# ON SPECTRAL CONCENTRATION OF SIGNALS ON THE 2-SPHERE UNDER A GENERALIZED MOMENT WEIGHTING CRITERION 

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#### Abstract

This paper formulates a spatially-limited signal energy concentration problem on the 2 -sphere using a generalized moment criterion in the spectral domain. The set of optimal signals with maximum concentration for general positive spherical harmonic coefficient frequency weightings is obtained. Numerically solving the resulting integral equation optimization shows that this set of functions not only decays slower but also has higher sidelobes than the set of spherical Slepian functions. This result on the 2 -sphere contrasts with the findings from the time-frequency analogy which compares the classical Slepian eigenfunctions with the minimum bandwidth basis functions for the fourth-moment bandwidth measure.


## 1. INTRODUCTION

According to the spherical harmonic transform, a spectrallylimited signal occupies the whole sphere, that is, considered as a function, the support of a signal is the whole 2sphere. However, in practice, only part of the signal restricted to some portion of the 2 -sphere is available, that is, a spatially-limited signal is often of interest. In this second case a spatially-limited signal will have infinite spectral content, that is, the spherical harmonic coefficients will be nonzero for arbitrarily high orders corresponding to arbitrarily high frequencies (decaying to zero).

How best to achieve or approximate a signal on the 2sphere of simultaneous limited spectrum and limited spatial extend is a problem of interest. In time and frequency, for signals on $\mathbb{R}$, this has been extensively studied. Of particularly interest is the formulation of by Slepian, Landau and Pollak [1,2] based on criteria of engineering interest. For the 2 -sphere, analogous results have been obtained leading two natural cases: 1) spectrally-limited spherical Slepian functions, and 2) spatially-limited spherical Slepian functions. It has proved to be a useful tool to analyze and represent a signal on the 2-sphere [3-6], though the spatially-limited Slepian signal case was emphasized less.

Our recent research [7] shows that the optimal spectrallylimited function with minimum globally (support is the whole 2 -sphere) $k$ th azimuthal moment weighting not only achieves good spatial concentration, but also has faster decaying tails than the spherical Slepian spectrally-limited function, which is a good alternative for the spherical filter design. Therefore, it is a natural question whether such an optimally spatially-limited function (complementary to

[^0]but not equivalent to the spectrally-limited function case) with spectral moment weighting exists, and whether this set of functions is a good option to represent and analyze a spatially-limited signals on the unit sphere.

In this paper, we formulate a spatially-limited signal energy concentration with a more general form of weighting based on a harmonic multiplication operation [8], but restricted to positive weights, in the spectral domain. A set of optimal spatially-limited functions is obtained and the characteristics of these functions are studied.

## 2. PRELIMINARIES

### 2.1 Notation

Let $\mathbb{S}^{2}=\left\{\boldsymbol{x} \in \mathbb{R}^{3}:\|\boldsymbol{x}\|=1\right\}$ denote the unit sphere in $\mathbb{R}^{3}$. $\boldsymbol{x} \equiv$ $(\theta, \phi) \triangleq(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \in \mathbb{R}^{3}$ denotes a point on the sphere where $\theta$ denotes the colatitude with $0 \leq \theta \leq$ $\pi$ and $\phi$ denotes the longitude with $0 \leq \phi<2 \pi$. Denote $d s(\boldsymbol{x})=\sin \theta d \theta d \phi$.

Let $L^{2}\left(\mathbb{S}^{2}, d s\right)$ be a complex Hilbert space containing all the square-integrable functions defined on the unit sphere $\mathbb{S}^{2}$, such that for $f, g \in L^{2}\left(\mathbb{S}^{2}, d s\right)$, the inner product is defined by

$$
\begin{align*}
\langle f, g\rangle & =\int_{\mathbb{S}^{2}} f(\boldsymbol{x}) \overline{g(\boldsymbol{x})} d s(\boldsymbol{x}) \\
& =\int_{0}^{\pi} \int_{0}^{2 \pi} f(\theta, \phi) \overline{g(\theta, \phi)} \sin \theta d \phi d \theta \tag{1}
\end{align*}
$$

### 2.2 Spherical Harmonic Representation

The spherical harmonics $Y_{n}^{m}(\boldsymbol{x})=Y_{n}^{m}(\theta, \phi)$ are defined as [9]

$$
\begin{gathered}
Y_{n}^{m}(\theta, \phi)=\sqrt{\frac{2 n+1}{4 \pi} \frac{(n-|m|)!}{(n+|m|)!}} P_{n}^{|m|}(\cos \theta) e^{i m \phi} \\
n=0,1, \cdots, m=-n, \cdots, n
\end{gathered}
$$

$P_{n}^{m}(\cdot)$ are the associated Legendre functions, $n$ is the angular (spectral) degree and $m(-n \leq m \leq n)$ is the angular order.

Any finite energy signal $f \in L^{2}\left(\mathbb{S}^{2}, d s\right)$ can be represented, in the sense of convergence in the mean with the norm induced by (1), by

$$
\begin{equation*}
f(\boldsymbol{x})=\sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{n}^{m} Y_{n}^{m}(\boldsymbol{x}) \tag{2}
\end{equation*}
$$

where the spherical harmonic coefficients $f_{n}^{m}$ are given by

$$
\begin{equation*}
f_{n}^{m}=\int_{\mathbb{S}^{2}} f(\boldsymbol{x}) \overline{Y_{n}^{m}(\boldsymbol{x})} d s(\boldsymbol{x}) . \tag{3}
\end{equation*}
$$

## 3. PROBLEM STATEMENT

In this paper, we study a spatially-limited signal energy concentration with positive spectral modes weighting in the spectral domain. The problem is stated that: for a spatiallylimited signal $f$ with support $\Gamma \subset \mathbb{S}^{2}$ and a spectrally-limited signal $v$ with positive spherical harmonic coefficients, the objective function to be extremized (generally maximized) is

$$
\begin{equation*}
\lambda=\frac{\langle v \odot f, v \odot f\rangle}{\langle f, f\rangle}=\text { maximum } \tag{4}
\end{equation*}
$$

where " $\odot$ " is the harmonic multiplication operation defined as [8],

$$
\begin{equation*}
(v \odot f)(\boldsymbol{x})=\sum_{n=0}^{N} \sum_{m=-n}^{n} v_{n}^{m} f_{n}^{m} Y_{n}^{m}(\boldsymbol{x}) \tag{5}
\end{equation*}
$$

$v_{n}^{m} \geq 0$ for all $n$ and $m$, and $N$ is the maximum spectral degree of $v$. Our aim is to find an optimal spatially-limited signal which achieves maximum spectral moment weighting measure.

## 4. FORMULATION

Substituting (5) and (3) into the objective function (4) and changing the summation and the integration, we have (6).

$$
\begin{align*}
& \lambda=\frac{\sum_{n=0}^{N} \sum_{m=-n}^{n}\left|v_{n}^{m} f_{n}^{m}\right|^{2}}{\sum_{n=0}^{\infty} \sum_{m=-n}^{n}\left|f_{n}^{m}\right|^{2}}= \\
& \iint_{\mathbb{S}^{2} \times \mathbb{S}^{2}} \overline{f(\boldsymbol{x})} f(\boldsymbol{y})\left[\sum_{n=0}^{N} \sum_{m=-n}^{n}\left|v_{n}^{m}\right|^{2} \overline{Y_{n}^{m}(\boldsymbol{y})} Y_{n}^{m}(\boldsymbol{x})\right] d s(\boldsymbol{x}) d s(\boldsymbol{y})  \tag{6}\\
& \int_{\mathbb{S}^{2}}|f(\boldsymbol{x})|^{2} d s(\boldsymbol{x})
\end{align*} .
$$

It is well known that to render the Rayleigh quotient (6) stationary, $f(\boldsymbol{x})$ is the solution of the Fredholm integral equation, (technically representing an integral operator on $\left.L^{2}(\Gamma, d s)\right)$,

$$
\begin{equation*}
\int_{\Gamma}\left[\sum_{n=0}^{N} \sum_{m=-n}^{n}\left|v_{n}^{m}\right|^{2} Y_{n}^{m}(\boldsymbol{x}) \overline{Y_{n}^{m}(\boldsymbol{y})}\right] f(\boldsymbol{y}) d s(\boldsymbol{y})=\lambda f(\boldsymbol{x}), \boldsymbol{x} \in \Gamma \tag{7}
\end{equation*}
$$

Denote the integral kernel as

$$
D(\boldsymbol{x}, \boldsymbol{y})=\sum_{n=0}^{N} \sum_{m=-n}^{n}\left|v_{n}^{m}\right|^{2} Y_{n}^{m}(\boldsymbol{x}) \overline{Y_{n}^{m}(\boldsymbol{y})} .
$$

In this paper, for concrete illustration, we restrict the region $\Gamma$ to be the polar cap $[0, \Theta]$, where $\Theta$ is the maximum colatitude of the region.

According to the separability of the spherical harmonics, we have

$$
\begin{equation*}
f(\theta, \phi)=\sum_{m=-N}^{N} e^{i m \phi} \sum_{n=|m|}^{N} f_{n}^{m} S_{n}^{m}(\theta)=\sum_{m=-N}^{N} e^{i m \phi} f_{m}(\theta) \tag{8}
\end{equation*}
$$

where

$$
\begin{aligned}
S_{n}^{m}(\theta) & =\sqrt{\frac{2 n+1}{4 \pi} \frac{(n-|m|)!}{(n+|m|)!}} P_{n}^{|m|}(\cos \theta), \\
f_{m}(\theta) & =\sum_{n=|m|}^{N} f_{n}^{m} S_{n}^{m}(\theta) .
\end{aligned}
$$

Similarly, for the kernel function, we have

$$
\begin{align*}
D(\boldsymbol{x}, \boldsymbol{y}) & =D\left(\theta, \phi, \theta^{\prime}, \phi^{\prime}\right) \\
& =\sum_{n=0}^{N} \sum_{m=-n}^{n}\left|v_{n}^{m}\right|^{2} S_{n}^{m}(\theta) e^{i m \phi} \overline{S_{n}^{m}\left(\theta^{\prime}\right) e^{i m \phi^{\prime}}} \\
& =\sum_{n=0}^{N} \sum_{m=-n}^{n}\left|v_{n}^{m}\right|^{2} S_{n}^{m}(\theta) \overline{S_{n}^{m}\left(\theta^{\prime}\right)} e^{i m \phi} e^{-i m \phi^{\prime}} \\
& =\sum_{m=-N}^{N} \sum_{n=|m|}^{N}\left|v_{n}^{m}\right|^{2} S_{n}^{m}(\theta) S_{n}^{m}\left(\theta^{\prime}\right) e^{i m \phi} e^{-i m \phi^{\prime}} \tag{9}
\end{align*}
$$

Substituting (8) and (9) into (7), we have

$$
\begin{aligned}
\sum_{m=-N}^{N} e^{i m \phi} \int_{0}^{\Theta} 2 \pi & \sum_{n=|m|}^{N}\left|v_{n}^{m}\right|^{2} S_{n}^{m}(\theta) S_{n}^{m}\left(\theta^{\prime}\right) f_{m}\left(\theta^{\prime}\right) \sin \theta^{\prime} d \theta^{\prime} \\
& =\lambda \sum_{m=-N}^{N} e^{i m \phi} f_{m}(\theta), \quad 0 \leq \theta \leq \Theta
\end{aligned}
$$

That is, (7) can be decomposed into a series of fixed-order, one-dimensional Fredholm eigenvalue equations [10],

$$
\begin{equation*}
\int_{0}^{\Theta} D\left(\theta, \theta^{\prime}\right) f_{m}\left(\theta^{\prime}\right) \sin \theta^{\prime} d \theta^{\prime}=\lambda f_{m}(\theta), \quad 0 \leq \theta \leq \Theta \tag{10}
\end{equation*}
$$

where $-N \leq m \leq N$ and the kernel

$$
D\left(\theta, \theta^{\prime}\right)=2 \pi \sum_{n=|m|}^{N}\left|v_{n}^{m}\right|^{2} S_{n}^{m}(\theta) S_{n}^{m}\left(\theta^{\prime}\right)
$$

Solving the integral equation (10), the optimally associated spatially-limited function $f_{m}(\theta)$ for fixed $m$ is obtained,

$$
\begin{equation*}
f_{m}(\theta, \phi)=f_{m}(\theta) e^{i m \phi} \tag{11}
\end{equation*}
$$

From here, we can conclude that the optimally spatiallylimited function that maximizes the concentration ratio (4) is only related to the spectral degree $n$ for fixed $m$, therefore, only the $n$ variable in the weighting function $v_{n}^{m}$ has some effect to the optimal eigenfunction $f_{m}(\theta)$.

## 5. SIMULATIONS

In the section, we formulate the spectrum moment weighting on the unit sphere. We take some special values of $v_{n}^{m}$ as examples to solve the corresponding optimal functions and discuss their properties.

## $5.1 v_{n}^{m}=1$ for all $0 \leq n \leq N$ and $-n \leq m \leq n$

This is the spherical Slepian concentration problem on the unit sphere, which has been well studied in [10]. The measure $\lambda$ is the maximum energy concentrated in the spectral interval $[0, N]$ for a spatially-limited function $f$. The obtained optimal signals are called the spherical prolate spheroidal wave functions (PSWFs), or the spherical Slepian functions.


Figure 1: The normalized eigenfunctions $f_{0}(\theta)$ with the maximum concentration ratio $\lambda_{k}=0.9999,0.7155,0.5802$ under different weighting $v_{n}^{m}=\left(\frac{n+1}{N+1}\right)^{k}$ for varied $k=0,1,2$.

$$
5.2 v_{n}^{m}=\left(\frac{n+1}{N+1}\right)^{k} \text { for all } 0 \leq n \leq N \text { and }-n \leq m \leq n
$$

Take

$$
\begin{equation*}
v_{n}^{m}=\left(\frac{n+1}{N+1}\right)^{k}, \quad k=1,2, \ldots \tag{12}
\end{equation*}
$$

to emulate the moment weighting in the analogy given in [11]. It should be noted that $k=0$ is the special case with $v_{n}^{m}=1$ and its function corresponds to the spherical Slepian function. The objective function is changed into

$$
\lambda=\frac{\sum_{n=0}^{N} \sum_{m=-n}^{n}\left(\frac{n+1}{N+1}\right)^{2 k}\left|f_{n}^{m}\right|^{2}}{\sum_{n=0}^{\infty} \sum_{m=-n}^{n}\left|f_{n}^{m}\right|^{2}}=\text { maximum }
$$

As we have proved before that it is equivalent to solve

$$
\begin{array}{r}
\int_{0}^{\Theta}\left(2 \pi \sum_{n=|m|}^{N}\left(\frac{n+1}{N+1}\right)^{2 k} S_{n}^{m}(\theta) S_{n}^{m}\left(\theta^{\prime}\right)\right) f_{m}\left(\theta^{\prime}\right) \sin \theta^{\prime} d \theta^{\prime} \\
=\lambda f_{m}(\theta), \quad 0 \leq \theta \leq \Theta \tag{13}
\end{array}
$$

We use Gaussian-legendre quadrature method [12] to numerically solve the above integral equation (13) and find the associated eigenfunctions.

### 5.3 Numerical Examples

Take $N=18$ and $\Theta=40^{\circ}$ for comparison with spherical Slepian functions presented in [10]. And we also take $k=0,1,2$ as examples to study the properties of the optimal functions.

Fig. 1 shows the normalized eigenfunctions $f_{m}(\theta)$ with maximum concentration ratio $\lambda$ for $m=0$ and its corresponding squared spherical harmonic coefficients $\left(f_{n}^{0}\right)^{2}$. Fig. 1 shows that: 1) the optimal waveform of $f_{0}(\theta)$ does not vary much as $k$ increases; 2) the peak value of the optimal functions moves to the right as $k$ increases; 3 ) most of the energy of the optimal function concentrated in the first spectral degree $[0,18]$ for all $k ; 4$ ) the spherical harmonic spectrum decays faster as $k$ increases, but the decaying rate is much slower than that of the spherical Slepian function, which differs from the time-frequency analogy [11]. The non-zero value for $f_{0}(\theta)$ to $k \geq 1$ at the boundary $\Theta=40$ also shows the limitation of Gaussian-legendre quadrature method to the inverse problems.

A similar situation is shown in Fig. 2 for $m=1$. However, the calculation error at this time is quite larger, for $f_{1}\left(40^{\circ}\right)=$ -0.0495 for $k=1$ and $f_{1}\left(40^{\circ}\right)=-0.0615$ for $k=2$.

Fig. 3 shows the normalized optimally associated spatially-limited functions $f_{m}(\theta, \phi)$ with the first four maximum concentration ratios $\lambda$ for $m=0$ and $m=1$. Obviously, these figures show that the functions obtained from the $v_{n}^{m}$ weighting have more sidelobes than the spherical Slepian function and the increasing $k$ has little effect to the spatiallylimited signals.

## 6. CONCLUSIONS

This paper formulated a spatially-limited signal energy concentration problem on the 2 -sphere using a generalized moment criterion in the spectral domain based on a harmonic multiplication operation. The set of optimal signals with maximum concentration for general positive spherical harmonic coefficient frequency weightings was obtained. Numerically solving the resulting integral equation optimization shows that this set of functions not only decays slower but also has higher sidelobes than the set of spherical Slepian functions.

The given formulation is very general and the moment weighting is a special case. To emulate the moment weighting analogous to [11], but for the 2 -sphere, we adopted the weighting (12). It remains an open question whether this is the true analogy for the moment weighting given the finite extent and the curvature on the 2 -sphere. If the analogy does differ from (12) our formulation using harmonic multiplication is general enough to deal with this case.

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Figure 2: The normalized eigenfunctions $f_{1}(\theta)$ with the maximum concentration ratio $\lambda_{k}=0.9999,0.7023,0.5836$ under different weighting $v_{n}^{m}=\left(\frac{n+1}{N+1}\right)^{k}$ with $k=0,1,2$.
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Figure 3: The normalized optimally associated spatiallylimited functions $f_{m}(\theta, \phi)$ with the first four maximum concentration ratio under different weighting $v_{n}^{m}=\left(\frac{n+1}{N+1}\right)^{k}$ for varied $k=0,1,2 . k=0$ for the top line, $k=1$ for the middle and $k=2$ for the bottom. (a) $m=0$ and (b) $m=1$.
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