DESIGN OF FRACTIONAL ORDER DIFFERENTIATOR USING DISCRETE HARTLEY TRANSFORM

Chien-Cheng Tseng[1] *and Su-Ling Lee*[2]

[1]Depart. of Computer and Communication Engineering
National Kaohsiung First University of Sci. and Tech.
Kaohsiung, Taiwan
tcc@ccms.nkfust.edu.tw

[2]Depart. of Computer Sci. and Information Engi.
Chung-Jung Christian University
Tainan, Taiwan
lilee@mail.cjcu.edu.tw

ABSTRACT

In this paper, the design of fractional order differentiator is investigated. First, the discrete Hartley transform (DHT) interpolation method is described. Then, the non-integer delay sample estimation of discrete-time sequence is derived by using DHT interpolation. Next, the Grünwald-Letnikov derivative and non-integer delay sample estimation are applied to obtain the transfer function of fractional order differentiator. Finally, some numerical comparisons with conventional fractional order differentiators are made to demonstrate the effectiveness of this new design approach.

1. INTRODUCTION

In recent years, the concepts of fractional operator and measure have been investigated extensively in many engineering applications and science. Four typical examples are described as follows: The first is that the fractal dimension is used to measure some real-world data such that coastline, clouds, dust in the air, and networks of neurons in the body [1]. The fractal dimension has been widely used in pattern recognition and classification. The second is that the fractional Fourier transform has been studied in the optical community and signal processing area [2]. The third is that fractional lower order moment has been used to analyze non-Gaussian signals, which is more realistic than the Gaussian model in signal processing applications [3]. The last is that fractional calculus has been received great attentions in many engineering applications and science including fluid flow, automatic control, electrical networks, electromagnetic theory and image processing [4][5].

In the research area of fractional calculus, the integer order n of derivative $D^n f(x) = \frac{d^n f(x)}{dx^n}$ of function $f(x)$ is generalized to fractional order $D^\upsilon f(x)$, where υ is a real number. One of important research topics in fractional calculus is to implement the fractional operator D^υ in continuous and discrete time domains. An excellent survey of this implementation has been presented in [6]. For continuous time case, some methods for obtaining an approximated rational function using evaluation, interpolation and curve fitting techniques have been studied. These methods include Carlson's method, Roy's method, Chareff's method and Oustaloup's method [6]. For discrete time case, there have been several methods presented to design FIR and IIR filters for implementing operator D^υ, including fractional differencing formula, Tustin method, Taylor series expansion, continued fraction, and least-squares method [7]-[10].

On the other hand, the Hartley transform was presented by Hartley for analyzing transmission problem in 1942 [11]. In 1983, Bracewell introduced the discrete Harley transform (DHT) and derived its fast computation algorithm [12][13]. Given the discrete-time sequence $x(0)$, $x(1)$, ..., $x(N-1)$, the DHT pairs are defined by

$$\begin{aligned} X(k) &= \sum_{m=0}^{N-1} x(m)\, cas\left(\tfrac{2\pi km}{N}\right) \\ x(m) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, cas\left(\tfrac{2\pi km}{N}\right) \end{aligned} \tag{1}$$

where $cas(\cdot) = \cos(\cdot) + \sin(\cdot)$. So far, DHT has been successfully applied to image processing, data interpolation, transform-domain adaptive filtering, data compression and fractional delay filter design [14][15]. In this paper, we will use DHT-based interpolation method and Grünwald-Letnikov derivative to design digital fractional order differentiator. The details are described in next sections.

2. DISCRETE HARTLEY TRANSFORM INTERPOLATION METHOD

In this section, the zero-padding in DHT domain will be applied to interpolate discrete-time signal $x(0)$, $x(1)$, …., $x(N-1)$.Without losing generality, we only consider the case of even-length N. Also, we assume that M is an integer multiple of N, say $M=NL$, where L is the interpolation factor. Given the DHT $X(k)$ in Eq.(1), let us define the zero-padded DHT as

$$X_d(k) = \begin{cases} LX(k) & k \in [0, \frac{N}{2}-1] \\ \frac{L}{2}X(\frac{N}{2}) & k = \frac{N}{2} \\ 0 & k \in [\frac{N}{2}+1, M-\frac{N}{2}-1] \\ \frac{L}{2}X(\frac{N}{2}) & k = M-\frac{N}{2} \\ LX(k-M+N) & k \in [M-\frac{N}{2}+1, M-1] \end{cases} \tag{2}$$

The above DHT has zero values at high frequency band. Now, the interpolated sequence $x_d(n)$ is defined as the length-M inverse DHT of $X_d(k)$, that is,

$$x_d(n) = \frac{1}{M}\sum_{k=0}^{M-1} X_d(k)cas\left(\tfrac{2\pi kn}{M}\right) \quad (3)$$

Substituting Eq.(2) into Eq.(3), we get

$$x_d(n) = \frac{1}{N}\left\{X(0) + \sum_{k=1}^{\frac{N}{2}-1} X(k)cas\left(\tfrac{2\pi kn}{M}\right)\right\} + \frac{1}{N}X\left(\tfrac{N}{2}\right)\cos\left(\tfrac{n\pi}{L}\right) + \frac{1}{N}\sum_{k=1}^{\frac{N}{2}-1} X(N-k)cas\left(\tfrac{-2\pi kn}{M}\right) \quad (4)$$

Using Eq.(1) and the following equality

$$cas(\theta_1)cas(\theta_2) + cas(-\theta_1)cas(-\theta_2) = 2\cos(\theta_1 - \theta_2) \quad (5)$$

then Eq.(4) can be rewritten as

$$x_d(n) = \frac{1}{N}\sum_{m=0}^{N-1} x(m)\left\{\begin{array}{l} 1 + (-1)^m \cos\left(\frac{n\pi}{L}\right) \\ + 2\sum_{k=1}^{\frac{N}{2}-1}\cos\left(\frac{2\pi k\left(m-\frac{n}{L}\right)}{N}\right)\end{array}\right\} \quad (6)$$

Obviously, the interpolated value of $x_d(n)$ is just the weighted average of the data $x(m)$ (m=0,1,...,N-1). Moreover, this interpolator will satisfy the following property:

$$x_d(iL) = x(i) \quad (7)$$

that is, the interpolation becomes an identity at the time points of the original length-N signal. Because $x_d(n)$ is the interpolated sequence of $x(n)$ with factor L, we have the following relation:

$$x_d(iL + p) \approx x\left(i + \tfrac{p}{L}\right) \quad (8)$$

for $0 \le p \le L-1$ and $0 \le i \le N-1$. When p=0, Eq.(8) reduces to Eq.(7). Combining Eq.(6) and Eq.(8), we have

$$x\left(i + \tfrac{p}{L}\right) \approx \sum_{m=0}^{N-1} x(m)b\left(m, i + \tfrac{p}{L}\right) \quad (9)$$

where interpolation basis is given by

$$b\left(m, i + \tfrac{p}{L}\right) = \frac{1}{N}\left\{\begin{array}{l} 1 + (-1)^m \cos\left(\pi\left(i + \frac{p}{L}\right)\right) \\ + 2\sum_{k=1}^{\frac{N}{2}-1}\cos\left(\frac{2\pi k\left(m-i-\frac{p}{L}\right)}{N}\right)\end{array}\right\} \quad (10)$$

Using the identities $\cos(\theta_1 - \theta_2) = \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)$, $\cos(m\pi) = (-1)^m$ and $\sin(m\pi) = 0$, we have

$$\begin{aligned}(-1)^m &\cos\left(\pi\left(i + \tfrac{p}{L}\right)\right) \\ &= \cos(m\pi)\cos\left(\pi\left(i + \tfrac{p}{L}\right)\right) \\ &= \cos\left(\frac{2\pi\left(i + \frac{p}{L} - m\right)\frac{N}{2}}{N}\right)\end{aligned} \quad (11)$$

Substituting Eq.(11) into Eq.(10), the basis is rewritten as

$$b\left(m, i + \tfrac{p}{L}\right) = \frac{1}{N}\sum_{k=0}^{\frac{N}{2}} \beta_k \cos\left(\frac{2\pi\left(i + \frac{p}{L} - m\right)k}{N}\right) \quad (12)$$

where

$$\beta_k = \begin{cases} 1 & k = 0, \frac{N}{2} \\ 2 & k = 1, 2, \cdots, \frac{N}{2} - 1\end{cases} \quad (13)$$

Let $t = i + \frac{p}{L}$, then the value of t can be any real number in [0,N) if factor L approaches infinity. Substituting $t = i + \frac{p}{L}$ into Eq.(9), we get

$$x(t) \approx \sum_{m=0}^{N-1} x(m)b(m,t) \quad (14)$$

This means that the continuous-time signal $x(t)$ can be approximately reconstructed from its samples $x(0)$, $x(1)$, ..., $x(N\text{-}1)$ in the range [0,N) by using continuous-time interpolation basis $b(m,t)$.

3. NON-INTEGER DELAY SAMPLE ESTIMATION

In this section, we will use DHT interpolation method to solve non-integer delay sample estimation problem because the proposed fractional order differentiator design method is based on this estimation method. The problem to be studied is how to estimate non-integer delay sample $s(n-I-d)$ from the given integer delay samples $s(n)$, $s(n-1)$, $s(n-2)$,..., $s(n-N+1)$, where I and N are integers and d is a real number in the interval $[0,1]$. And, I is usually chosen in the range $[0, N-1]$. In this paper, we use the weighted average approach to achieve the purpose, that is, non-integer delay sample is estimated by

$$s(n-I-d) = \sum_{r=0}^{N-1} w(r, I+d)s(n-r) \quad (15)$$

Now, the remaining problem is how to use the DHT interpolation method in the preceding section to determine the weights $w(r, I+d)$. To solve this problem, we choose

$$x(t) = s(n-(N-1)+t) \quad (16)$$

Substituting Eq.(16) into Eq.(14), we get

$$s(n-(N-1)+t) \approx \sum_{m=0}^{N-1} s(n-(N-1)+m)b(m,t) \quad (17)$$

Let $m = N-1-r$, this expression becomes

$$s(n-(N-1)+t) \approx \sum_{r=0}^{N-1} s(n-r)b(N-1-r,t) \quad (18)$$

Replacing t by $N-1-I-d$, the above equation can be rewritten as

$$s(n-I-d) \approx \sum_{r=0}^{N-1} b(N-1-r, N-1-I-d)s(n-r) \quad (19)$$

Comparing Eq.(15) with Eq.(19), we get

$$\begin{aligned} w(r, I+d) &= b(N-1-r, N-1-I-d) \\ &= \frac{1}{N}\sum_{k=0}^{\frac{N}{2}} \beta_k \cos\left(\frac{2\pi(r-I-d)k}{N}\right)\end{aligned} \quad (20)$$

Finally, given integer N, and delay $I+d$, the procedure to estimate non-integer delay sample $s(n-I-d)$ from the given integer delay samples $s(n)$, $s(n-1)$, $s(n-2)$,..., $s(n-N+1)$ is summarized below:

Step 1: Use Eq.(20) to compute the weights $w(r, I+d)$.

Step 2: The non-integer delay sample is estimated by $s(n-I-d)=\sum_{r=0}^{N-1} w(r,I+d)s(n-r)$.

4. DESIGN OF FRACTIONAL ORDER DIFFERENTIATOR

In the literature, there are several definitions of fractional derivative and integral such as the Riemann-Liouville, the Grünwald-Letnikov and the Caputo definitions [5]. In this paper, we will use the Grünwald-Letnikov derivative whose definition is given by

$$D^{\upsilon}s(t)=\lim_{h\to 0}\sum_{k=0}^{\infty}\frac{(-1)^k C_k^{\upsilon}}{h^{\upsilon}}s(t-kh) \tag{21}$$

where coefficient C_k^{υ} is given by

$$C_k^{\upsilon}=\begin{cases}1 & k=0\\ \dfrac{\upsilon(\upsilon-1)(\upsilon-2)\cdots(\upsilon-k+1)}{1\cdot 2\cdot 3\cdots k} & k\ge 1\end{cases} \tag{22}$$

Based on this definition, it can be shown that the fractional derivatives of exponential and sinusoidal signals are given by

$$D^{\upsilon}e^{\alpha t}=\alpha^{\upsilon}e^{\alpha t} \tag{23a}$$

$$D^{\upsilon}A\sin(\omega t+\phi)=A\omega^{\upsilon}\sin(\omega t+\phi+\tfrac{\pi}{2}\upsilon) \tag{23b}$$

Now, let us study the fractional derivative in frequency-domain below. It is well-known that the continuous time Fourier transform pair of signal $s(t)$ is defined by

$$S(\omega)=\int_{-\infty}^{\infty}s(t)e^{-j\omega t}dt \tag{24a}$$

$$s(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(\omega)e^{j\omega t}d\omega \tag{24b}$$

Taking the fractional derivative at both sides of Eq.(24b), we get

$$\begin{aligned}D^{\upsilon}s(t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(\omega)D^{\upsilon}[e^{j\omega t}]\ d\omega\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}(j\omega)^{\upsilon}S(\omega)e^{j\omega t}\ d\omega\end{aligned} \tag{25}$$

Thus, the Fourier transform of fractional derivative $D^{\upsilon}s(t)$ is $(j\omega)^{\upsilon}S(\omega)$. This means that when a signal $s(t)$ passes through a differentiator with frequency response $(j\omega)^{\upsilon}$, then the output of differentiator is fractional derivative $D^{\upsilon}s(t)$. Thus, the ideal frequency response of fractional order differentiator is $(j\omega)^{\upsilon}$. So far, the definition of fractional derivative has been described. In what follows, let us use the DHT interpolation method and Grünwald-Letnikov derivative in Eq.(21) to design a digital fractional order differentiator that approximates the following frequency domain specification as well as possible:

$$H_d(\omega)=(j\omega)^{\upsilon}e^{-j\omega I} \tag{26}$$

where I is a prescribed delay value. First, let us define coefficients $a(k)$ below

$$a(k)=(-1)^k C_k^{\upsilon} \tag{27}$$

then the fractional derivative in Eq.(21) can be rewritten as

$$D^{\upsilon}s(t)=\lim_{h\to 0}\sum_{k=0}^{\infty}\frac{a(k)}{h^{\upsilon}}s(t-kh) \tag{28}$$

Fig.1 shows the coefficient sequence $a(k)$ for various order υ. It is clear that the $a(k)$ is a rapidly decaying sequence for various order υ. Thus, by truncation, $D^{\upsilon}s(t)$ in Eq.(28) can be approximated by

$$D^{\upsilon}s(t)\approx\lim_{h\to 0}\sum_{k=0}^{K}\frac{a(k)}{h^{\upsilon}}s(t-kh) \tag{29}$$

where K is truncation order. Moreover, by removing limit, the $D^{\upsilon}s(t)$ can be further approximated by

$$D^{\upsilon}s(t)\approx\sum_{k=0}^{K}\frac{a(k)}{h^{\upsilon}}s(t-kh) \tag{30}$$

Obviously, the smaller h is, the better approximation in Eq.(30) has. By taking $t=n-I$, the discrete-time derivative signal $D^{\upsilon}s(n-I)$ can be obtained as

$$D^{\upsilon}s(n-I)\approx\sum_{k=0}^{K}\frac{a(k)}{h^{\upsilon}}s(n-I-kh) \tag{31}$$

Because $s(n-I-kh)$ are non-integer delay samples of signal $s(n)$, the $s(n-I-kh)$ needs to be estimated by using the formula in Eq.(15):

$$s(n-I-kh)=\sum_{r=0}^{N-1}w(r,I+kh)s(n-r) \tag{32}$$

Substituting Eq.(32) into Eq.(31), we have

$$\begin{aligned}D^{\upsilon}s(n-I)&\approx\sum_{k=0}^{K}\frac{a(k)}{h^{\upsilon}}\sum_{r=0}^{N-1}w(r,I+kh)s(n-r)\\ &=\sum_{r=0}^{N-1}\left[\frac{1}{h^{\upsilon}}\sum_{k=0}^{K}a(k)w(r,I+kh)\right]s(n-r)\end{aligned} \tag{33}$$

Defining coefficient

$$g(r)=\frac{1}{h^{\upsilon}}\sum_{k=0}^{K}a(k)w(r,I+kh) \tag{34}$$

then Eq.(33) can be rewritten as the following convolution form:

$$\begin{aligned}D^{\upsilon}s(n-I)&\approx\sum_{r=0}^{N-1}g(r)s(n-r)\\ &=g(n)*s(n)\end{aligned} \tag{35}$$

where * denotes the convolution sum operator. Taking z-transform at both sides of Eq.(35), we get

$$Y(z)=\left(\sum_{r=0}^{N-1}g(r)z^{-r}\right)S(z) \tag{36}$$

where $Y(z)$ is z-transform of $D^{\upsilon}s(n-I)$ and $S(z)$ is z-transform of $s(n)$. Let FIR filter be defined as

$$G(z)=\sum_{r=0}^{N-1}g(r)z^{-r} \tag{37}$$

then $G(z)$ is the transfer function of the designed fractional order differentiator which will approximate ideal frequency response $(j\omega)^{\upsilon} e^{-j\omega I}$ well. Now, given integer N, fractional order υ, delay I, integer K and small positive number h, the procedure to design fractional order differentiator $G(z)$ is summarized below:

Step 1: Use Eq.(20) to compute the weights $w(r, I + kh)$.

Step 2: Compute coefficients $a(k)$ by using Eq.(27).

Step 3: Use Eq.(34) to calculate coefficients $g(r)$.

Step 4: The transfer function of the designed fractional order differentiator is given by $G(z) = \sum_{r=0}^{N-1} g(r) z^{-r}$.

Finally, some remarks are made as follows: First, a large integer K needs to be chosen for reducing truncation error which occurs in Eq.(29). Second, a smaller positive number h needs to be chosen for reducing the approximation error which occurs in Eq.(30). Third, if N is large, the designed fractional order differentiator is a long-length FIR filter. To reduce implementation complexity, the Prony method in [16] can be used to approximate long-length FIR filter $G(z)$ by an IIR filter below:

$$\overline{G}(z) = \frac{\sum_{n=0}^{N_1} g_1(n) z^{-n}}{1 + \sum_{n=1}^{N_1} g_2(n) z^{-n}} \tag{38}$$

5. DESIGN EXAMPLES AND COMPARISONS

In this section, we will study the design error of the proposed DHT-based fractional order differentiator and compare it with conventional methods. To evaluate the performance, the integral squares error of frequency response is defined by

$$E = \sqrt{\int_0^{\lambda\pi} \left| G(e^{j\omega}) - H_d(\omega) \right|^2 d\omega} \tag{39}$$

Obviously, the smaller the error E is, the better performance of design method has.

Example 1: In this example, the design parameters of the proposed method are chosen as $N = 100$, $I = 40$, $\upsilon = 0.5$, $K = 1000$ and $h = 0.02$. Fig.2(a) depicts the magnitude responses (solid line) of the $G(z)$. The dashed line is the ideal magnitude response ω^{υ}. So, the specification is fitted well except the region near $\omega = \pi$. Fig.2(b) shows the phase response $90 * [angle(G(e^{j\omega})) + \omega I] / 0.5\pi$ in degree. The dashed line is the ideal response 90υ. It can be observed that the specification is approximated well. Now, let us compare the proposed method with the conventional fractional delay method in [9]. When the design parameters are chosen as $N = 100$, $I = 40$ and $\upsilon = 0.5$, the conventional fractional order differentiator is designed by the Lagrange fractional delay method used in Fig.4 of [9]. Fig.2(c)(d) show the designed results (solid line) of this method. The dashed line is ideal response. It can be seen that the actual response does not fit the ideal response well in high frequency region. If $\lambda = 0.9$ is chosen, the error E of conventional fractional delay method in [9] is 0.1255, and the error E of proposed DHT method is 0.0287. Thus, the proposed method has smaller design error than the conventional method in [9].

Example 2: In this example, let us compare the proposed method with the conventional time domain least-squares method in [10] whose design procedure is described below:

Step 1: Expand the fractional order Tustin differentiator $[U(z)]^{\upsilon}$ as the following power series form:

$$\begin{aligned} [U(z)]^{\upsilon} &= \left(2 \frac{1 - z^{-1}}{1 + z^{-1}} \right)^{\upsilon} \\ &= 2^{\upsilon} \left[\sum_{k=0}^{\infty} C_k^{\upsilon} (-z^{-1})^k \right] \left[\sum_{k=0}^{\infty} C_k^{-\upsilon} z^{-k} \right] \\ &= 2^{\upsilon} \left(1 + \sum_{k=1}^{\infty} u(k) z^{-k} \right) \end{aligned} \tag{40}$$

where filter coefficient $u(k)$ is the convolution sum of $(-1)^k C_k^{\upsilon}$ and $C_k^{-\upsilon}$. After truncating the high-order terms, $[U(z)]^{\upsilon}$ can be approximated by FIR filter

$$\overline{U}(z) = 2^{\upsilon} \left(1 + \sum_{k=1}^{N_c - 1} u(k) z^{-k} \right) \tag{41}$$

where N_c is the truncation length.

Step 2: Using the Prony method, the long-length FIR filter $\overline{U}(z)$ can be approximated by the IIR filter below:

$$\hat{U}(z) = \frac{\sum_{n=0}^{N_2} u_1(n) z^{-n}}{1 + \sum_{n=1}^{N_2} u_2(n) z^{-n}} \tag{42}$$

Then, the frequency response of filter $z^{-I}\hat{U}(z)$ will approximate the ideal response $H_d(\omega) = (j\omega)^{\upsilon} e^{-j\omega I}$ well. Now, one example is used to compare this conventional design with the proposed design in Eq.(38). The parameters in conventional design are chosen as $N_c = 60$, $N_2 = 10$ and $\upsilon = 0.5$. Fig.3(a)(b) show the magnitude and phase responses (solid line) of the designed differentiator $\hat{U}(z)$. The dashed line is ideal response. The maximum pole radius is 0.9941, so IIR filter $\hat{U}(z)$ is stable. From this result, it is clear that the error of phase is very small, but the magnitude error at high frequency band is very large. After $G(e^{j\omega})$ in Eq.(39) is changed to $e^{-j\omega I}\hat{U}(e^{j\omega})$, the error E with $\lambda = 0.9$ is 13.4803 for this traditional design. For comparison, the designed results of proposed DHT method are reported below. The design parameters are chosen as $N = 60$, $I = 9$, $K = 1000$, $h = 0.02$, $N_1 = 10$, and $\upsilon = 0.5$. Fig.3(c)(d) show the magnitude and phase responses (solid line) of the designed IIR differentiator $\overline{G}(z)$ in Eq.(38). The dashed line is ideal response. The maximum pole radius is 0.8169, so IIR filter $\overline{G}(z)$ is stable. Compared Fig.3(a)(b) with Fig.3(c)(d), it can be observed that the proposed DHT method has better magnitude response than conventional method. However, the phase response error of conventional approach is smaller than the proposed method. After $G(e^{j\omega})$ in Eq.(39) is changed to $\overline{G}(e^{j\omega})$, the error E with $\lambda = 0.9$ is 2.0015 for the DHT design.

6. CONCLUSIONS

In this paper, the design of fractional order differentiator has been presented. First, the DHT interpolation method is described. Then, non-integer delay sample estimation of discrete-time sequence is derived by using DHT interpolation approach. Next, the Grünwald-Letnikov derivative and non-integer delay sample estimation are applied to obtain the transfer function of fractional order differentiator. Finally, the numerical examples are studied to show the usefulness of this new design approach. However, only one-dimensional fractional order differentiator design is studied here. Thus, it is interesting to extend the proposed DHT interpolation method to design multi-dimensional fractional order differentiators in the future.

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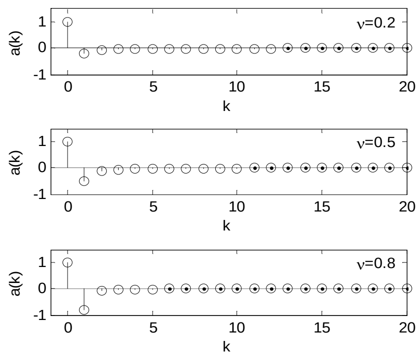

Fig.1 The coefficient sequence $a(k)$ for various order υ.

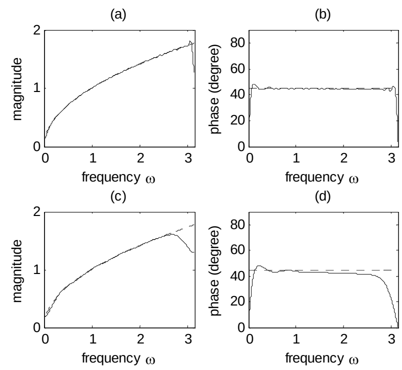

Fig.2 The designed results (solid line) of the fractional order FIR differentiator. (a)(b) The results of the proposed DHT method. (c)(d) The results of the fractional delay method in [9]. The dashed line is the ideal response.

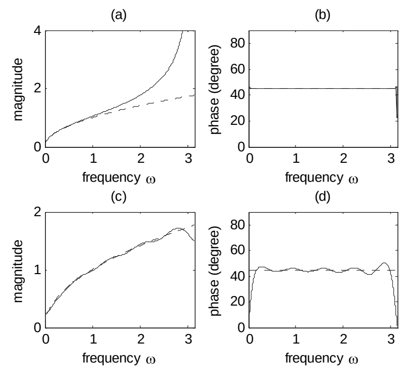

Fig.3 The designed results (solid line) of the fractional order IIR differentiator. (a)(b) The results of $\hat{U}(z)$ in conventional method. (c)(d) The results of $\overline{G}(z)$ in proposed DHT method. The dashed line is the ideal response.