 3}$ | $e^{-j\frac{2\pi}{6}\cdot 4}$ | $e^{-j\frac{2\pi}{6}\cdot 5}$ |
|---|---|---|---|---|---|---|---|
| eigenvectors | | $\mathbf{h}_0$ | $\mathbf{h}_1$ | $\mathbf{h}_2$ | $\mathbf{h}_3$ | $\mathbf{h}_4$ | $\mathbf{h}_5$ |
| | | $\mathbf{h}_6$ | $\mathbf{h}_7$ | $\mathbf{h}_8$ | $\mathbf{h}_9$ | $\mathbf{h}_{10}$ | $\mathbf{h}_{11}$ |
| | | $\mathbf{h}_{12}$ | $\mathbf{h}_{13}$ | $\mathbf{h}_{14}$ | | $\mathbf{h}_{16}$ | |

TABLE IV EIGENVALUE MULTIPLICITIES OF $N\times N$ $\mathbf{F}^{4/6}$

| $\lambda$ | $e^{-j\frac{2\pi}{6}\cdot 0}$ | $e^{-j\frac{2\pi}{6}\cdot 1}$ | $e^{-j\frac{2\pi}{6}\cdot 2}$ | $e^{-j\frac{2\pi}{6}\cdot 3}$ | $e^{-j\frac{2\pi}{6}\cdot 4}$ | $e^{-j\frac{2\pi}{6}\cdot 5}$ |
|---|---|---|---|---|---|---|
| $N=6m$ | $m+1$ | $m$ | $m$ | $m$ | $m$ | $m-1$ |
| $N=6m+1$ | $m+1$ | $m$ | $m$ | $m$ | $m$ | $m$ |
| $N=6m+2$ | $m+1$ | $m$ | $m+1$ | $m$ | $m$ | $m$ |
| $N=6m+3$ | $m+1$ | $m+1$ | $m+1$ | $m$ | $m$ | $m$ |
| $N=6m+4$ | $m+1$ | $m+1$ | $m+1$ | $m$ | $m+1$ | $m$ |
| $N=6m+5$ | $m+1$ | $m+1$ | $m+1$ | $m+1$ | $m+1$ | $m$ |

TABLE V EIGENVALUE MULTIPLICITIES OF $N\times N$ $\mathbf{F}^{4/3}$

| $\lambda$ | $e^{-j\frac{2\pi}{3}\cdot 0}$ | $e^{-j\frac{2\pi}{3}\cdot 1}$ | $e^{-j\frac{2\pi}{3}\cdot 2}$ |
|---|---|---|---|
| $N=6m$ | $2m+1$ | $2m$ | $2m-1$ |
| $N=6m+1$ | $2m+1$ | $2m$ | $2m$ |
| $N=6m+2$ | $2m+1$ | $2m$ | $2m+1$ |
| $N=6m+3$ | $2m+1$ | $2m+1$ | $2m+1$ |
| $N=6m+4$ | $2m+1$ | $2m+2$ | $2m+1$ |
| $N=6m+5$ | $2m+2$ | $2m+2$ | $2m+1$ |

### 3.2. Eigendecomposition of $\mathbf{F}^{4/p}$

We will first discuss eigendecomposition of the RODFRFT $\mathbf{F}^{4/p}$ which approximates its corresponding FRT:

$$\mathbf{F}^{4/p}=\begin{cases}\sum_{k=0}^{N-1}\left(e^{-j\frac{\pi}{2}k}\right)^{4/p}\mathbf{h}_k\mathbf{h}_k^T, \text{ if } N \text{ is odd}\\ \sum_{k=0}^{N-2}\left(e^{-j\frac{\pi}{2}k}\right)^{4/p}\mathbf{h}_k\mathbf{h}_k^T+e^{-j\frac{\pi}{2}N\cdot(4/p)}\mathbf{h}_N\mathbf{h}_N^T, \text{ if } N \text{ is even.}\end{cases} \tag{9}$$

*Property 2:* If $k=q$ (mod $p$), then the discrete HGF DFT eigenvector $\mathbf{h}_k$ in (9) is the eigenvector of $\mathbf{F}^{4/p}$ with corresponding eigenvalue $\exp(-j\frac{2\pi}{p}\cdot q)$.

*Proof:* From (9), $\mathbf{h}_k$ is the eigenvector of $\mathbf{F}^{4/p}$. The corresponding eigenvalue of $\mathbf{h}_k$ is:

$\exp(-j\frac{2\pi}{p}\cdot k)=\exp(-j\frac{2\pi}{p}\cdot(np+q))=\exp(-j\frac{2\pi}{p}\cdot q)$ Q.E.D.

From *Property 2* and (9), we can collect eigenvectors with the same eigenvalues and rewrite (9) as:

$$\mathbf{F}^{4/p}=\sum_{q=0}^{p-1}\left(\sum_{n=0}e^{-j\frac{2\pi}{p}q}\mathbf{h}_{np+q}\mathbf{h}_{np+q}^{T}\right), \qquad (10)$$

where the upper bound in inner summation with index $n$ is equal to the largest $n$ value such that $np+q$ is an element of {0, 1, 2, …, ($N$-1)} if $N$ is odd or {0, 1, 2, …, ($N$-2), $N$ } if $N$ is even.

From (6), the period of $\mathbf{F}^{4/p}$ is $p$ and $\mathbf{F}^{4/p}$ has $p$ distinct eigenvalues. Consequently, from (10), eigenvalues and their corresponding discrete HGF DFT eigenvectors are conceptually shown in TABLE II. As an example, eigenvalues and eigenvectors of 16×16 RODFRFT $\mathbf{F}^{4/6}$ can be obtained as in TABLE III. In TABLE III, we should note that there is a jump in the largest and second largest orders of the HGF DFT eigenvectors since its $N$ is even. The HGF DFT eigenvectors of the largest order of 16×16 $\mathbf{F}^{4/6}$ is $\mathbf{h}_{16}$ and its corresponding eigenvalue is $\exp(-j\frac{2\pi}{6}\cdot(16 \bmod 6))$. We can generalize this example and use TABLE II to derive eigenvalue multiplicities of $N$×$N$ $\mathbf{F}^{4/p}$ as follows:

1) If $p$ is even: We first observe that there are four cases of eigenvalue multiplicities for the DFT **F** in TABLE I, since **F** has 4 distinct eigenvalues. Similarly, because $\mathbf{F}^{4/p}$ has $p$ distinct eigenvalues, we need to consider $p$ cases of $N$ to derive eigenvalue multiplicities of $\mathbf{F}^{4/p}$. These $p$ cases are for $N$=$pm$, $pm$+1, $pm$+2, …, $pm$+($p$-1). For each case, the eigenvalue multiplicities can be obtained by first finishing the DFT HGF eigenvectors table in TABLE II and then the eigenvalue multiplicities are equal to their corresponding numbers of different DFT HGF eigenvectors. For example, from TABLE III, the eigenvalues of 16×16 $\mathbf{F}^{4/6}$ are { $\exp(-j\frac{2\pi}{6}\cdot 0)$ , $\exp(-j\frac{2\pi}{6}\cdot 1)$ , …, $\exp(-j\frac{2\pi}{6}\cdot 5)$ } and their corresponding multiplicities are {3,3,3,2,3,2}. The complete eigenvalue multiplicities table of $N$×$N$ $\mathbf{F}^{4/6}$ is shown in TABLE IV. It is easy to generalize TABLE IV for $N$×$N$ $\mathbf{F}^{4/6}$ to the general case for $N$×$N$ $\mathbf{F}^{4/p}$. To avoid the unnecessary complexity, the details are omitted. It is interesting that the eigenvalue multiplicities of the DFT **F** in TABLE I can also be obtained by the method for this case.

2) If $p$ is odd: For this case, to derive eigenvalue multiplicities of $N$×$N$ $\mathbf{F}^{4/p}$, it should be noted that we cannot simply consider only the $p$ cases of $N$=$pm$, $pm$+1, $pm$+2, …, $pm$+($p$-1), because for odd $p$ the even or odd property of $N$=$pm$, $pm$+1, $pm$+2, …, $pm$+($p$-1) is ambiguous and TABLE III for this case cannot be constructed. However, if we consider instead the 2$p$ cases $N$=2$pm$, 2$pm$+1, 2$pm$+2, …, 2$pm$+(2$p$-1), then the corresponding table of $N$×$N$ $\mathbf{F}^{4/p}$ can be constructed for each $N$ and its eigenvalue multiplicities can be derived using the previous method described. As an example, eigenvalue multiplicities of $N$×$N$ $\mathbf{F}^{4/3}$ are given in TABLE V.

After understanding eigenvalue multiplicities for the RODFRFT $\mathbf{F}^{4/p}$, we will discuss eigendecomposition property for the general RODFRFT $\mathbf{F}^{s/r}$ as follows.

### 3.3. Eigendecomposition of $\mathbf{F}^{s/r}$

Similar to discussing periodicity of $\mathbf{F}^{s/r}$, there are three cases for eigenvalue multiplicities of $\mathbf{F}^{s/r}$:

1) When $s$ is odd: From (10), we have

$$\mathbf{F}^{s/r}=\mathbf{F}^{\frac{4s}{4r}}=(\mathbf{F}^{\frac{4}{4r}})^{s}=(\mathbf{F}^{\frac{4}{p}})^{s}=\sum_{q=0}^{p-1}\left(\sum_{n=0}e^{-j\frac{2\pi}{p}q\cdot s}\mathbf{h}_{np+q}\mathbf{h}_{np+q}^{T}\right), \qquad (11)$$

where $p$≡4$r$ is the period of $\mathbf{F}^{s/r}$ and ($p$,$s$)=(4$r$,$s$)=1. Comparing (11) and (10), we can see that the eigenvalue multiplicities table of $\mathbf{F}^{s/r}$ is the same as that of $\mathbf{F}^{4/(4r)}\equiv\mathbf{F}^{4/p}$ , but with corresponding eigenvalues of $\exp(-j\frac{2\pi}{p}q)$, $q$=0, 1, 2, …, ($p$-1), for $\mathbf{F}^{4/p}$ being replaced by $\exp(-j\frac{2\pi}{p}qs)$ ,$q$=0, 1, 2, …, ($p$-1), for $\mathbf{F}^{s/r}$. Since $p$ and $s$ are coprime, the result is that eigenvalues of $\mathbf{F}^{s/r}$ are appropriate permutations of eigenvalues of $\mathbf{F}^{4/p}$ . For example, since $\mathbf{F}^{3/2}=\mathbf{F}^{12/8}=(\mathbf{F}^{4/8})^{3}$ , if we want to construct eigenvalue multiplicities table of $N$×$N$ $\mathbf{F}^{3/2}$, we can first construct the eigenvalue multiplicity table of $N$×$N$ $\mathbf{F}^{4/8}$ using the method described in Subsection 3.2 and then map the resulting eigenvalues $\exp(-j\frac{2\pi}{8}q)$ , $q$=0, 1, 2, …, 7, of $\mathbf{F}^{4/8}$ to eigenvalues $\exp(-j\frac{2\pi}{8}q\cdot 3)$ , $q$=0, 1, 2, …, 7, of $\mathbf{F}^{3/2}$ to get the eigenvalue multiplicity table of $(\mathbf{F}^{4/8})^{3}=\mathbf{F}^{3/2}$ . The resulting eigenvalue multiplicities tables for $N$×$N$ $\mathbf{F}^{4/8}$ and $\mathbf{F}^{3/2}$ are given in TABLE VI.

2) When $s$=2$b$ with $b$ being odd: For this case,

$$\mathbf{F}^{s/r}=\mathbf{F}^{\frac{2s}{2r}}=(\mathbf{F}^{\frac{4}{2r}})^{\frac{s}{2}}=(\mathbf{F}^{\frac{4}{p}})^{\frac{s}{2}}, \qquad (12)$$

where $p$≡2$r$ is the period of $\mathbf{F}^{s/r}$. One should note that in (12), (2$r$,$s$/2)=($p$,$s$/2)=1. As a consequence, for this case, the eigenvalue multiplicities table of $\mathbf{F}^{s/r}$ is the same as that of $\mathbf{F}^{4/2r}\equiv\mathbf{F}^{4/p}$, but with the eigenvalues being appropriately permuted. For example, since $\mathbf{F}^{6/5}=\mathbf{F}^{12/10}=(\mathbf{F}^{4/10})^{3}$, the eigenvalue multiplicity table of $N$×$N$ $\mathbf{F}^{6/5}$ and $\mathbf{F}^{4/10}$ are the same, but with the eigenvalues $\exp(-j\frac{2\pi}{10}q)$ , $q$=0, 1, 2, …, 9, for $\mathbf{F}^{4/10}$ being replaced by their corresponding powers of 3 (that is, $\exp(-j\frac{2\pi}{10}q\cdot 3)$, $q$=0, 1, 2, …, 9) for eigenvalues of $\mathbf{F}^{6/5}$.

TABLE VI EIGENVALUE MULTIPLICITIES OF $N\times N$ $\mathbf{F}^{4/8}$ AND $\mathbf{F}^{3/2}$

| $\lambda$ (of $\mathbf{F}^{4/8}$) | $e^{-j\frac{2\pi}{8}\cdot 0}$ | $e^{-j\frac{2\pi}{8}\cdot 1}$ | $e^{-j\frac{2\pi}{8}\cdot 2}$ | $e^{-j\frac{2\pi}{8}\cdot 3}$ | $e^{-j\frac{2\pi}{8}\cdot 4}$ | $e^{-j\frac{2\pi}{8}\cdot 5}$ | $e^{-j\frac{2\pi}{8}\cdot 6}$ | $e^{-j\frac{2\pi}{8}\cdot 7}$ |
|---|---|---|---|---|---|---|---|---|
| $\lambda$ (of $\mathbf{F}^{3/2}$) | $e^{-j\frac{2\pi}{8}\cdot 0}$ | $e^{-j\frac{2\pi}{8}\cdot 3}$ | $e^{-j\frac{2\pi}{8}\cdot 6}$ | $e^{-j\frac{2\pi}{8}\cdot 1}$ | $e^{-j\frac{2\pi}{8}\cdot 4}$ | $e^{-j\frac{2\pi}{8}\cdot 7}$ | $e^{-j\frac{2\pi}{8}\cdot 2}$ | $e^{-j\frac{2\pi}{8}\cdot 5}$ |
| $N$=8$m$ | $m$+1 | $m$ | $m$ | $m$ | $m$ | $m$ | $m$ | $m$-1 |
| $N$=8$m$+1 | $m$+1 | $m$ | $m$ | $m$ | $m$ | $m$ | $m$ | $m$ |
| $N$=8$m$+2 | $m$+1 | $m$ | $m$+1 | $m$ | $m$ | $m$ | $m$ | $m$ |
| $N$=8$m$+3 | $m$+1 | $m$+1 | $m$+1 | $m$ | $m$ | $m$ | $m$ | $m$ |
| $N$=8$m$+4 | $m$+1 | $m$+1 | $m$+1 | $m$ | $m$+1 | $m$ | $m$ | $m$ |
| $N$=8$m$+5 | $m$+1 | $m$+1 | $m$+1 | $m$+1 | $m$+1 | $m$ | $m$ | $m$ |
| $N$=8$m$+6 | $m$+1 | $m$+1 | $m$+1 | $m$+1 | $m$+1 | $m$ | $m$+1 | $m$ |
| $N$=8$m$+7 | $m$+1 | $m$+1 | $m$+1 | $m$+1 | $m$+1 | $m$+1 | $m$+1 | $m$ |

3) When $s$=4$b$: For this case,

$$\mathbf{F}^{s/r} = (\mathbf{F}^{4/r})^{s/4}, \qquad (13)$$

where $(r,s/4)$=1. In (13), $r \equiv p$ is the period of $\mathbf{F}^{s/r}$. Therefore, in this case, the eigenvalue multiplicities table of $\mathbf{F}^{s/r}$ is the same as that for $\mathbf{F}^{4/r} \equiv \mathbf{F}^{4/p}$, but with eigenvalues of $\mathbf{F}^{s/r}$ being powers of $s$/4 for those of $\mathbf{F}^{4/r}$. For example, since $\mathbf{F}^{16/5}$ = $(\mathbf{F}^{4/5})^4$, the eigenvalue multiplicities table of $\mathbf{F}^{16/5}$ is the same as that of $\mathbf{F}^{4/5}$, but with eigenvalues $\exp(-j\frac{2\pi}{5}q)$, $q$=0, 1, 2, 3, 4, for $\mathbf{F}^{4/5}$ being respectively replaced by $\exp(-j\frac{2\pi}{5}q\cdot 4)$, $q$=0, 1, 2, 3, 4, for eigenvalues of $\mathbf{F}^{16/5}$.

## 4. CONCLUSION

We find that the RODFRFTs $\mathbf{F}^{s/r}$ are periodic for all integers $r$ and $s$. All of the periods of $\mathbf{F}^{s/r}$ are derived from the fact that the DFT matrix $\mathbf{F}$ is of period 4. From eigendecomposition of $\mathbf{F}^{s/r}$ which approximates its corresponding continuous FRT, eigenvalue multiplicities of $N\times N$ $\mathbf{F}^{4/p}$ with $p$ being the period is first derived for both cases of even and odd $p$. Eigenvalue multiplicities of general $\mathbf{F}^{s/r}$ are then derived from $\mathbf{F}^{4/p}$ according to three cases of periods for $\mathbf{F}^{s/r}$. It is known that the RODFRFT can be applied as an alternative method to compute the DFRFT of any irrational orders [11]. Except for periodicity and eigendecomposition properties of the RODFRFT, we believe that there should be other special properties and applications of the RODFRFT which are valuable for further researches.


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