

IMPROVED SUBSPACE-BASED FREQUENCY ESTIMATION FOR REAL-VALUED DATA USING ANGLES BETWEEN SUBSPACES

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ABSTRACT

A multitude of applications contain signals that can be well described as being formed as a sum of sinusoidal components corrupted by noise, and, as a result, the literature contains a large variety of estimation algorithms tailored for this problem. Many of these estimators assume a complex-valued signal model, typically formed using the discrete-time Hilbert transform. For a large number of observations, or with frequencies being neither too high nor too low, this approach works well, whereas it might well cause considerable problems otherwise. One way to handle these situations is to instead form the frequency estimation algorithm assuming a real-valued signal. In this paper, we show how the principle of angles between subspaces can be applied to this problem to alleviate some of the shortcomings of subspace-based frequency estimation using the MUSIC algorithm and demonstrate the resulting attractive properties via computer simulations.

1. INTRODUCTION

When dealing with the problem of estimating the parameters of sinusoidal components from noisy observations, one often employ a complex-valued model of the observed signal, typically converting the measured real-valued signal to its discrete-time analytic counterpart using the Hilbert transform [1, 2]. There are several reasons for doing this. Firstly, it is generally more convenient from a mathematical point of view to manipulate the complex-valued model. Secondly, it can lead to a reduction of the computational complexity of the estimator as only half the number of complex exponentials have to be considered, as well as only half the number of samples, although these now being complex-valued. Thirdly, the estimated parameters are identical to those obtained using a real model under certain conditions, namely that the frequencies of the sinusoids are neither too low nor too high relative to the number of observed samples. However, if this is not the case, there will be significant interaction between the positive and negative sides of the spectrum, which may lead to biased estimates, and for applications where this may happen, one should consider using real signal models instead.

The specific problem considered here can be stated as follows: a signal consisting of real-valued sinusoids having frequencies $\{\omega_l\}$ is corrupted by additive, real-valued, white noise, $\varepsilon(n)$, having variance σ^2 , for $n = 0, \dots, N-1$, i.e.,

$$x(n) = \sum_{l=1}^L A_l \cos(\omega_l n + \phi_l) + \varepsilon(n), \quad (1)$$

where $A_l > 0$ and ϕ_l are the amplitude and the phase of the l 'th sinusoid, respectively. The problem of interest is thus how one should appropriately estimate the frequencies $\{\omega_l\}$ from $\{x(n)\}$, taking into account the fact that the signal is real-valued. Obviously, many methods have been proposed for doing exactly this, with some being more straightforward than others. More specifically, several adaptations of well-known estimators have been proposed, for example for the MUSIC [3], subspace fitting [4], ESPRIT [5], Capon's [6], Pisarenko's [7], and the linear prediction [8] methods. Recently, it has been shown that the underlying principle of MUSIC, i.e., the subspace orthogonality, can be reformulated and interpreted using the principal angles between the signal and noise subspaces, and that this concept can also be successfully applied to order estimation [9] (see also [10] for another application). Moreover, it has been shown that the original MUSIC cost function can be obtained as a special case of this framework, more specifically as an asymptotically valid approximation. The aim of this paper is to explore whether the angles between subspaces can also be applied to the problem of estimating the frequencies of real-valued sinusoids from real-valued measurements.

The remainder of this paper is organized as follows. First, we review the covariance matrix model along with some basic results in Section 2. Then, in Section 3, we introduce the concept of angles between subspaces and its application to estimation. Finally, we then apply it to the problem at hand in Section 4, before presenting some results in Section 5, and concluding on our work in Section 6.

2. COVARIANCE MATRIX MODEL

Before proceeding to address the stated problem, we will first introduce some basic notation and results. We define $\mathbf{x}(n)$ as

$$\mathbf{x}(n) = [x(n) \ x(n+1) \ \cdots \ x(n+M-1)]^T, \quad (2)$$

with $(\cdot)^T$ denoting the transpose. Assuming that the phases of the sinusoids are independent and uniformly distributed on the interval $(-\pi, \pi]$, the covariance matrix $\mathbf{R} \in \mathbb{C}^{M \times M}$ of the signal in (1) can be written as [11]

$$\mathbf{R} = \mathbf{E}\{\mathbf{x}(n)\mathbf{x}^H(n)\} = \mathbf{A}\mathbf{P}\mathbf{A}^H + \sigma^2\mathbf{I}, \quad (3)$$

where $\mathbf{E}\{\cdot\}$, $(\cdot)^H$, and \mathbf{I} denote the statistical expectation, the conjugate transpose, and the identity matrix, respectively. The diagonal matrix \mathbf{P} contains the squared amplitudes on the diagonal, i.e.,

$$\mathbf{P} = \text{diag}([A_1^2 \ A_1^2 \ \cdots \ A_L^2 \ A_L^2]), \quad (4)$$

and $\mathbf{A} \in \mathbb{C}^{M \times 2L}$ is a Vandermonde matrix defined as

$$\mathbf{A} = [\mathbf{a}(\omega_1) \quad \mathbf{a}^*(\omega_1) \quad \cdots \quad \mathbf{a}(\omega_L) \quad \mathbf{a}^*(\omega_L)], \quad (5)$$

with

$$\mathbf{a}(\omega) = [1 \quad e^{j\omega} \quad \cdots \quad e^{j\omega(M-1)}]^T. \quad (6)$$

Assuming that the frequencies $\{\omega_l\}$ are distinct, the columns of \mathbf{A} are linearly independent, and \mathbf{A} and $\mathbf{A}\mathbf{A}^H$ have rank $2L$. It should be noted that it is assumed that $2L < M < N$. Let $\mathbf{R} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H$ be the eigenvalue decomposition (EVD) of the covariance matrix. Then, \mathbf{Q} contains the M orthonormal eigenvectors of \mathbf{R} , i.e.,

$$\mathbf{Q} = [\mathbf{q}_1 \quad \cdots \quad \mathbf{q}_M] \quad (7)$$

and $\mathbf{\Lambda}$ is a diagonal matrix containing the corresponding eigenvalues, λ_k , with $\lambda_1 \geq \cdots \geq \lambda_M$. Let \mathbf{S} be formed from the eigenvectors corresponding to the $2L$ most significant eigenvalues, i.e.,

$$\mathbf{S} = [\mathbf{q}_1 \quad \cdots \quad \mathbf{q}_{2L}]. \quad (8)$$

We denote the signal subspace, i.e., the space spanned by the columns of \mathbf{S} , as $\mathcal{R}(\mathbf{S})$. Similarly, let \mathbf{G} be formed from the eigenvectors corresponding to the $M - 2L$ least significant eigenvalues, i.e.,

$$\mathbf{G} = [\mathbf{q}_{2L+1} \quad \cdots \quad \mathbf{q}_M], \quad (9)$$

and, as a consequence, $\mathcal{R}(\mathbf{G})$ is referred to as the noise subspace. It can then easily be shown that the columns of \mathbf{A} , the sinusoids, span the same space as the columns of \mathbf{S} , and that \mathbf{A} therefore also must be orthogonal to \mathbf{G} , i.e.,

$$\mathbf{A}^H \mathbf{G} = \mathbf{0}, \quad (10)$$

which is the basic result used in the MUSIC algorithm [12, 13, 14]. The MUSIC frequency estimate is formed by finding the parameters for which the candidate model \mathbf{A} is the closest to being orthogonal to \mathbf{G} .

3. ANGLES BETWEEN SUBSPACES

As seen above, the orthogonality property states that for the true parameters, the matrix \mathbf{A} is orthogonal to the noise subspace eigenvectors in \mathbf{G} . However, due to finite sample effects, one cannot expect that the vectors will be exactly orthogonal, implying that one needs a measure for determining the extent to which the orthogonality holds. In the original MUSIC algorithm, this measure was formed using the Frobenius norm. As an alternative, one may instead note that the concept of orthogonality is, of course, closely related to the concept of angles, in this case the (principal) angles between the signal and noise subspaces. We will now briefly introduce the theory behind angles between subspaces and its application to estimation. Let Π_G be the projection matrix for the subspace $\mathcal{R}(\mathbf{G})$ and Π_A the projection matrix for the subspace $\mathcal{R}(\mathbf{A})$. The principal, and non-trivial, angles $\{\theta_k\}$ between the two subspaces are defined recursively, for $k = 1, \dots, K$, as (see, e.g., [15])

$$\cos(\theta_k) = \max_{\mathbf{y} \in \mathbb{C}^M} \max_{\mathbf{z} \in \mathbb{C}^M} \frac{\mathbf{y}^H \Pi_A \Pi_G \mathbf{z}}{\|\mathbf{y}\|_2 \|\mathbf{z}\|_2} \quad (11)$$

$$\triangleq \mathbf{y}_k^H \Pi_A \Pi_G \mathbf{z}_k = \sigma_k, \quad (12)$$

where K is the minimal dimension of the two subspaces, i.e.,

$$K = \min\{L, M - L\}, \quad (13)$$

which is also the number of non-trivial angles between the two subspaces. Moreover, the vectors \mathbf{y} and \mathbf{z} are restricted to being orthogonal in the sense that $\mathbf{y}^H \mathbf{y}_i = 0$ and $\mathbf{z}^H \mathbf{z}_i = 0$ for $i = 1, \dots, k-1$. It then follows that $\{\sigma_k\}$ are the singular values of the matrix product $\Pi_A \Pi_G$. The singular values can be related to the Frobenius norm of the product $\Pi_A \Pi_G$ as $\|\Pi_A \Pi_G\|_F^2 = \sum_{k=1}^K \sigma_k^2$ and, thus, also to the angles between the subspaces, i.e.,

$$\sum_{k=1}^K \cos^2(\theta_k) = \|\Pi_A \Pi_G\|_F^2. \quad (14)$$

Additionally, the Frobenius norm of the product $\Pi_A \Pi_G$ can be expressed as

$$\|\Pi_A \Pi_G\|_F^2 = \text{Tr} \left\{ \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{G} \mathbf{G}^H \right\}. \quad (15)$$

This expression, being somewhat complicated, can be simplified in the following way: the columns of \mathbf{A} consist of complex sinusoids, and for any distinct set of frequencies, these are asymptotically orthogonal, i.e.,

$$\lim_{M \rightarrow \infty} M \Pi_A = \lim_{M \rightarrow \infty} M \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \quad (16)$$

$$= \mathbf{A} \mathbf{A}^H. \quad (17)$$

Based on this approximation, we may now write (15) into a form similar to that of MUSIC i.e.,

$$\|\Pi_A \Pi_G\|_F^2 \approx \frac{1}{M} \text{Tr} \left\{ \mathbf{A}^H \mathbf{G} \mathbf{G}^H \mathbf{A} \right\} = \frac{1}{M} \|\mathbf{A}^H \mathbf{G}\|_F^2, \quad (18)$$

which only differs from the classical MUSIC estimator in the scaling. From this, we note that

$$\frac{1}{M} \|\mathbf{A}^H \mathbf{G}\|_F^2 \approx \sum_{k=1}^K \cos^2(\theta_k), \quad (19)$$

which, interestingly, shows that the original MUSIC cost function can be explained as an approximation to the angles between the subspaces. Here, it must be emphasized that this interpretation only holds for signal models consisting of vectors that are orthogonal or asymptotically orthogonal; the result will thus hold for sinusoids, but not for damped sinusoids. In the MUSIC algorithm, the set of frequencies $\{\omega_l\}$ are found by minimizing the cost function

$$\{\hat{\omega}_l\} = \arg \min_{\{\omega_l\}} \|\mathbf{A}^H \mathbf{G}\|_F^2. \quad (20)$$

As the squared Frobenius norm is additive over the columns of \mathbf{A} , it facilitates finding the individual frequencies as

$$\hat{\omega}_l = \arg \min_{\omega_l} \|\mathbf{a}^H(\omega_l) \mathbf{G}\|_F^2, \quad (21)$$

with the requirements that the frequencies are distinct and fulfill the two following conditions:

$$\frac{\partial \|\mathbf{a}^H(\omega_l) \mathbf{G}\|_F^2}{\partial \omega_l} = 0 \quad \text{and} \quad \frac{\partial^2 \|\mathbf{a}^H(\omega_l) \mathbf{G}\|_F^2}{\partial \omega_l^2} > 0. \quad (22)$$

4. APPLICATION TO REAL-VALUED DATA

The question is now the following: in applying the results from the previous section to the problem of estimating frequencies from real-valued data, what error are we making, if any? The answer can be found in the approximation (17). For sinusoids that are well-separated in frequency (relative to N and M), the approximation can be expected to be a good one. However, for low frequencies, the complex exponentials and their conjugate counterparts cannot be expected to be well-separated. This happens under two circumstances. Firstly, if for a given N , the frequencies are too low or too high, or, secondly, if for a given frequency, N is too low. To take these cases into account, we proceed as follows. Let the matrix \mathbf{A} be partitioned as

$$\mathbf{A} = [\tilde{\mathbf{A}}_1 \quad \dots \quad \tilde{\mathbf{A}}_L] \quad (23)$$

where

$$\tilde{\mathbf{A}}_l = [\mathbf{a}(\omega_l) \quad \mathbf{a}^*(\omega_l)] \quad (24)$$

contains a complex sinusoid and its complex conjugate. Now, ideally, one would use the exact expression for the projection matrix and then solve for the unknowns, i.e., $\{\omega_l\}$. However, this generally results in a useless estimator as the complexity of doing so would be prohibitive—it results in a multidimensional nonlinear optimization problem. Instead, we will proceed by assuming that the approximation in (17) is valid for the sinusoidal components with different frequencies, but not for a complex sinusoids and its complex conjugate. To estimate ω_l , one now has to compute

$$\sum_{k=1}^K \cos^2(\theta_k) = \|\Pi_{\tilde{\mathbf{A}}_l} \Pi_G\|_F^2 \quad (25)$$

without using the approximation in (17). This expression depends only on one unknown parameter, ω_l . The projection matrix is given by

$$\Pi_{\tilde{\mathbf{A}}_l} = \tilde{\mathbf{A}}_l (\tilde{\mathbf{A}}_l^H \tilde{\mathbf{A}}_l)^{-1} \tilde{\mathbf{A}}_l^H \quad (26)$$

which can be shown to equal

$$\Pi_{\tilde{\mathbf{A}}_l} = \frac{1}{\gamma} \text{Re} \{ M \mathbf{a}(\omega_l) \mathbf{a}^H(\omega_l) - \beta \mathbf{a}^*(\omega_l) \mathbf{a}^H(\omega_l) \}, \quad (27)$$

with

$$\beta = \mathbf{a}^H(\omega_l) \mathbf{a}^*(\omega_l), \quad (28)$$

and

$$\gamma = \frac{M^2 - |\beta|^2}{2}. \quad (29)$$

It should be stressed that both β and γ are frequency dependent, but, in the interest of simplicity, we will here omit this dependence in the notation. The above quantities can be used for understanding the problem at hand. What happens when the frequency ω_l is far from zero is that $\beta \approx 0$ (in which case the approximation in (17) is accurate), whereas $\beta \rightarrow M$ as $\omega_l \rightarrow 0$, in which case the matrix $\tilde{\mathbf{A}}_l^H \tilde{\mathbf{A}}_l$ is singular. We remark that similar conclusions apply when ω_l is near π . The cases of interest here are the ones where $\beta \neq 0$. Clearly, the unknown frequencies can be estimated as

$$\hat{\omega}_l = \arg \min_{\omega_l} \|\Pi_{\tilde{\mathbf{A}}_l} \Pi_G\|_F^2. \quad (30)$$

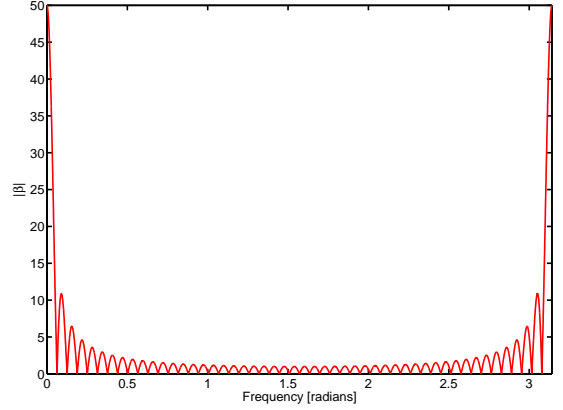


Figure 1: The magnitude of the quantity $\beta = \mathbf{a}^H(\omega_l) \mathbf{a}^*(\omega_l)$, which determines the accuracy of the complex model, shown as a function of frequency.

Using (27), we can rewrite the involved expression as

$$\|\Pi_{\tilde{\mathbf{A}}_l} \Pi_G\|_F^2 = \frac{1}{\gamma} (M \mathbf{a}^H(\omega_l) \mathbf{G} \mathbf{G}^H \mathbf{a}(\omega_l) \quad (31)$$

$$- \text{Re} \{ \beta \mathbf{a}^H(\omega_l) \mathbf{G} \mathbf{G}^H \mathbf{a}^*(\omega_l) \}), \quad (32)$$

where it should be noted that $\mathbf{a}^H(\omega_l) \mathbf{G} \mathbf{G}^H \mathbf{a}(\omega_l)$ is just the usual MUSIC cost function. This shows that the proposed methodology leads to a simple modification of the well-known MUSIC principle; in fact, we can now redefine the MUSIC pseudo-spectrum as

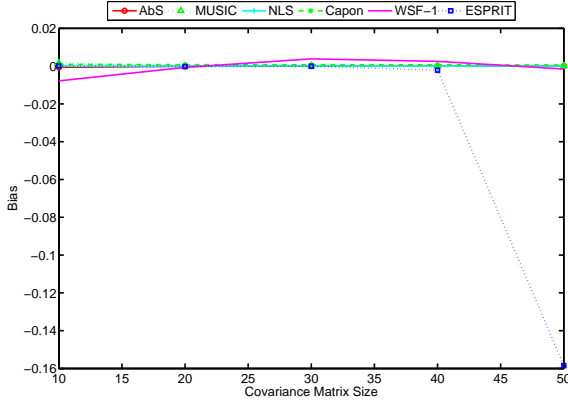
$$P(\omega_l) = \frac{\gamma}{\text{Re} \{ \mathbf{a}^H(\omega_l) \mathbf{G} \mathbf{G}^H (M \mathbf{a}(\omega_l) - \beta \mathbf{a}^*(\omega_l)) \}}, \quad (33)$$

which can then be used to estimate frequencies as

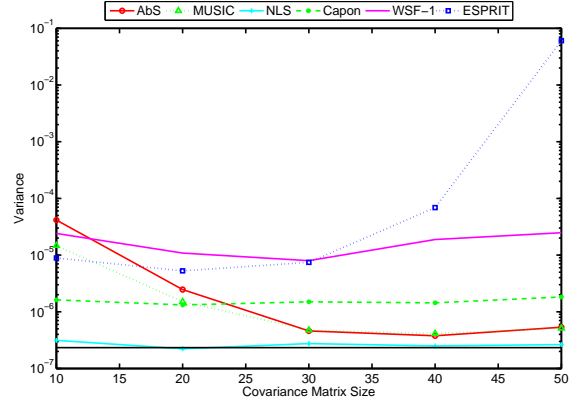
$$\hat{\omega}_l = \arg \max_{\omega_l} P(\omega_l). \quad (34)$$

To summarize:

- (i) The estimator in (34) takes the interaction between complex sinusoids and their complex conjugates into account.
- (ii) It does not take the interaction between different complex sinusoids into account (doing so leads to an intractable problem). Rather, it is based on the usual assumption regarding these.
- (iii) Consequently, it is expected to lead to improved estimates for sinusoids having very low or high frequencies, and/or low M and N , but not for closely spaced sinusoids.
- (iv) Interestingly, the proposed improvement leads to a simple augmentation of the original MUSIC cost function, wherefrom a modified pseudo-spectrum can easily be defined.



(a)



(b)

Figure 2: Performance of various estimators, measured in terms of bias and variance, as a function of the covariance matrix size M , for $N = 101$ and an SNR of 20 dB. Note that the asymptotic CRLB (solid) is shown in Figure 2(b).

5. EXPERIMENTAL RESULTS

We now proceed to demonstrate the properties of the introduced method using simulated data. First, however, we will provide an example of how the quantity $|\beta|$ varies as a function of the frequency. This is shown in Figure 1, with $M = 50$. When β is significantly different from zero, one can expect a discrepancies between the complex- and real-valued models. As can be seen, β attains the value M in both extremes, near 0 and π , which causes a division by zero in (27).

Next, we will examine the accuracy of the proposed method as compared to the original MUSIC algorithm (we note that it is equivalent to the adaptation to real-valued data in [3] with no weighting). We will do this using Monte Carlo simulations and then measure the bias and variance of the obtained frequency estimates. Additionally, we also compare to a number of other frequency estimators proposed for real-valued data, namely some of those of [4, 5, 6], as well as a real-valued non-linear least-squares (NLS) estimator. The NLS method is simply an estimator based on minimizing the squared error, which is equivalent to maximizing $\mathbf{x}^H(n)\Pi_{\tilde{\mathbf{A}}_l}\mathbf{x}(n)$, with $M = N$. As is well-known, under the assumption of white Gaussian noise, this estimator is equivalent to the maximum likelihood estimator.

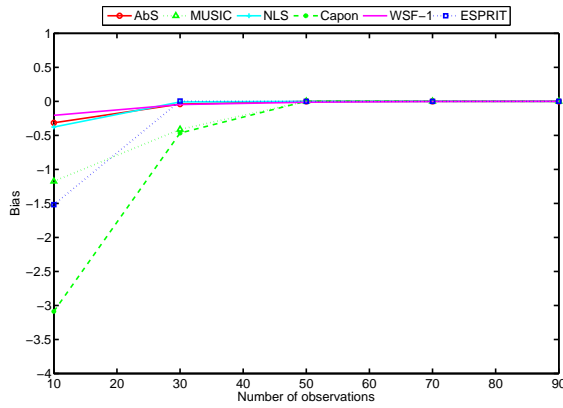
The experimental setup is as follows: 100 Monte Carlo runs are used for each data point and a real sinusoid having frequency $\omega_1 = 0.0523$, i.e., a fairly low frequency, having unit amplitude and uniformly distributed phase is generated using $N = 101$. The signal is then corrupted by a white Gaussian noise with an SNR of 20 dB. In the following figures, the proposed method is referred to as AbS, whereas Capon is the method introduced in [6], WSF-1 is the method from [4], ESPRIT is the real version proposed in [5], and NLS is the above mentioned nonlinear least-squares method.

First, we investigate the importance of the covariance matrix size for the various methods. The results are shown in Figures 2(a) and 2(b) in the form of bias and variance of the estimated frequencies. Also shown in the figure is the asymptotic Crámer-Rao lower bound (CRLB). Note that this bound is not valid for very low frequencies and/or number of sam-

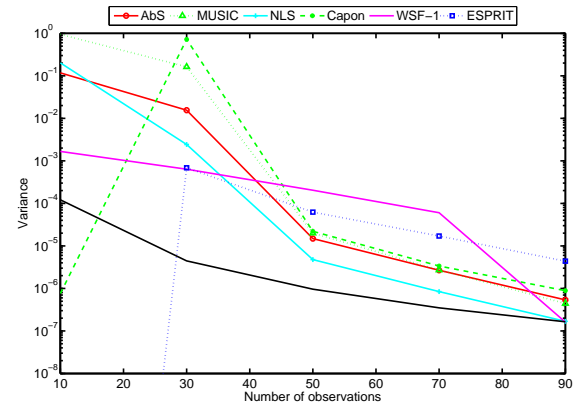
ples. As can be seen, the methods exhibit different dependencies on the covariance matrix size. Some of the methods work poorly for low M , whereas others work poorly when M is too high. Consequently, for the MUSIC-like methods, we use a covariance matrix of size 50×50 in the experiment that follow, while for the methods of [4, 5, 6], we use 25×25 covariance matrices. We remark that the NLS estimator does not require any covariance matrices and its performance does thus not depend on M . In the final experiment, which is the most important one, we will seek to quantify the improvements that the proposed method offers over the original MUSIC algorithm. We do this by varying the number of samples N . As N gets smaller, β will become larger, meaning the interaction between the positive and negative sides of the spectrum gets larger and so, presumably, does the discrepancy between the estimates obtained using the real and complex models. The results are shown in Figures 3(a) and 3(b), as before, in terms of the bias and the variance of the obtained estimates. As can be seen, the proposed method greatly reduces the bias of the MUSIC method for low N , and also leads to a somewhat lower variance for a high number of observations. Note that the reason that some of the estimators exhibit variances lower than the CRLB is that they produce biased estimates for low N . It can also be observed that the proposed estimator performs better under these conditions than the real-valued Capon, ESPRIT and WSF-1 methods. At this point, it should be stressed that the MUSIC method is not statistically efficient, so it will not attain the CRLB, unlike the NLS method. Generally, the experiments demonstrate the ability of the proposed modifications of MUSIC to compensate for some of the classical technique's shortcomings for real signals.

6. CONCLUSION

In this paper, we have considered the problem of estimating the frequencies of real-valued sinusoidal signals corrupted by white additive noise. We have proposed to improve the performance of the classical MUSIC algorithm for this problem using the theory of angles between subspaces. More specifi-



(a)



(b)

Figure 3: Performance of various estimators, measured in terms of bias and variance, as a function of the number of samples N for $N = 101$ and an SNR of 20 dB. Note that the asymptotic CRLB is shown in Figure 3(b).

cally, we have derived a modified pseudo-spectrum that takes the interaction between negative and positive sides of the spectrum into account. Our results confirm that the proposed method does indeed reduce the bias of the MUSIC method under adverse condition, i.e., for low or high frequencies, and/or for a low number of samples. In closing, we remark that it is quite possible that the results reported here can be improved further by adapting the proposed scheme using the alternative covariance matrix model proposed in [5]. Similarly, it is also possible that the technique could benefit from the various weighted approaches used in weighted subspace fitting [3, 4, 5], but these suggestions improve other aspects of the subspaces-methods, for which reason they were not further considered herein.

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