ROBUST COHERENCE ANALYSIS IN THE FREQUENCY DOMAIN

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ABSTRACT

In this paper, a new cross periodogram, called Laplace cross periodogram, is introduced for robust coherence analysis of multiple time series in the frequency domain. It is derived by replacing the ordinary Fourier transform that defines the ordinary cross periodogram with what we call the Laplace-Fourier transform obtained from trigonometric least-absolute-deviations (LAD) regression. Under certain stationarity assumptions, the Laplace cross periodogram is found through an asymptotic analysis to be associated with what we call the Laplace cross spectrum, a function proportional to the Fourier transform of cross zero-crossing rates. Robustness of the Laplace cross periodogram and the corresponding coherency estimator is demonstrated by numerical examples.

1. INTRODUCTION

The objective of coherence analysis is to discover and quantify the mutual dependence of multiple time series. Coherence analysis in the frequency domain is traditionally done through the cross periodogram that can be regarded as an estimate of the cross spectrum defined as the Fourier transform of the lagged covariances for stationary multiple random processes [1]. Although effective and powerful under regular conditions, the cross periodogram lacks robustness for analyzing signals contaminated by outliers or time series with heavy-tailed distributions. Recently, a new periodogram, called Laplace periodogram, is introduced in [2] as a robust alternative to the ordinary periodogram for analyzing single time series. In this paper, we extend this methodology to multiple time series and introduce a robust cross periodogram called Laplace cross periodogram and a robust coherency called Laplace coherency.

For each $k = 1, \ldots, p$, let $\{Y_{tk}\}$ $(t = 1, \ldots, n)$ be a real-valued time series of length n and let

$$Z_{nk}(\omega) := n^{-1/2} \sum_{t=1}^{n} Y_{tk} \exp(-it\omega)$$

be the Fourier transform of $\{Y_{tk}\}$. The p-by-p matrix of periodogram and cross periodogram for these time series is defined as [3]

$$\mathbf{G}_n(\omega) := [G_{njk}(\omega)]_{j,k=1}^{p} = \mathbf{Z}_n(\omega)\mathbf{Z}_n^H(\omega), \qquad (1)$$

where $\mathbf{Z}_n(\omega) := [Z_{n1}(\omega), \ldots, Z_{np}(\omega)]^T$ and $G_{njk}(\omega) := Z_{nj}(\omega) Z_{nk}^*(\omega)$. Under the assumption that $\{Y_{tj}\}$ and $\{Y_{tk}\}$ are jointly stationary in second moments with mean zero and cross covariances $r_{jk}(\tau) := E\{Y_{t+\tau,j} Y_{tk}\}$, a smoothed cross periodogram, denoted as $\tilde{G}_{njk}(\omega)$, can be used to estimate the cross spectrum defined by

$$g_{jk}(\omega) := \sum_{\tau=-\infty}^{\infty} r_{jk}(\tau) \exp(-i\tau\omega).$$

Moreover, for any $j \neq k$, the coherency between $\{Y_{nj}\}$ and $\{Y_{nk}\}$ at frequency ω, defined as $c_{jk}(\omega) := g_{jk}(\omega)/\sqrt{g_{jj}(\omega) g_{kk}(\omega)}$, can be estimated by [3]

$$C_{njk}(\omega) := \frac{\tilde{G}_{njk}(\omega)}{\sqrt{\tilde{G}_{njj}(\omega)\tilde{G}_{nkk}(\omega)}}. \qquad (2)$$

The coherency $c_{jk}(\omega)$ is an ensemble measure of correlation between the trigonometric components of $\{Y_{nj}\}$ and $\{Y_{nk}\}$ at frequency ω.

The cross periodogram defined by (1) and the coherency estimator defined by (2) lack sufficient robustness against outliers because the Fourier transform is a linear function of the data points. To improve the robustness, we introduce in the next section an alternative transformation motivated by a relationship between the ordinary cross periodogram and the least-squares (LS) trigonometric regression.

2. LAPLACE CROSS PERIODOGRAM

Let $\tilde{\boldsymbol{\beta}}_{nk}(\omega) := [\tilde{\beta}_{nk1}(\omega), \tilde{\beta}_{nk2}(\omega)]^T$ denote the LS solution of the following trigonometric regression problem:

$$\tilde{\boldsymbol{\beta}}_{nk}(\omega) := \arg\min_{\boldsymbol{\beta} \in \mathbb{R}^2} \sum_{t=1}^{n} |Y_{tk} - \boldsymbol{c}_t^T(\omega)\boldsymbol{\beta}|^2, \qquad (3)$$

where $\boldsymbol{c}_t(\omega) := [\cos(\omega t), \sin(\omega t)]^T$. If ω is a Fourier frequency, i.e., an integer multiple of $2\pi/n$, then it is easy to show that $\sum_{t=1}^{n} \boldsymbol{c}_t(\omega)\boldsymbol{c}_t^T(\omega) = n/2$. In this case, $\tilde{\boldsymbol{\beta}}_{nk}(\omega) = 2n^{-1}\sum_{t=1}^{n} Y_{tk}\boldsymbol{c}_t(\omega)$ and hence

$$Z_{nk}(\omega) = \tfrac{1}{2}\sqrt{n}\{\tilde{\beta}_{nk1}(\omega) - i\tilde{\beta}_{nk2}(\omega)\}. \tag{4}$$

For improved robustness, we replace the LS solution in (3) by the least-absolute-deviations (LAD) solution

$$\begin{aligned} \boldsymbol{\beta}_{nk}(\omega) &:= [\beta_{nk1}(\omega), \beta_{nk2}(\omega)]^T \\ &:= \arg\min_{\boldsymbol{\beta}\in\mathbb{R}^2} \sum_{t=1}^{n} |Y_{tk} - \boldsymbol{c}_t^T(\omega)\boldsymbol{\beta}| \end{aligned} \tag{5}$$

and define a new transform in the same way as (4):

$$U_{nk}(\omega) := \tfrac{1}{2}\sqrt{n}\{\beta_{nk1}(\omega) - i\beta_{nk2}(\omega)\}. \tag{6}$$

We call $U_{nk}(\omega)$ the *Laplace-Fourier transform* of $\{Y_{tk}\}$. The LAD method is known to produce more robust solutions for linear regression problems than the LS method [4]. The Laplace-Fourier transform $U_{nk}(\omega)$ is expected to have the same benefit.

With the Laplace-Fourier transform in place of the ordinary Fourier transform in (1), we obtain a new matrix of periodogram and cross periodogram

$$\mathbf{L}_n(\omega) := [L_{njk}(\omega)]_{j,k=1}^{p} = \mathbf{U}_n(\omega)\mathbf{U}_n^H(\omega), \tag{7}$$

where $\mathbf{U}_n(\omega) := [U_{n1}(\omega), \ldots, U_{np}(\omega)]^T$. We call

$$L_{njk}(\omega) := U_{nj}(\omega)U_{nk}^*(\omega)$$

the *Laplace cross periodogram* of $\{Y_{tj}\}$ and $\{Y_{tk}\}$. It is easy to see that in the special case of $j = k$, we have

$$L_{nkk}(\omega) = |U_{nk}(\omega)|^2 = \tfrac{1}{4}n\|\boldsymbol{\beta}_{nk}(\omega)\|^2.$$

This is nothing but the Laplace periodogram of $\{Y_{tk}\}$ discussed in [2]. Therefore, the Laplace cross periodogram is an extension of the Laplace periodogram for multiple time series.

Unlike the ordinary cross periodogram which is related to the ordinary cross spectrum $g_{jk}(\omega)$, the Laplace cross periodogram is associated with what we call the Laplace cross spectrum. This relationship is revealed through an asymptotic analysis in the next section.

3. ASYMPTOTIC DISTRIBUTION

Assume that for each k the random process $\{Y_{tk}\}$ has zero median and is stationary in zero-crossings [2]. Then, it can be shown [2] that as $n \to \infty$,

$$U_{nk}(\omega) = (1/\dot{F}_k(0))\xi_{nk}(\omega) + o_P(1), \tag{8}$$

where $\dot{F}_k(0)$ denotes the density of $\{Y_{tk}\}$ at zero and

$$\xi_{nk}(\omega) := n^{-1/2}\sum_{t=1}^{n} \psi(Y_{tk})\exp(-it\omega)$$

with $\psi(u) := 1/2 - I(u < 0) = (1/2)\,\mathrm{sgn}(u)$.

For $j \neq k$, assume that $\{Y_{tj}\}$ and $\{Y_{tk}\}$ are jointly stationary in zero-crossings in the sense that there exists a function $\gamma_{jk}(\tau)$ such that for all t and τ,

$$P\{Y_{t+\tau,j}Y_{tk} < 0\} = \gamma_{jk}(\tau).$$

We call $\gamma_{jk}(\tau)$ the *lag-τ cross zero-crossing rate* between $\{Y_{tj}\}$ and $\{Y_{tk}\}$. We refer to the Fourier transform of the cross zero-crossing rates, defined by

$$f_{jk}(\omega) := \sum_{\tau=-\infty}^{\infty} \{1 - 2\gamma_{jk}(\tau)\}\exp(-i\tau\omega),$$

as the *cross zero-crossing spectrum*. Note that $\gamma_{kk}(\tau)$ and $f_{kk}(\omega)$ are nothing but the zero-crossing rates and the zero-crossing spectrum of $\{Y_{tk}\}$ discussed in [2].

Consider $\boldsymbol{\xi}_n(\omega) := [\xi_{n1}(\omega), \ldots, \xi_{np}(\omega)]^T$. Because $E\{\psi(Y_{t+\tau,j})\psi(Y_{tk})\} = \frac{1}{4}\{1 - 2\gamma_{jk}(\tau)\}$, it follows that

$$\lim_{n\to\infty} E\{\boldsymbol{\xi}_n(\omega)\boldsymbol{\xi}_n^H(\omega)\} = \tfrac{1}{4}\boldsymbol{\Lambda}(\omega),$$

where $\boldsymbol{\Lambda}(\omega) := [f_{jk}(\omega)]_{j,k=1}^{p}$. This, together with a central limit theorem [2], yields

$$\boldsymbol{\xi}_n(\omega) \xrightarrow{D} N_c(\mathbf{0}, \tfrac{1}{4}\boldsymbol{\Lambda}(\omega)), \tag{9}$$

where $N_c(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes a complex Gaussian distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Moreover, let $\eta_k := 1/\{2\dot{F}_k(0)\}$ and $\mathbf{S}(\omega) := [\eta_j\eta_k f_{jk}(\omega)]_{j,k=1}^{p}$. Then, by combining (9) with (8), we obtain

$$\mathbf{U}_n(\omega) \xrightarrow{D} N_c(\mathbf{0}, \mathbf{S}(\omega)). \tag{10}$$

As can be seen from this result, the asymptotic distribution of $\mathbf{U}_n(\omega)$ is determined completely by

$$\ell_{jk}(\omega) := \eta_j\eta_k f_{jk}(\omega). \tag{11}$$

We call $\ell_{jk}(\omega)$ the *Laplace cross spectrum*. By definition, the Laplace cross spectrum is proportional to the cross zero-crossing spectrum, where the proportionality is determined by the densities of the zero-median time series at zero. With this notation, $\mathbf{S}(\omega) = [\ell_{jk}(\omega)]_{j,k=1}^{p}$ is simply the Laplace spectral matrix.

According to (10), the Laplace periodogram matrix $\mathbf{L}_n(\omega)$ is asymptotically distributed as $\mathbf{X}(\omega)\mathbf{X}^H(\omega)$, which we denote as

$$\mathbf{L}_n(\omega) \overset{A}{\sim} \mathbf{X}(\omega)\mathbf{X}^H(\omega), \tag{12}$$

where $\mathbf{X}(\omega) := [X_1(\omega), \ldots, X_p(\omega)]^T$ is a complex Gaussian random vector with distribution $N_c(\mathbf{0}, \mathbf{S}(\omega))$. This result is the counterpart of a classical assertion for the ordinary periodogram matrix [3, Theorem 11.7.1]. Note that (12) holds even if the time series do not have finite moments. This is a manifestation of robustness to heavy-tailed distributions.

Because $E\{X_j(\omega)X_k^*(\omega)\} = \ell_{jk}(\omega)$, it follows from (12) that the asymptotic mean of $L_{njk}(\omega)$ is equal to $\ell_{jk}(\omega)$. This observation suggests that the Laplace cross spectrum can be estimated by smoothing the raw Laplace cross periodogram across the frequency:

$$\tilde{L}_{njk}(\omega_l) := \sum_{l'=-m}^{m} w_{l'} L_{njk}(\omega_{l-l'}),$$

where the $\omega_l := 2\pi l/n$ are Fourier frequencies and the w_l are nonnegative weights that sum up to unity. As usual, the bias-variance tradeoff applies when choosing the smoothing parameters such as m [3].

By inserting the smoothed Laplace periodogram and cross periodogram in (2), we obtain

$$R_{njk}(\omega) := \frac{\tilde{L}_{njk}(\omega)}{\sqrt{\tilde{L}_{njj}(\omega)\tilde{L}_{nkk}(\omega)}}. \quad (13)$$

This function can be regarded as an estimate of what we call the *Laplace coherency* between $\{Y_{nj}\}$ and $\{Y_{nk}\}$ at frequency ω, defined as

$$\begin{aligned} \rho_{jk}(\omega) &:= \frac{\ell_{jk}(\omega)}{\sqrt{\ell_{jj}(\omega)\ell_{kk}(\omega)}} \\ &= \frac{f_{jk}(\omega)}{\sqrt{f_{jj}(\omega)f_{kk}(\omega)}}. \end{aligned}$$

Because it does not require the existence of finite moments, the Laplace coherency $\rho_{jk}(\omega)$ serves as a robust alternative to the ordinary coherency $c_{jk}(\omega)$.

4. NUMERICAL EXAMPLES

In this section, we provide two numerical examples. The first example demonstrates the robustness of the Laplace cross periodogram against outliers.

Example 1. Consider the time series shown in Figure 1. The first two series are real EEG recordings of a subject when performing the task of identifying the alternative image in an ambiguous figure. Highly coherent activities during such tasks have been found and reported in [5] as evidence of cognitive binding.

Figure 2 depicts the amplitude of the ordinary and Laplace cross periodograms together with their smoothed versions, where smoothing is done by using the Parzen window $w_l \propto \{\sin(\frac{1}{2}\lfloor n/m \rfloor \omega_l)/\sin(\frac{1}{2}\omega_l)\}^4$ with $m = 26$ ($n = 256$). As can be seen, both periodograms portray a similar lowpass spectral pattern.

The solid line in Figure 3 depicts the estimated absolute coherency, showing the existence of very high coherence between the two time series in the low frequency region and the decay of the coherence as the frequency increases. In addition, the dashed line in Figure 3 depicts the estimated absolute coherency when the second series is contaminated by a patch of outliers as shown in Figure 1 (series 3). The ordinary coherency changes dramatically because of the outliers, whereas the Laplace coherency remains nearly intact. □

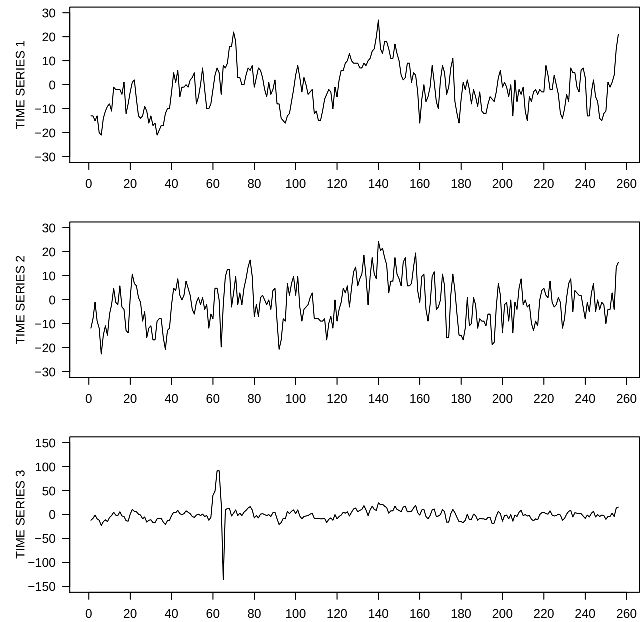

Figure 1: Series 1 and 2 are real EEG recordings; Series 3 is the same as series 2 except for the outlier contamination.

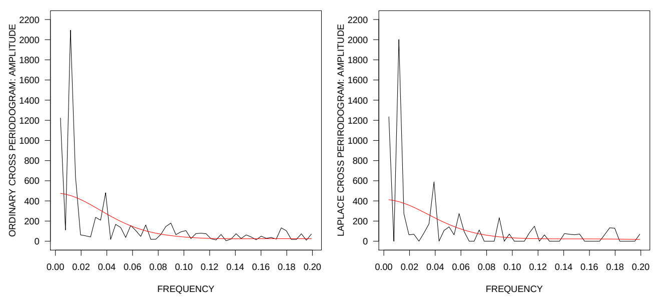

Figure 2: Amplitude plot of raw and smoothed cross periodograms for series 1 and 2: left, ordinary; right, Laplace.

The next example demonstrates the superior performance of the Laplace cross periodogram over the ordinary cross periodogram in detecting hidden coherence from noisy data with heavy-tailed distributions.

Example 2. Let $\{X_t\}$ be a unit-variance narrow-band AR(2) process satisfying

$$X_t + \phi_1 X_{t-1} + \phi_2 X_{t-1} = \zeta_t,$$

where $\phi_1 = -1.375$, $\phi_2 = 0.7225$, and $\{\zeta_t\}$ is Gaussian white noise. Consider the following time series:

$$\begin{aligned} Y_{t1} &:= c_1(X_t + \varepsilon_{t1}), \\ Y_{t2} &:= c_2(aX_{t-5} + \varepsilon_{t2}), \end{aligned}$$

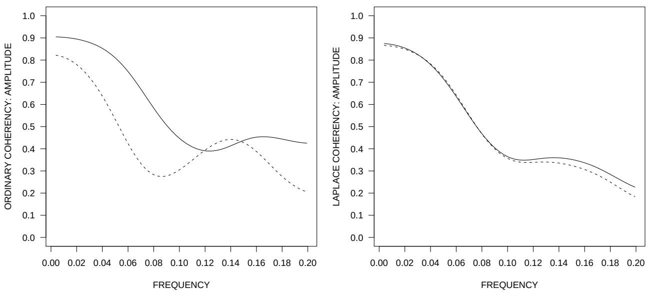

Figure 3: Amplitude plot of estimated coherency for series 1 and 2 (solid line) and for series 1 and 3 (dashed line): left, ordinary; right, Laplace.

where $\{\varepsilon_{t1}\}$ and $\{\varepsilon_{t2}\}$ are mutually independent i.i.d. white noise with variance $\sigma^2 = 3.162$ and independent of the unobservable process $\{X_t\}$. The scale parameters c_1 and c_2 are chosen to make the variance of $\{Y_{t1}\}$ and $\{Y_{t2}\}$ equal to unity. The problem under consideration is to detect the presence of the common component in these series through coherence analysis. This can be done by testing the hypotheses $H_0 : a = 0$ versus $H_1 : a \neq 0$ using the maximum absolute value of the estimated ordinary or Laplace coherency at Fourier frequencies. The tests reject H_0 in favor of H_1 if the maximum absolute coherency is higher than a threshold. We call these tests the ordinary and Laplace maximum absolute coherency tests, respectively.

Figure 4 depicts the ROC curves of the tests under four noise distributions: Gaussian ($a = 1$), double exponential (or Laplace; $a = 0.6$), Student's T distribution with 2.1 degrees of freedom (or $T_{2.1}$; $a = 0.2$), and Cauchy with scale parameter 0.25 ($c_1 = c_2 = 1$; $a = 0.3$). Note that all three non-Gaussian distributions have heavier tails than the Gaussian distribution.

As can be seen, the Laplace maximum absolute coherency test has greater detection power in all but the Gaussian case. The performance in the Cauchy case is particularly impressive: the ordinary coherency is essentially useless in this case, whereas the Laplace coherency is able to detect the coherence with high probability. As with any robust method, the efficiency loss in the Gaussian case is the unavoidable tradeoff for the robustness gain in the case of heavy-tailed noise. □

5. CONCLUDING REMARKS

We have introduced a robust cross periodogram, called the Laplace cross periodogram, for coherence analysis of multiple time series in the frequency domain. We have derived the asymptotic distribution of the Laplace cross periodogram and discovered the important relationship with the cross zero-crossing spectrum through the mean of the asymptotic distribution. We have also demonstrated via two examples the robustness of the Laplace cross periodogram for coherence estimation detection under the condition of outliers and heavy-tailed noise. The robustness advantage of the Laplace cross periodogram over the ordinary cross periodogram is confirmed in these examples.

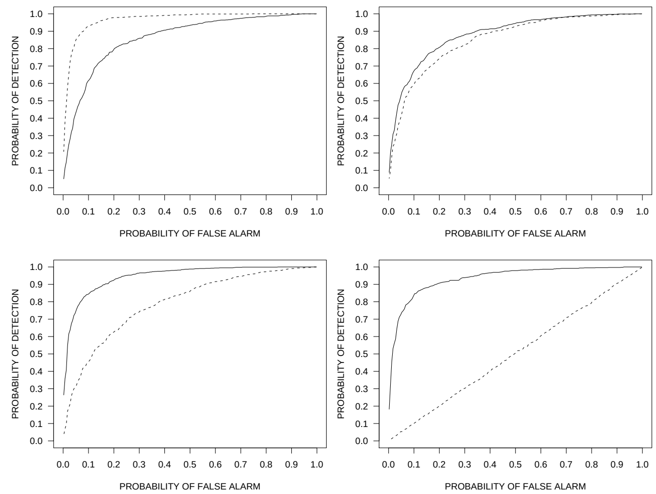

Figure 4: ROC curves of the maximum absolute coherency test based on the Laplace cross periodogram (solid line) and the ordinary cross periodogram (dashed line) under different noise distributions: top left, Gaussian; top right, double exponential; bottom left, Student's $T_{2.1}$; bottom right, Cauchy. Results are based on 1,000 Monte Carlo runs.

The methodology described in this paper can be generalized, for more flexibility in the efficiency-robustness tradeoff, by replacing the LAD criterion with the L_p-norm criterion for any $p \in (1,2)$ along the lines of [6], or with any convex criterion function commonly used in robust estimation [7].

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