# DESIGN AND APPLICATIONS OF 2D DIRECTIONAL FILTERS BASED ON FREQUENCY TRANSFORMATIONS 

Radu Matei ${ }^{1}$ and Daniela Matei ${ }^{2}$<br>${ }^{1}$ Faculty of Electronics, Telecommunications and Information Technology, Technical University of Iasi Bd. Carol I nr.11, 700506, Iasi, Romania<br>phone: + (40) 232 217720, fax: + (40) 232 217720, email: rmatei@etti.tuiasi.ro<br>${ }^{2}$ Faculty of Biomedical Engineering, University of Medicine and Pharmacy, Iasi, Romania


#### Abstract

We present a new approach to the design of 2D recursive filters, with a directionally-selective frequency response. Two analytical design methods are proposed, based on digital prototype filters and complex frequency transformations, determined using rational approximations and the bilinear transform. Design examples are presented for given specifications. The resulted filters are efficient, of low complexity and relatively high selectivity. They find useful applications in image processing, like detecting lines with a given orientation from an image.


## 1. INTRODUCTION

The currently-used design methods of 2D recursive filters rely in a large measure on 1D digital filter prototypes, using spectral transformations from $s$ to $z$ plane via bilinear or Euler transformations followed by $z$ to $\left(z_{1}, z_{2}\right)$ transformations [1], [2], [6]-[10]. There are 2D filters with various shapes and different applications in image processing. Several classes of filters have an orientation-selective frequency response [4], [5], useful in image processing tasks like edge detection, motion analysis etc. An important class are the steerable filters, synthesized as a linear combination of a set of basis filters [3].
The frequency transformation technique is a classical approach in obtaining desired 2D filters from 1D simple prototype filters [1], [2]. A major reference on this topic is [7], where a technique for rotating the frequency response of separable filters is developed. The method considers transfer functions in rational powers of $z$, realized by input-output signal array interpolations. For each specific method the filter stability must also be ensured. In [9], an efficient technique was given for stable IIR filters of various shapes and without interpolation. Other important papers in this field are [6], [8], [10].
In this paper we propose a design method for implementing a class of 2D spatially oriented low-pass filters which select narrow domains along specified directions in the frequency plane ( $\omega_{1}, \omega_{2}$ ). Such filters with specified orientation and selectivity can be used in selecting lines with a given orientation from an input image. Our approach is based on the design in the spatial frequency domain, starting from a 1D prototype filter. Since we envisage designing filters of minimum order, we will use recursive filters as prototypes,
and the 2D oriented filters will result recursive as well. Section 2 begins with the 1D prototype filters used and their frequency response; then two types of directional filters are presented, based on specific spectral transformations which also use the bilinear transform as an intermediate step. The first filter type starts directly from a very selective low-pass prototype. The second type of filter is more general and has an elliptical shape in the frequency plane, with a specified orientation. A selective directional filter may be obtained as a particular case by setting the adequate parameters. A zerophase version of the elliptically-shaped filter was approached in [11] and uses a real frequency transformation. Here we treat the general case using a complex frequency transformation. Other related methods for directional filter design were proposed in [12].

## 2. 2D DIRECTIONAL FILTERS DERIVED FROM 1D PROTOTYPES

This section introduces the 1D prototype filters and the general form of the 2D directionally-selective filters obtained through frequency transformations.

### 2.1 1D Prototype Filters

The idea behind the proposed design methods is to start from a 1D digital filter of a common type (maximally-flat, equiripple etc.) and to transform its transfer function using specific frequency mappings in order to derive the transfer function of the 2D filter with the desired shape. The advantage is that the prototype selectivity and stability properties are inherited by the designed 2D filter.
We refer throughout this paper only to recursive filters, which are known to be more efficient than FIR filters, although generally more difficult to design. Let us consider a recursive digital filter $H_{P}(z)$ of order $N$ with the transfer function:

$$
\begin{equation*}
H_{P}(z)=\frac{P(z)}{Q(z)}=\sum_{i=0}^{M} p_{i} \cdot z^{i} / \sum_{j=0}^{N} q_{j} \cdot z^{j} \tag{1}
\end{equation*}
$$

We consider now this general transfer function with $M=N$ factorized into simpler rational functions of first and second order. For an odd order filter $H_{P}(z)$ will have at least one first order factor of the form:

$$
\begin{equation*}
H_{1}(z)=\left(b_{1} z+b_{0}\right) /\left(z+a_{0}\right) \tag{2}
\end{equation*}
$$

The transfer function also contains second-order factor functions referred to as biquad functions:

$$
\begin{equation*}
H_{2}(z)=\left(b_{2} z^{2}+b_{1} z+b_{0}\right) /\left(z^{2}+a_{1} z+a_{0}\right) \tag{3}
\end{equation*}
$$

where in general the second-order polynomials at the numerator and denominator have complex-conjugated roots. The main issue approached in this paper is to find the transfer function of the desired 2D filter $H_{2 D}\left(z_{1}, z_{2}\right)$ using appropriate frequency transformations of the form $z \rightarrow F_{Z}\left(z_{1}, z_{2}\right)$. Since the transformations used map the real frequency variable $\omega$ onto the plane $\left(\omega_{1}, \omega_{2}\right)$, first a mapping of the form $\omega \rightarrow F\left(\omega_{1}, \omega_{2}\right)$ will be found.
Let us now consider the elementary transfer functions (2) and (3); these can be turned into the following form of complex numbers with real and imaginary parts:

$$
\begin{gather*}
H_{1}(j \omega)=\frac{b_{0}+b_{1} \cos \omega+j b_{1} \sin \omega}{a_{0}+\cos \omega+j \sin \omega}  \tag{4}\\
H_{2}(j \omega)=\frac{b_{0}+\left(b_{2}+b_{0}\right) \cos \omega+j\left(b_{2}-b_{0}\right) \sin \omega}{a_{1}+\left(1+a_{0}\right) \cos \omega+j\left(1-a_{0}\right) \sin \omega}=\frac{P(\omega)}{Q(\omega)} \tag{5}
\end{gather*}
$$

We notice that the first- and second-order functions have a similar form when expressed as a ratio of complex numbers. Therefore, as we will see further, the corresponding 2D transfer functions will be implemented with convolution kernels of the same size.

### 2.2 Design of Directional Filters Using Frequency Transformations

The next step starts from the expressions (4) and (5) of the frequency response and uses of the following accurate rational approximations for sine and cosine on $[-\pi, \pi]$ :

$$
\begin{align*}
& \cos \omega \cong \frac{1-0.435949 \cdot \omega^{2}+0.011319 \cdot \omega^{4}}{1+0.06095 \cdot \omega^{2}+0.0037557 \cdot \omega^{4}}=\frac{C(\omega)}{M(\omega)}  \tag{6}\\
& \sin \omega \cong \frac{\omega \cdot\left(1-0.101046 \cdot \omega^{2}\right)}{1+0.06095 \cdot \omega^{2}+0.0037557 \cdot \omega^{4}}=\frac{S(\omega)}{M(\omega)} \tag{7}
\end{align*}
$$

The above expressions were obtained through the Chebyshev-Padé approximation, which can be found for a large class of functions using a symbolic computation software like MAPLE.
The advantage of these rational approximations is that they have the same denominator and therefore they can be directly substituted into the expressions (4) and (5), yielding a loworder rational expression of the frequency response $H\left(e^{j \omega}\right)$. Given a prototype filter $H\left(\omega_{1}\right)$ (which varies on one axis only), a 2D orientation-selective (directional) filter may be obtained by rotating the axes of the plane $\left(\omega_{1}, \omega_{2}\right)$ by an angle $\varphi$, through the linear transformation:

$$
\left[\begin{array}{l}
\omega_{1}  \tag{8}\\
\omega_{2}
\end{array}\right]=\left[\begin{array}{cc}
\cos \varphi & \sin \varphi \\
-\sin \varphi & \cos \varphi
\end{array}\right] \cdot\left[\begin{array}{l}
\bar{\omega}_{1} \\
\bar{\omega}_{2}
\end{array}\right]
$$

where $\omega_{1}, \omega_{2}$ are the original frequency variables and $\bar{\omega}_{1}, \bar{\omega}_{2}$ the rotated ones [1]. The spatial orientation is specified by an angle $\varphi$ with respect to $\omega_{1}$-axis and is therefore defined by
the following 1D to 2D spatial frequency transformation in the frequency response $H\left(\omega_{1}, \omega_{2}\right)$ :

$$
\begin{equation*}
\omega \rightarrow \omega_{1} \cos \varphi+\omega_{2} \sin \varphi \tag{9}
\end{equation*}
$$

By substitution, we obtain the transfer function of the oriented filter $H_{\varphi}\left(\omega_{1}, \omega_{2}\right)$ :

$$
\begin{equation*}
H_{\varphi}\left(\omega_{1}, \omega_{2}\right)=H\left(\omega_{1} \cos \varphi+\omega_{2} \sin \varphi\right) \tag{10}
\end{equation*}
$$

The 2D filter $H_{\varphi}\left(\omega_{1}, \omega_{2}\right)$ has the magnitude section along the line $\omega_{1} \cos \varphi+\omega_{2} \sin \varphi=0$ identical with the prototype $H(\omega)$, and constant along the filter longitudinal axis: $\omega_{1} \sin \varphi-\omega_{2} \cos \varphi=0$.
We determine a 1D to 2D complex transformation which allows an oriented 2D filter to be obtained from a 1D prototype filter.

In this paper the term template will be used, used in the field of cellular neural networks, referring to the coefficient matrices corresponding to the numerator and denominator of a 2D filter transfer function $H\left(z_{1}, z_{2}\right)$ (the originals of the 2D Z transform). We will use here both odd-sized $(3 \times 3,5 \times 5)$ and even-sized ( $2 \times 2$ ) templates.
The following step is to find the discrete approximation of the frequency transformation (9), in other words a mapping of the frequency variable $\omega$ into the complex plane ( $z_{1}, z_{2}$ ). In order to find the discrete approximation of these functions, the bilinear transform will be used. The sample interval in our case can be taken $T=1$ so the bilinear transform for $s_{1}$ and $s_{2}$ in the complex plane ( $s_{1}, s_{2}$ ) has the form:

$$
\begin{equation*}
s_{1}=2\left(z_{1}-1\right) /\left(z_{1}+1\right) \quad s_{2}=2\left(z_{2}-1\right) /\left(z_{2}+1\right) \tag{11}
\end{equation*}
$$

Next we have to find discrete expressions for $\omega$ and $\omega^{2}$ in order to be used in the approximations (6) and (7). Since $s_{1}=j \omega_{1}$ and $s_{2}=j \omega_{2}$ using expressions (11) we obtain:

$$
\begin{equation*}
\omega \rightarrow \omega_{1} \cos \varphi+\omega_{2} \sin \varphi=-j\left(s_{1} \cos \varphi+s_{2} \sin \varphi\right) \tag{12}
\end{equation*}
$$

$$
\left.\omega \rightarrow \frac{\left[\begin{array}{ll}
1 & z_{1}
\end{array}\right] \times \mathbf{F}_{\varphi} \times\left[\begin{array}{ll}
1 & z_{2}
\end{array}\right]^{T}}{\left[\begin{array}{ll}
1 & z_{1} \tag{13}
\end{array}\right] \times \mathbf{G} \times[1} z_{2}\right]^{T}
$$

which corresponds to the $2 \times 2$ templates $j \mathbf{F}_{\varphi}$ and $\mathbf{G}$ :

$$
\begin{gather*}
j \mathbf{F}_{\varphi}=2 j\left[\begin{array}{cc}
\sin \varphi+\cos \varphi & \sin \varphi-\cos \varphi \\
\cos \varphi-\sin \varphi & -\sin \varphi-\cos \varphi
\end{array}\right] G=\left[\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right]  \tag{14}\\
\omega^{2} \rightarrow-\left(s_{1} \cos \varphi+s_{2} \sin \varphi\right)^{2}  \tag{15}\\
=-0.5\left(s_{1}^{2}(1+\cos 2 \varphi)+s_{2}^{2}(1-\cos 2 \varphi)+2 s_{1} s_{2} \sin 2 \varphi\right) \\
\omega^{2} \rightarrow-\left(\mathbf{z}_{1} \times \mathbf{P} \times \mathbf{z}_{2}^{T}\right) /\left(\mathbf{z}_{1} \times \mathbf{Q} \times \mathbf{z}_{2}^{T}\right) \tag{16}
\end{gather*}
$$

where

$$
\begin{gather*}
\mathbf{P}=\left[\begin{array}{ccc}
1+\sin 2 \varphi & -2 \cos 2 \varphi & 1-\sin 2 \varphi \\
2 \cos 2 \varphi & -4 & 2 \cos 2 \varphi \\
1-\sin 2 \varphi & -2 \cos 2 \varphi & 1+\sin 2 \varphi
\end{array}\right] \mathbf{Q}=\left[\begin{array}{lll}
1 & 2 & 1 \\
2 & 4 & 2 \\
1 & 2 & 1
\end{array}\right]  \tag{17}\\
\mathbf{z}_{1}=\left[\begin{array}{lll}
1 & z_{1} & z_{1}^{2}
\end{array}\right], \mathbf{z}_{2}=\left[\begin{array}{lll}
1 & z_{2} & z_{2}^{2}
\end{array}\right] \tag{18}
\end{gather*}
$$

The matrix $\mathbf{P}$ depends on the orientation angle $\varphi$. It can be easily checked that $\mathbf{G} * \mathbf{G}=\mathbf{Q}$. Having the discrete approximations (13) and (16) for $\omega$ and $\omega^{2}$, the matrices $\mathbf{C}$,


Figure 1 - 1D very selective low-pass prototype filter.
$j \cdot \mathbf{S}$ and $\mathbf{M}$ of size $5 \times 5$ corresponding to the polynomials $C(\omega), S(\omega)$ and $M(\omega)$ result as:

$$
\begin{align*}
& \mathbf{C}=\mathbf{Q} * \mathbf{Q}-0.435949 \cdot \mathbf{P} * \mathbf{Q}+0.011319 \cdot \mathbf{P} * \mathbf{P}  \tag{19}\\
& j \cdot \mathbf{S}=j \cdot \mathbf{F}_{\varphi} * \mathbf{G} *(\mathbf{Q}+0.101046 \cdot \mathbf{P})  \tag{20}\\
& \mathbf{M}=\mathbf{Q} * \mathbf{Q}+0.06095 \cdot \mathbf{P} * \mathbf{Q}+0.0037557 \cdot \mathbf{P} * \mathbf{P} \tag{21}
\end{align*}
$$

The discrete mappings of $\cos \omega$ and $\sin \omega$ are therefore:

$$
\begin{align*}
& \cos \omega \rightarrow\left(\mathbf{z}_{1} \times \mathbf{C} \times \mathbf{z}_{2}^{T}\right) /\left(\mathbf{z}_{1} \times \mathbf{M} \times \mathbf{z}_{2}^{T}\right)  \tag{22}\\
& \sin \omega \rightarrow\left(\mathbf{z}_{1} \times \mathbf{S} \times \mathbf{z}_{2}^{T}\right) /\left(\mathbf{z}_{1} \times \mathbf{M} \times \mathbf{z}_{2}^{T}\right)
\end{align*}
$$

where

$$
\mathbf{z}_{1}=\left[\begin{array}{lllll}
1 & z_{1} & z_{1}^{2} & z_{1}^{3} & z_{1}^{4}
\end{array}\right], \mathbf{z}_{2}=\left[\begin{array}{lllll}
1 & z_{2} & z_{2}^{2} & z_{2}^{3} & z_{2}^{4} \tag{23}
\end{array}\right]
$$

Substituting the above mappings into the biquad expression
(5) we obtain the following filter templates of size $5 \times 5$ :

$$
\begin{align*}
& \mathbf{B}=b_{0} \cdot \mathbf{M}+\left(b_{0}+b_{2}\right) \cdot \mathbf{C}+\left(b_{0}-b_{2}\right) \cdot \mathbf{S}  \tag{24}\\
& \mathbf{A}=a_{1} \cdot \mathbf{M}+\left(1+a_{0}\right) \cdot \mathbf{C}+\left(a_{0}-1\right) \cdot \mathbf{S}
\end{align*}
$$

We notice that finally the filter templates result with real elements.
Design example: We will next present the design of a very selective directional filter. We will use the simplest possible 1D prototype, for instance a first order Butterworth low-pass filter with a cut-off frequency $\omega_{c}=0.025$, the value 1.0 corresponding to half the sample rate. The frequency response magnitude is shown in Fig.1. The transfer function in $z$ has the general form (2), with $b_{1}=b_{0}$. For these specifications the coefficients are: $b_{1}=b_{0}=0.037804$, $a_{0}=-0.92439$.
Given this 1D prototype, the 2D oriented filter results by simply by substituting the frequency mappings (22) into (4). The 2D transfer function results as:

$$
\begin{equation*}
H_{\varphi}\left(z_{1}, z_{2}\right)=\left(\mathbf{z}_{1} \times \mathbf{B} \times \mathbf{z}_{2}^{T}\right) /\left(\mathbf{z}_{1} \times \mathbf{A} \times \mathbf{z}_{2}^{T}\right) \tag{25}
\end{equation*}
$$

where the $5 \times 5$ matrices $\mathbf{B}$ and $\mathbf{A}$ are given in this case by:

$$
\begin{align*}
& \mathbf{B}=b_{0} \cdot(\mathbf{M}+\mathbf{C})-b_{0} \cdot \mathbf{S}  \tag{26}\\
& \mathbf{A}=a_{0} \cdot \mathbf{M}+\mathbf{C}-\mathbf{S}
\end{align*}
$$

These simple expressions corresponding to the frequency response (4) are simpler than the more general set given by (24), which correspond to a biquad section like (5).

## 3. DESIGN OF ELLIPTICALLY-SHAPED FILTERS

We study in this section a class of 2D low-pass filters having an elliptically-shaped horizontal section. These filters will be specified by imposing the values of the semi-axes of the ellipse, and the orientation is given by the angle of the large axis with respect to $\omega_{2}$-axis. Starting from the frequency response for a 1D filter given by (4) or (5), we can derive a 2D elliptically-shaped filter using the frequency transformation $\omega \rightarrow \sqrt{E_{\varphi}\left(\omega_{1}, \omega_{2}\right)}$ :

$$
\begin{equation*}
E_{\varphi}\left(\omega_{1}, \omega_{2}\right)=\omega_{1}^{2}\left(\frac{\cos ^{2} \varphi}{E^{2}}+\frac{\sin ^{2} \varphi}{F^{2}}\right)+\omega_{2}^{2}\left(\frac{\sin ^{2} \varphi}{E^{2}}+\frac{\cos ^{2} \varphi}{F^{2}}\right) \tag{27}
\end{equation*}
$$

$$
+\omega_{1} \omega_{2} \sin (2 \varphi)\left(\frac{1}{F^{2}}-\frac{1}{E^{2}}\right)=a \cdot \omega_{1}^{2}+b \cdot \omega_{2}^{2}+c \cdot \omega_{1} \omega_{2}
$$

The elliptically-shaped filter can be considered as derived from a circular filter through the linear transformation:

$$
\left[\begin{array}{l}
\omega_{1}  \tag{28}\\
\omega_{2}
\end{array}\right]=\left[\begin{array}{cc}
E & 0 \\
0 & F
\end{array}\right] \cdot\left[\begin{array}{cc}
\cos \varphi & -\sin \varphi \\
\sin \varphi & \cos \varphi
\end{array}\right] \cdot\left[\begin{array}{l}
\omega_{1}^{\prime} \\
\omega_{2}^{\prime}
\end{array}\right]
$$

where usually we consider $E>F$; in (28), $\left(\omega_{1}, \omega_{2}\right)$ are the current coordinates and ( $\omega_{1}^{\prime}, \omega_{2}^{\prime}$ ) are the former (rotated) coordinates. Thus, the unit circle is stretched along the axes $\omega_{1}$ and $\omega_{2}$ with factors $E$ and $F$, then counter-clockwise rotated with an angle $\varphi$, becoming an oriented ellipse, which is the shape of the proposed filter in horizontal section.
Consequently, given a 1D prototype filter of the general form (4) or (5), we can obtain a corresponding 2D filter with an elliptical section, specified by the above-mentioned parameters which impose the shape and the orientation using the substitution:

$$
\begin{equation*}
\omega \rightarrow \sqrt{E_{\varphi}\left(\omega_{1}, \omega_{2}\right)}=\sqrt{a \cdot \omega_{1}^{2}+b \cdot \omega_{2}^{2}+c \cdot \omega_{1} \omega_{2}} \tag{29}
\end{equation*}
$$

However, using (29) together with (6) and (7) leads to a more complicate design. Instead, we will find rational expressions for the functions $\cos \sqrt{\omega}$ and $\sin \sqrt{\omega}$ and then make the substitution:

$$
\begin{equation*}
\omega \rightarrow E_{\varphi}\left(\omega_{1}, \omega_{2}\right)=a \cdot \omega_{1}^{2}+b \cdot \omega_{2}^{2}+c \cdot \omega_{1} \omega_{2} \tag{30}
\end{equation*}
$$

which is more convenient than (29).
Replacing the real frequency variables $\omega_{1}$ and $\omega_{2}$ by the complex variables $s_{1}=j \omega_{1}$ and $s_{2}=j \omega_{2}$, the function $E_{\varphi}\left(\omega_{1}, \omega_{2}\right)$ can be written in the 2D Laplace domain:

$$
\begin{equation*}
\omega \rightarrow E_{\varphi}\left(s_{1}, s_{2}\right)=-\left(a \cdot s_{1}^{2}+b \cdot s_{2}^{2}+c \cdot s_{1} s_{2}\right) \tag{31}
\end{equation*}
$$

The next step is to find the discrete approximation $E_{\varphi}\left(z_{1}, z_{2}\right)$ of (31). This can be achieved either using the forward or backward Euler approximations, or otherwise the bilinear transform, which in principle gives better accuracy. Using the Chebyshev-Padé method we find the following approximations for the functions $\cos \sqrt{\omega}$ and $\sin \sqrt{\omega}$, with the same denominator:

$$
\begin{equation*}
\cos \sqrt{\omega} \cong \frac{1.05595-0.086514 \cdot \omega-0.13045 \cdot \omega^{2}}{1+0.75 \cdot \omega-0.110583 \cdot \omega^{2}}=\frac{C_{S}(\omega)}{A_{S}(\omega)} \tag{32}
\end{equation*}
$$

$\sin \sqrt{\omega} \cong \frac{0.167+1.46287 \cdot \omega-0.259815 \cdot \omega^{2}}{1+0.75 \cdot \omega-0.110583 \cdot \omega^{2}}=\frac{S_{S}(\omega)}{A_{S}(\omega)}$
which are sufficiently accurate on the range $\omega \in[0, \pi]$. Since these functions are developed on the range $[0, \pi]$, their approximations result neither odd nor even. The above expressions were obtained through the Chebyshev-Padé approximation, which can be found for a large class of functions using a symbolic computation software.
For the elementary functions $H_{1}(j \omega)$ and $H_{2}(j \omega)$ we find the corresponding functions of the 2D elliptically-shaped filter using the frequency mapping (27). Therefore we need discrete approximations for the functions $C\left(\omega_{1}, \omega_{2}\right)=\cos \sqrt{E_{\varphi}\left(\omega_{1}, \omega_{2}\right)}, S\left(\omega_{1}, \omega_{2}\right)=\sin \sqrt{E_{\varphi}\left(\omega_{1}, \omega_{2}\right)}$. Substituting in (31) and (32) $\omega$ by $E_{\varphi}\left(\omega_{1}, \omega_{2}\right)$, we find the expressions $C\left(\omega_{1}, \omega_{2}\right), S\left(\omega_{1}, \omega_{2}\right)$ in $\omega_{1}^{2}, \omega_{2}^{2}$.
In order to find the discrete approximation of these functions, the bilinear transform will be used, as in the previous section. Substituting expressions (11) into (31), the frequency transformation $\omega \rightarrow E_{\varphi}\left(\omega_{1}, \omega_{2}\right)$ in matrix form is:

$$
\begin{equation*}
\omega \rightarrow-4\left(\mathbf{z}_{1} \times \mathbf{B} \times \mathbf{z}_{2}^{T}\right) /\left(\mathbf{z}_{1} \times \mathbf{A} \times \mathbf{z}_{2}^{T}\right) \tag{34}
\end{equation*}
$$

where:
$\mathbf{B}=2 \alpha \cos 2 \varphi\left[\begin{array}{rrr}0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0\end{array}\right]+\alpha \sin 2 \varphi\left[\begin{array}{rrr}-1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1\end{array}\right]+\beta\left[\begin{array}{rrr}1 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & 1\end{array}\right]$
$=2 \alpha \cos 2 \varphi \cdot \mathbf{M}+\alpha \sin 2 \varphi \cdot \mathbf{N}+\beta \cdot \mathbf{P}$

$$
\begin{gather*}
\mathbf{A}=\left[\begin{array}{lll}
1 & 2 & 1 \\
2 & 4 & 2 \\
1 & 2 & 1
\end{array}\right]  \tag{36}\\
\mathbf{z}_{1}=\left[\begin{array}{lll}
1 & z_{1} & z_{1}^{2}
\end{array}\right], \mathbf{z}_{2}=\left[\begin{array}{lll}
1 & z_{2} & z_{2}^{2}
\end{array}\right]
\end{gather*}
$$

In (35) the following notations were made: $\alpha=1 / E^{2}-1 / F^{2}$, $\beta=1 / E^{2}+1 / F^{2}$.
Using this approximation, the templates corresponding to the polynomials $C_{S}(\omega), S_{S}(\omega)$ and $A_{S}(\omega)$ from (32) and (32), namely $\mathbf{C}_{S}, \mathbf{S}_{S}$ and $\mathbf{A}_{S}$ of size $5 \times 5$ result as:
$\mathbf{C}_{S}=1.055951 \cdot \mathbf{A} * \mathbf{A}+0.346056 \cdot \mathbf{A} * \mathbf{B}-2.0872 \cdot \mathbf{B} * \mathbf{B}$ (38)
$\mathbf{S}_{S}=0.167 \cdot \mathbf{A} * \mathbf{A}-5.85148 \cdot \mathbf{A} * \mathbf{B}-4.15704 \cdot \mathbf{B} * \mathbf{B}$

$$
\begin{equation*}
\mathbf{A}_{S}=\mathbf{A} * \mathbf{A}-3 \cdot \mathbf{A} * \mathbf{B}-1.769328 \cdot \mathbf{B} * \mathbf{B} \tag{39}
\end{equation*}
$$

The final filter templates will be also of size $5 \times 5$ and result taking into account (5) as:

$$
\begin{align*}
\mathbf{P} & =b_{0} \cdot \mathbf{A}_{S}+\left(b_{0}+b_{2}\right) \cdot \mathbf{C}_{S}+j\left(b_{2}-b_{0}\right) \cdot \mathbf{S}_{S}=\mathbf{P}_{r e}+j \cdot \mathbf{P}_{i m}  \tag{41}\\
\mathbf{Q} & =a_{1} \cdot \mathbf{A}_{S}+\left(1+a_{0}\right) \cdot \mathbf{C}_{S}+j\left(1-a_{0}\right) \cdot \mathbf{S}_{s}=\mathbf{Q}_{r e}+j \cdot \mathbf{Q}_{i m} \tag{42}
\end{align*}
$$

Design example:
We consider a second-order low-pass Butterworth prototype filter with passband-edge frequency $\omega_{p}=0.9$; its transfer function has the general form (3) with the parameter values: $b_{2}=b_{0}=0.800592, b_{1}=2 b_{0}=1.601184, a_{1}=1.561018$, $a_{0}=0.641351$. The magnitude characteristics of this


Figure 2 - 1D low-pass maximally-flat Butterworth prototype filter.


Figure 3. (a) Frequency response and contour plot of an ellipticallyshaped filter with $\varphi=\pi / 3, E=2.8, F=1$; (b) Frequency response of directional filter with $\varphi=25^{\circ}, E=3.4, F=0.1$
maximally-flat low-pass filter is shown in Fig.2. The frequency characteristic and contour plot of an ellipticallyshaped filter using this prototype is shown in Fig.3(a), for the specified parameters.
Let us design an elliptically-shaped filter with a large ratio $E / F$, for instance $E=3.4, F=0.1$; in this case we get a very selective directional filter which forms the angle $\varphi$ with $\omega_{2}$-axis; we consider $\varphi=25^{\circ}=0.1389 \mathrm{rad}$. The frequency response for this filter is plotted in Fig.3(b). We get a complex template $\mathbf{Q}$ and a real matrix $\mathbf{P}$ with the form:

$$
\mathbf{P}=\left[\begin{array}{rrrrr}
0.10806 & 0.15743 & 0.08640 & 0.02117 & 0.00195  \tag{43}\\
-0.15649 & -0.60253 & -0.53342 & -0.17973 & -0.02106 \\
0.08571 & 0.53226 & 1.00000 & 0.53226 & 0.08571 \\
-0.02106 & -0.17973 & -0.53342 & -0.60253 & -0.15649 \\
0.00195 & 0.02117 & 0.08640 & 0.15743 & 0.10806
\end{array}\right]
$$

Even if the stability of the 2 D filters resulted through the presented methods is not analyzed here, it can be shown that the proposed frequency transformations preserve the stability of the 1D prototype filter since they are based on the bilinear transform and accurate approximations. Therefore, the only issue would be to ensure the stability of the prototype filter. The derived 2D filter could become unstable only if the


Figure 4 - (a) Test image; (b) directionally filtered image with parameters: (a) $\varphi=2.5$ rad., $E=6.4, F=0.5$;
numerical approximations used introduce large errors. In this case we would have to increase the approximation precision by taking more higher order terms, which would increase the filter complexity.

## 4. APPLICATIONS AND SIMULATION RESULTS

We present an example of detecting straight lines with a given inclination from an image, by means of a filtering with a 2D IIR oriented LP filter. The spectrum of a straight line is oriented in the plane $\left(\omega_{1}, \omega_{2}\right)$ at an angle of $\pi / 2$ with respect to the line direction. The image in Fig.4(a) contains straight lines oriented at various angles, and is filtered with a directional filter with $\varphi=0.44 \mathrm{rad}$., designed using the method from section 3. In the filtered image from Fig.4(b), only the lines which have the spectrum oriented more or less along the filter characteristic, remain practically unchanged, while all the other lines are low-pass filtered.

## 5. CONCLUSION

We proposed two design methods for 2D orientationselective filters based on 1D selective LP prototypes and on complex frequency transformations. The methods are more general and can be applied also to other types of filters. The developed frequency transformations are based on efficient rational approximations and on the bilinear transform. The resulted filters are efficient and inherit the selectivity of their 1 D counterparts. The methods are versatile in the sense that once determined the adequate frequency transformation, the prototype specifications can be changed and different 2D filters will be obtained. Possible applications of these orientation-selective filters were suggested through simulation results on a test image. As design examples we considered filters of minimum order for the sake of simplicity and efficiency. Further research also envisages an efficient implementation of this class of filters.

## ACKNOWLEDGMENT

This paper is supported by the National University Research Council under Grant PN2 - ID_310 "Algorithms and parallel architectures for signal acquisition, compression and processing".

## REFERENCES

[1] D. E. Dudgeon and R. M. Mersereau, Multidimensional Digital Signal Processing, Englewood Cliffs, NJ: PrenticeHall, 1984
[2] W. S. Lu and A. Antoniou, Two-Dimensional Digital Filters, CRC Press, 1992
[3] W. T. Freeman and E. H. Adelson, "The design and use of steerable filters", IEEE Transactions PAMI, Vol.13, No.9, September 1991
[4] P. E. Danielsson, "Rotation-invariant linear operators with directional response", 5th ICPR Conference, Miami, Dec. 1980
[5] R. H. Bamberger and M. J. T. Smith, "A filter bank for the directional decomposition of images: theory and design", IEEE Trans. Signal Process. Volume 40 pp. 882-893
[6] N. A. Pendergrass, S. K. Mitra and E. I. Jury, "Spectral transformations for two-dimensional digital filters", IEEE Trans. Circuit Syst., vol. CAS-23, pp.26-35, Jan. 1976
[7] H. Chang and J. K. Aggarwal, "Design of twodimensional recursive filters by interpolation", IEEE Trans. Circ. Syst. CAS-24, pp.281-291, 1977.
[8] K. Hirano and J. K. Aggarwal, "Design of twodimensional recursive digital filters", IEEE Trans. Circ. Syst., CAS-25, pp.1066-1076, Dec. 1978
[9] L. Harn and B. A. Shenoi, "Design of stable twodimensional IIR filters using digital spectral transformations", IEEE Trans. Circuit Syst., vol. CAS-33, pp. 483-490, May 1986.
[10] H. Chang and J. K. Aggarwal, "Design of twodimensional semicausal recursive filters", IEEE Trans. Circuit Syst., CAS-25, pp. 1051-1059
[11] R. Matei and P. Ungureanu - "Image processing using elliptically-shaped filters", Proc. of IEEE International Symposium on Signals, Circuits and Systems, ISSCS 2009, July 9-10, 2009, Iasi, Romania, Vol.2, pp. 337-340
[12] R. Matei and D. Matei, "Orientation-Selective 2D Recursive Filter Design Based on Frequency Transformations", IEEE Region 8 EUROCON 2009 Conference, St. Petersburg, Russia, May 18-23, 2009, pp. 1320-1327

