POLYNOMIAL-BASED DIGITAL FILTERS AS PROTOTYPE FILTERS IN DFT MODULATED FILTER BANKS

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ABSTRACT

In this paper, we investigate the possibility to use polynomial-based digital FIR filters as prototype filters in DFT and cosine modulated filter banks. In order to apply the FIR filter with piecewise polynomial response as prototype filter in the filter bank, it is beneficial to find expressions for polyphase components of the filter. In the paper it is shown that it is possible to construct the two following polyphase decompositions of polynomial-based digital filters for implementation; (i) polyphase decomposition based on the prolonged Farrow structure, and (ii) polyphase decomposition based on the transposed Farrow structure. The paper shows that both polyphase structures have the same multiplication rate, while the polyphase decomposition based on the prolonged Farrow structure has a considerable smaller number of multipliers. Both structures are equivalent in terms of filter performance in the frequency domain, and can be used as prototype filter in DFT and cosine modulated filter bank.

1. INTRODUCTION

Among various classes of uniform multi-channel filter banks, cosine modulated and complex modulated filter banks become very popular in many applications due to the following two main reasons [1], [2]. First, these banks can be generated using a single prototype finite-impulse response (FIR) filter by exploiting a proper transformation, enabling one to simultaneously implement all the filters in the analysis and synthesis bank. This leads to efficient implementation structures compared to the case where each filter in the analysis and synthesis part is separately realized. Second, the overall synthesis can concentrate on optimizing only the prototype filter, making the design easier and more straightforward. As has been pointed out in [2], the same prototype filter only differently scaled can be used for both aforementioned types of filter banks. Due to this fact, this paper concentrates only on designing complex modulated filter banks.

The main drawback of FIR filters is a higher number of multipliers needed in a conventional implementation when a narrow transition band is required [3]. The dominant reason is the fact that the filter order of an FIR filter is roughly inversely proportional to transition bandwidth. An efficient approach to overcome the above-mentioned cost problem is to synthesize linear-phase FIR filters so that their impulse responses are piecewise approximated by polynomials, and their implementation applies efficient structures [4]–[10]. The arithmetic complexity of these filters is proportional to the number of impulse-response pieces N and the order M of the polynomials rather than to the actual filter order.

This paper shows that it is possible to perform polyphase decompositions of polynomial-based digital filters. We have found the two following polyphase decompositions: polyphase decomposition based on the prolonged Farrow structure, and polyphase decomposition based on the transposed Farrow structure. This paper also shows that both polyphase structures have the same multiplication rate, while the polyphase decomposition based on the prolonged Farrow structure has considerable smaller number of embodies multipliers. Further, we show that a polynomial-based digital filter can be used as a prototype filter in a polyphase modulated filter bank.

2. FIR FILTER WITH PIECEWISE POLYNOMIAL IMPULSE RESPONSE

In the direct form implementation of FIR filters, each multiplier coefficient determines the value of one impulse response sample independently of the other samples. However, in practical frequency selective filters there is a relatively strong correlation between neighbouring impulse response values. By developing structures that exploit this correlation, the number of multipliers in the implementation can be significantly reduced [8].

An efficient approach to exploit the above-mentioned correlation is to synthesize a linear-phase FIR filter so that its impulse response is approximated by N polynomial segments of order M [8], [9]. It can be shown that any FIR filter transfer function of order $N_{FIR}=NL$-1 is expressible as

$$H(z)=\sum_{m=0}^{M} C_m(z^L)F_m(z)\,. \tag{1}$$

provided that $F_m(z)$ is properly selected. Here the length N impulse responses $C_m(z^L)$ are sparse with only every L^{th} sam-

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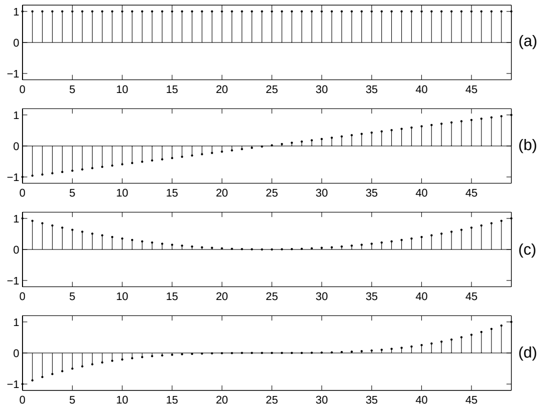

Figure 1 – Basis FIR filters $F_m(z)$ for (a) $m = 0$, (b) $m = 1$, (c) $m = 2$, and (d) $m = 3$. The basis filter is symmetric for even m and anti-symmetric for odd m.

ple being nonzero. The basis filter $F_m(z)$ can be selected in such manner that the overall impulse response can be divided into N blocks of L samples, and in each block the impulse responses are polynomials of the relatively low order M.

Let the transfer function of a symmetric FIR filter be

$$H(z) = \sum_{i=0}^{N_{FIR}} h(i) z^{-i}, \tag{2}$$

where $N_{FIR}=NL-1$. Based on the above discussion, the impulse response can be expressed as [8], [9]

$$h(i) = \sum_{n=0}^{N-1} h_n(i), \tag{3}$$

where

$$h_n(i) = \sum_{m=0}^{M} c_m(n) f_m(i-nL), \tag{4}$$

and

$$f_m(i) = \left[\frac{i-(L-1)/2}{(L-1)/2}\right]^m \text{ for } i=0, 1,\ldots, L-1. \tag{5}$$

In general $f_m(i)$ is an m^{th} order polynomial in i. Further, $f_m(i)$ is symmetric for m even and achieves the value of one both for i=0 and $i=L-1$. The basis FIR filters $F_m(z)$ are depicted in Fig.1 for m=0, 1, 2 and 3. For m odd, $f_m(i)$ is anti-symmetric and achieves the value of minus one at i=0, and one at $i=L-1$. Alternatively and using (3) – (5), $h(i)$ can be rewritten as

$$h(i) = \sum_{m=0}^{M}\sum_{n=0}^{N-1} c_m(n) f_m(i-nL), \tag{6}$$

with

$$f_m(i-nL) = \begin{cases} \left[\dfrac{i-nL-L_2}{L_2}\right]^m & \text{for } nL \le i \le (n+1)L-1 \\ 0 & \text{otherwise} \end{cases}, \tag{7}$$

The corresponding transfer function can be expressed as

$$H(z) = \sum_{m=0}^{M} C_m(z^L) F_m(z), \tag{8}$$

with

$$C_m(z) = \sum_{n=0}^{N-1} c_m(n) z^{-n}, \tag{9}$$

and

$$F_m(z) = \sum_{l}^{L-1} f_m(l) z^{-l}. \tag{10}$$

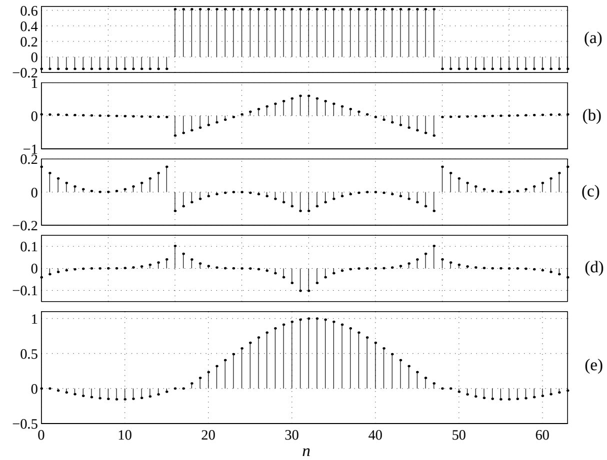

Figure 2 – Construction of the overall impulse response $h(n)$ for $N = 4$, $M = 3$, and L=16. $\bar{h}(i,m) = \sum_{n=0}^{N-1} c_m(n) f_m(i-nL)$ for (a) $m = 0$, (b) $m = 1$, (c) $m = 2$, (d) $m = 3$. (e) The resulting overall impulse response $h(i) = \sum_{m=0}^{M} \bar{h}(i,m)$.

The generation of the impulse response of the overall filter $H(z)$ is illustrated in Fig. 2, where there are four polynomial segments (N=4), a polynomial order of M=3, and L=16 taps per segment. The frequency response of the filter can be determined as [8], [9]:

$$H(e^{j\omega}) = \sum_{m=0}^{M}\sum_{n=0}^{N-1} c_m(n) e^{-j\omega nL} F_m(e^{j\omega}). \tag{11}$$

Since all the basis filters $F_m(z)$ are symmetric for m even, and anti-symmetric for m odd with respect to $(L-1)/2$, the frequency response of the basis filter $F_m(z)$ can be expressed in terms of the constant delay and the zero-phase frequency response as we suppose that L is even. However, the above relations (6)-(9) for the impulse response can be represented in matrix form as [10]:

$$h(i) = \mathbf{C}(\lfloor i/N \rfloor) \cdot \mathbf{F}(i - \lfloor i/N \rfloor), \tag{12}$$

Here $\mathbf{C}(n)$= $[c_0(n)\ c_1(n)\ldots\ c_M(n)]$ is a vector of polynomial coefficients of the polynomial n, and $\mathbf{F}(i)$ is a vector of the basis filter coefficients, i.e. $\mathbf{F}(i)=[f_0(i)\ f_1(i)\ldots\ f_M(i)]^{\mathrm{T}}$. If $\mathbf{h}$ is a vector of FIR filter coefficients, then we can express it as:

$$\mathbf{h} = \mathbf{C} \cdot \hat{\mathbf{F}}, \tag{13}$$

where vector $\mathbf{C}$=[$\mathbf{C}(0)$ $\mathbf{C}(1)\ldots$ $\mathbf{C}(N)$], and $\mathbf{F}$ is a matrix of the following form

$$\hat{\mathbf{F}} = \begin{bmatrix} \mathbf{F} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{F} & \mathbf{0} & \mathbf{0} \\ \vdots & \mathbf{0} & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{F} \end{bmatrix}, \tag{14}$$

with $\mathbf{F}$= [$\mathbf{F}(0)$ $\mathbf{F}(1)\ldots$ $\mathbf{F}(L)$]. If the FIR filter order is N_{FIR}, and if the impulse response is divided into N segments of L taps, and polynomial order is M, then length of vector $\mathbf{C}$ is $N(M+1)$, and matrix $\mathbf{F}$ has $N(M+1)$ rows and $NL= N_{FIR}+1$ columns. Using transformation (13), it is possible to deal

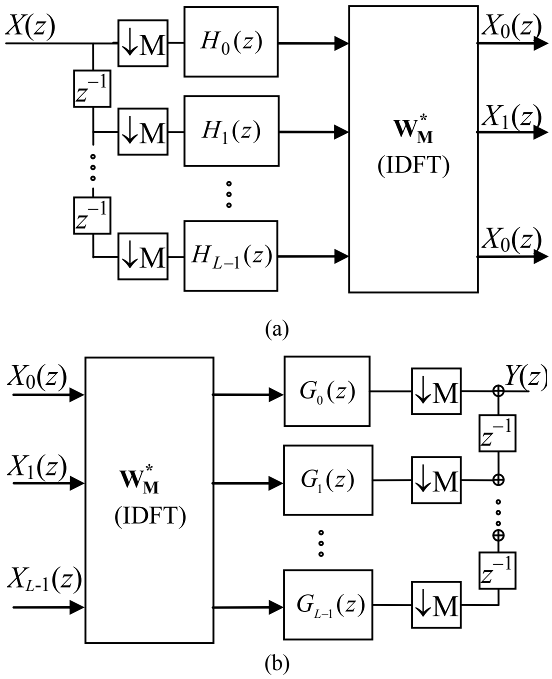

Figure 3 – Polyphase complex modulated filter bank; (a) analysis part, and (b) synthesis part.

with the piecewise polynomial filters as ordinary FIR filters with special properties. This fact and the transformation (13) will be exploited for the filter design.

3. EFFICIENT IMPLEMENTATIONS OF FIR FILTERS WITH PIECEWISE IMPULSE RESPONSES AND THE FARROW STRUCTURE

In this paragraph we consider efficient implementation structures for FIR filters with piecewise polynomial impulse responses both for non-decimating and decimation filters.

If a narrowband FIR filter with a piecewise polynomial impulse response is used in applications where the input sampling frequency is not changed, it is beneficial to use efficient implementation structures [8], [9]. These structures exploit the (anti-)symmetry properties of the basis filters $F_m(z)$ and their well defined impulse responses (see Fig. 1) and, thus, significantly reduce the overall number of multiplications compared to that of conventional FIR implementations. The structure consists of M+1 FIR branch filters $C_m(z^L)$. The impulse responses of these branch filters are sparse with every Lth sample being nonzero, and altogether there are N nonzero filter taps in each branch filter. The role of basis FIR filters $F_m(z)$ is to fill in missing samples. The filters $F_m(z)$ are implemented using recursive relations, reducing number of multipliers [9].

If such a narrowband FIR filter with piecewise polynomial impulse response is used in applications where the input sampling frequency is decreased (the main conclusions are likewise valid for the dual interpolation case), the resulting implementation is based on the Farrow structure, the transposed Farrow or the prolonged Farrow structure [6], [7]. The choice of implementation depends on the overall sampling rate conversion factor $R=F_{in}/F_{out}$. In case of a decimation factor of the form L/K, where L is the number of taps per polynomial segment and K an arbitrary positive integer ($K<L$), the transposed Farrow structure [7] represents the most suitable implementation. We will use this fact in order to define FIR filters with piecewise polynomial impulse responses as prototype filter for DFT modulated filter banks decimating by L.

4. FIR FILTERS WITH PIECEWISE POLYNOMIAL IMPULSE RESPONSES AS PROTOTYPE FILTERS IN DFT MODULATED FILTER BANKS

In order to apply the FIR filter with piecewise polynomial impulse response as a DFT filter bank prototype filter, it is beneficial to find expressions for polyphase components of the filter [1]. When decimating by factor of K, the first step is to express the overall transfer function as

$$H(z)=\sum_{k=0}^{K-1} z^{-k} H_k(z^K) \qquad (15)$$

The transfer function is a sum of K branch filters, with the transfer functions

$$H_k(z^K)=\sum_{m=0}^{M} C_m(z^K) f_m(k) \qquad (16)$$

If it is required that the number of polyphase components K is equal to the number of samples per polynomial segment L, i.e. K=L then we can write:

$$H_k(z^L)=\sum_{m=0}^{M} C_m(z^L) f_m(k)\,. \qquad (17)$$

A single nth tap in the kth polyphase branch can be expressed as [10], [11]:

$$h_k(n)=\sum_{m=0}^{M} c_m(n) f_m(k) \qquad (18)$$

Equation (15) can be rewritten in the following form

$$H(z)=\sum_{k=0}^{K-1} z^{-k} \sum_{m=0}^{M} f_m(k) \sum_{n=0}^{N-1} c_m(n) z^{-nL}\,, \qquad (19)$$

and the polyphase branch accordingly

$$H_k(z^L)=\sum_{m=0}^{M} f_m(k) \sum_{n=0}^{N-1} c_m(n) z^{-nL} \qquad (20)$$

Based on the above equations, it is possible to construct the two following polyphase decompositions for implementation: (i) polyphase decomposition based on the prolonged Farrow structure, and (ii) polyphase decomposition based on the transposed Farrow structure.

The polyphase decomposition based on the prolonged Farrow structure is shown in Fig. 4. As it can be seen, this structure has M+1 branch filters $C_m(z^L)$, as explained in Chapter 3 and in (18)-(20), operating at the high input sampling rate F_{in}. The branch filters $C_m(z^L)$ are followed by the network of the M+1 basis filters $F_m(z)$, which are decomposed into L polyphase branches and are operated at low output sampling rate F_{out}. In fact, each filter $F_m(z)$ has only one tap in each polyphase branch. Furthermore, filter $F_0(z)$ has all taps equal to unity, while all others have very simple coefficients that can be realized with a small number of bits each. The total implementation cost of the structure can be expressed in terms of the multiplication rate, which depends

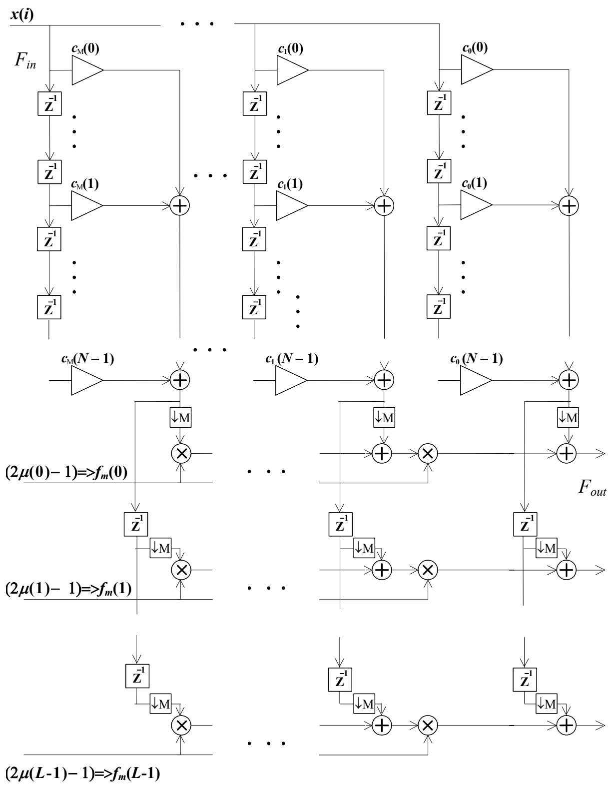

Figure 4 – The polyphase decomposition of the polynomial-based filter based on the prolonged Farrow structure

on the overall number of multipliers and the respective multiplier clock. The overall number of coefficients of this structure is $N(M+1)+LM$. The multiplication rate of the polyphase decomposition based on the prolonged Farrow structure can be expressed as:

$$S_{PF} = N(M+1)F_{in} + LMF_{out} \tag{21}$$

The polyphase decomposition based on the transposed Farrow structure is obtained from previous structure by transposition, using noble identities and shifting all operations to the low output sampling rate. A single k^{th} polyphase branch, $H_k(z)$ in Fig 3, is shown in Fig. 5. One can observe that there is a transposed Farrow structure replicated in each polyphase branch. The transposed Farrow structure consists of M+1 branch filters $C_m(z)$, with each branch filter multiplied by the corresponding tap of basis filter $f_m(i)$. The overall number of coefficients for this structure is $L[N(M+1)+M]$. The multiplication rate for the polyphase decomposition based on the transposed Farrow structure is given by:

$$S_{TF} = F_{out}[LN(M+1) + LM]. \tag{22}$$

By comparing S_{PF} of (21) and S_{TF} of (22), we can conclude that both polyphase structures have the same multiplication rates, since $F_{in}=LF_{out}$, while the polyphase decomposition based on the prolonged Farrow structure has a considerably smaller number of multipliers. Both structures are equivalent in terms of filter performance in the frequency domain.

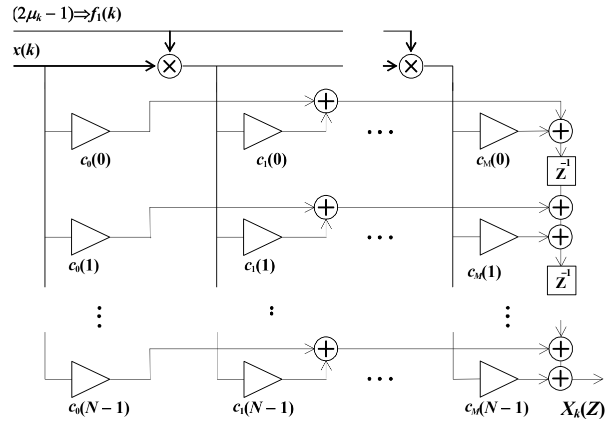

Figure 5 – The k^{th} polyphase branch $H_k(z)$of the polyphase decomposition of the polynomial-based filter based on the transposed Farrow structure

5. DESIGN EXAMPLES

Let us now consider the properties of a complex modulated 16-channel filter bank, thus K=L=16. For the prototype filter, we will use a polynomial-based filter having N=8 polynomial segments, with polynomial order M=5. The polynomial-based filter is optimised in the least squares sense [4], and it is converted to an FIR filter using the transformation of (13). In each segment, there are L=K=16 taps, thus N_{FIR}=NL=160 coefficients are used for the FIR prototype. Fig. 6 shows the frequency response $H(e^{j\Omega})$ of the prototype filter in the interval $0 \leq \Omega \leq \pi$. It can be seen that the attenuation in the stopband $\Omega \geq 2\pi/16$ exceeds 60 dB. Fig. 7 shows the actual frequency response of the designed 16-channel complex modulated filter bank.

The reconstruction error (linear distortion, aliasing) of a subband coder filter bank pair can be taken as a measure for the quality of the filter bank [1]. Figure 8 gives a quantitative view of the two distortions of the filter bank as function of frequency. Figure 8(a) shows the linear distortion of the obtained filter bank, while Fig. 8(b) shows the aliasing function.

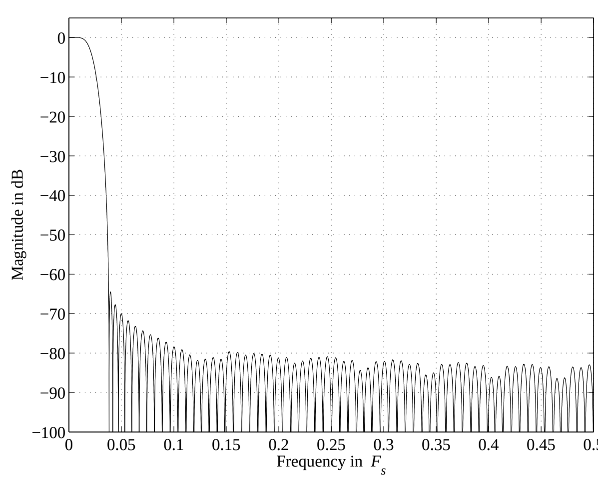

Figure 6 – The magnitude response of the polynomial-based prototype filter.

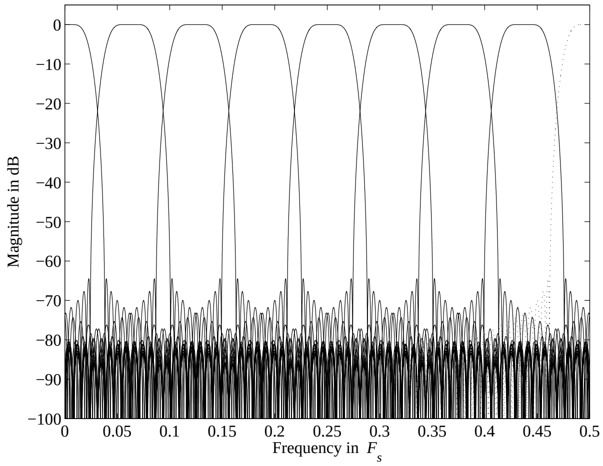

Figure 7 – Frequency response of a complex modulated filter bank, and frequency shifted channel filters.

This is always less than 60dB, and has the same order of the magnitude as the signals in the stopband of the prototype.

6. CONCLUSIONS

In this paper, the polyphase decomposition of polynomial-based FIR filters has been derived. We have found two polyphase decompositions: (*i*) polyphase decomposition applied to the prolonged Farrow structure, and (*ii*) polyphase decomposition based on the transposed Farrow structure. We have also shown that polynomial-based FIR filters can be used as prototype filters in polyphase modulated filter banks. The remaining task is to find a suitable design method which will be used to determine the polynomial coefficients. The polynomial coefficients shall be determined in such way that the respective subband coder filter bank achieves nearly perfect reconstruction.

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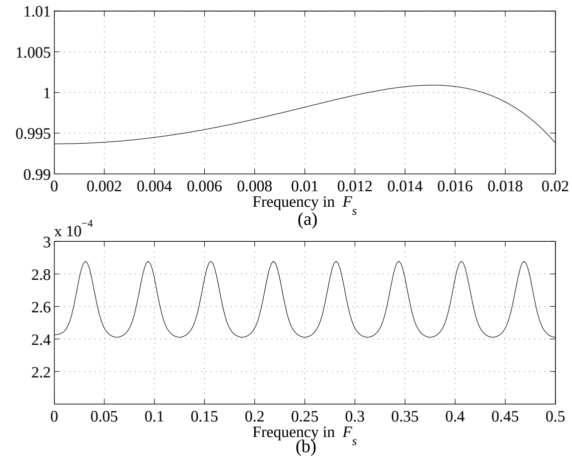

Figure 8 – Reconstruction errors of the 16-channel filter bank: (a) linear distortion, and (b) aliasing.

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