ANGLE OF ARRIVAL DETECTION USING COMPRESSIVE SENSING

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ABSTRACT

The conventional method for determining target angle of arrival with an array of sensors is to digitize the output of each sensor at the Nyquist sampling rate for the system bandwidth and use digital signal processing algorithms such as maximum likelihood estimation or multiple signals classification (MUSIC). Here we show that if the targets sparsely populate the angle/frequency domain, the angles and frequencies can be obtained from a much smaller number of measurements by randomly summing sensor outputs and by randomly sampling in time through the use of a novel application of recently developed compressive sensing algorithms.

1. INTRODUCTION

Angle-of-arrival (AOA) determination using an array of sensors is an important topic across a wide range of disciplines [1-3]. Recently, several researchers have applied the new paradigm of compressive sensing (CS) [4-6] to AOA determination [7-10]. In this prior work, the authors have exploited sparsity in either the time *or* angle domains to ease receiver constraints while retaining AOA resolving capability. Here we take advantage of sparsity in both angle and frequency to formulate the conventional AOA problem [1] in the conventional CS format. Some of this work is related to that presented in [4-6] but our formulation shows how to treat frequency domain and angle domain sparsity on an equal footing and how to morph the doubly-sparse AOA problem into the conventional CS format. The major contribution of our work is, however, the use of a super-resolution algorithm with our CS AOA formulation to determine angle. We present, for the first time to our knowledge, cumulative probability distributions as a function of angle for CS with variable compression ratios and compare these results to AOA determination with reduced numbers of elements.

2. COMPRESSIVE SENSING

In the conventional formulation for compressive sensing [3], a sparse vector s of dimension j can be recovered from a measured vector y of dimension k ($k << j$) after transformation by a sensing matrix Θ as shown in eq. (1)

$$y = \Theta s + w \qquad (1)$$

where w is a noise vector. Often, Θ is factored into two matrices, $\Theta = \Phi\Psi$ where Φ is a "random" mixing matrix and Ψ is a Hermitian matrix with columns that form a basis in which the input vector is sparse. A canonical example is the case in which the input is a time series with samples taken from a single sinusoid with an integer number of periods. These data are not sparse but are transformed into a sparse vector by the discrete Fourier transform (DFT). Note that although Θ is not square and hence not invertible, Ψ is both square and invertible. Work in compressive sensing has shown that under quite general conditions, all j components of s may be recovered from the much smaller number of measurements of y. With no noise ($w = 0$) recovery proceeds by minimizing the ℓ_1 norm of a test vector s' (the sum of the absolute values of the elements of s') subject to the constraint $y = \Theta s'$. In the presence of noise, recovery proceeds by minimizing a linear combination of the ℓ_1 norm of the target vector and the ℓ_2 norm of the residual vector given by $y - \Theta s$

$$s'(\lambda) = \arg \min_s(\lambda\|s\|_1 + \| y - \Theta s \|^2) \qquad (2)$$

where the parameter λ is chosen such that the signal is optimally recovered [11].

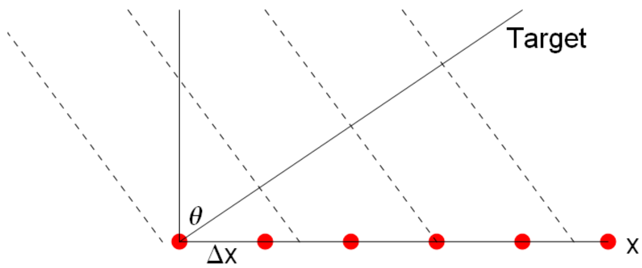

Figure 1 – Target-antenna element array geometry.

3. APPLYING CS TO AOA ESTIMATION ON THE GRID

Here we consider angle of arrival estimation using the array shown in Fig. 1. For simplicity, we consider a linear array of antenna elements and a set of targets located far from the sensors. Thus the incoming signals are plane waves and the relative delay at the individual elements gives the angle of arrival directly. Conventional algorithms for determining the angle of arrival are discussed in the pioneering paper on Multiple Signal Classification (MUSIC) and include beamforming, maximum likelihood, maximum entropy as well as

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MUSIC itself [1]. Following Schmidt, we consider the matrix formulation of the angle-of-arrival problem given by

$$X(t) = AF(t) + W(t) \qquad (3)$$

where $X(t)$ is the vector of signals received at the N antenna elements, $F(t)$ is the vector of the target signals at J locations in angle space, and $W(t)$ is a noise vector. For a single frequency f, the elements a_{nj} of the N x J matrix A are given by $\exp(i\, k\, \Delta_{nj})$ where $k = 2\pi f/c$ (c is the speed of light) and Δ_{nj} is given by the product of the location of the n^{th} antenna element with the unit vector corresponding to the j^{th} angle location [1]. In a conventional system, the output of each antenna element is digitized at the Nyquist rate and digital signal processing is used to obtain the target angle and frequency. Denote the matrix formed by the output of N antenna elements for M time steps D. In many systems, it is neither possible nor desirable to make and/or process all the measurements required to form D. For example, the N Nyquist-rate ADCs, one at each antenna element, may consume too much power [7] or the communication rate between elements may be limited [9]. This motivates us to determine if we can achieve the desired target information (amplitude, angle and frequency) from a smaller number of measurements using techniques described in the well-known papers on compressive sensing [1-3].

Angle of arrival estimation can be converted to the conventional compressive sensing format as follows. We assume that the number of targets is small compared to the total number of elements in the D matrix. Thus, even though D is not measured, its 2-dimensional Fourier transform in the sine-of-the-angle (sine-angle) and the frequency domain is sparse. This assumption might break down for wideband targets such as chirp radar reflections or cluttered environments with chaff at many angles. The dimension of the angle space is determined by the number of antenna elements. Therefore, before compression the number of sine-angle bins is taken to be equal to the number of elements in the array; likewise, the number of frequency bins is equal to the number of time samples. The matrix D can be transformed from the antenna-element/time domain to the sine-angle/frequency domain with Fourier transforms to obtain a sparse matrix S given by $S = F_t D F_a$ where F_t is the Fourier transform from time to frequency and F_a is the Fourier transform from array position to sine-angle position (F_t and F_a are square matrices compatible with D). In our sytem, neither the measurements of D, nor the transforms F_t and F_a are performed at the receiver. The inverse transforms are, however, required during the recovery of D, since $D = F_t^{-1}\, S\, F_a^{-1}$ where F^{-1} is the inverse Fourier transform. Therefore, after determining S through compressive sensing techniques, the original data matrix D can be recovered as if it were measured at the Nyquist rate at each array element, and then D could be processed with traditional techniques.

The next step is to mix the matrix D along the lines of conventional compressive sensing. D is compressed in the antenna-array dimension by a "wide" pseudo-random matrix L and in the time dimension by a "skinny" pseudo-random matrix R such that L is of dimension n x N and R is of dimension M x m. L and R are known matrices that we generate and store. The new matrix C, given by $C = LDR$, is of dimension n x m. In our system C is digitized with mn/MN fewer resources (sampling steps per unit time) than would be required for a similar measurement of D. Note that neither the sampling rate nor the array dimension changes as a result of measuring C; only the total number of measurements is reduced.

The sparse representation S can be recovered from the compressed measurements C as follows. First, write $C = L\, D\, R = L\, F_t^{-1}\, S\, F_a^{-1}\, R$. Second, minimize the matrix ℓ_1 norm of S subject to the constraint $C = L\, F_t^{-1}\, S\, F_a^{-1}\, R$ in the absence of noise or in the presence of noise.

$$S'(\lambda) = \arg\min_S \left(\lambda\|S\|_1 + \| C - L\, F_t^{-1}\, S\, F_a^{-1}\, R\|_2^2\right) \qquad (4)$$

where as in eq. (2) we use the double bar notation to indicate the "entry-wise" norm of the enclosed matrix and the choice of the penalty parameter λ allows optimal recovery of S with minimal noise [11].

There are two differences between eq. (4) and eq. (2). First, in eq. (4), the target quantity S is a matrix. This can be fixed by "flattening" the N x M dimensional matrix S to a vector s of length MN. Second, the matrix multiplications in the least-squares term of eq. (4) must be rewritten in terms of a single matrix on the left-hand side of s. That is, since both the 2D Fourier transform and the transformations by the left and right mixing matrices are linear, we can express eq. (4) in the form:

$$G.F_2\, s = \text{Flatten}(L\, F_t^{-1}\, S\, F_a^{-1}\, R) \qquad (5)$$

where G performs the left-right mixing matrix transformations and F_2 performs the 2D discrete Fourier transformation. A brief derivation of G and F_2 in terms of L, R, F_t, and F_t (or L, R, F_t^{-1} and F_a^{-1}) is given in the Appendix. One final step is needed before most CS packages can be used. All matrices in eq. (4) are complex (including the mixing matrices) while most packages use real numbers. We rewrite G as $\{\{G_R, -G_I\}, \{G_I, G_R\}\}$ and s as $\{s_R, s_I\}$ where the subscripts R and I indicate the real and imaginary parts of G and s. Although this formulation is a slight departure from the strict minimization of the ℓ_1 norm for complex vectors, we have shown with extensive calculations that this recovery technique works well without modification to existing codes.

Fig. 2 shows the mean square error in the recovered matrix S as a function of the small dimension of the mixing matrix ($m = n$ in these calculations) with a 32-element array and 32-time steps and 3 targets with the target angles and frequencies chosen such that S is sparse. The sharp dependence of CS recovery on the small dimension of the mixing matrix is characteristic of CS systems.

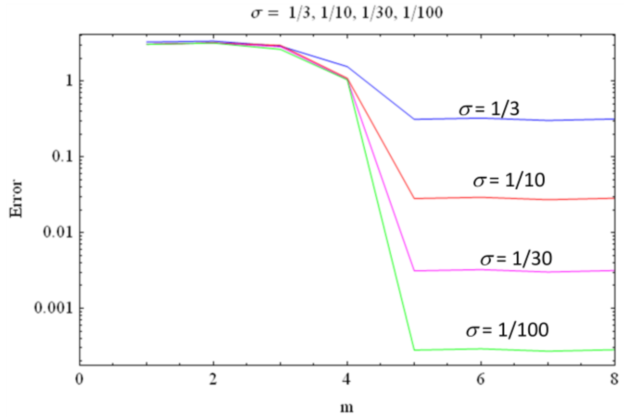

Figure 2 – Residual error as a function of the small dimension m or the mixing matrices use in space and time with noise level σ as a parameter. Residual error is defined as the mean-square of the difference between the elements of the S matrix calculated from the penalized ℓ_1 norm and the S matrix calculated directly from transforming D. There are 32 antenna elements and 32 time points sampled at each ADC. Three targets are present with angles and frequencies on the grids.

4. RESOLVING AOA OFF THE GRIDS

The critical issue in using compressive sensing for determining AOA is formatting the recovery such that the flattened *S* matrix is sparse. But the Fourier transform of a digital replica of a single frequency sine wave is sparse if and only if the duration of digital replica is exactly an integer number of periods. If the duration of the time window is not an exact number of periods, then the discrete Fourier transform will not be sparse because of the extra frequency components introduced by truncating the waveform at a fraction of a period. For AOA determination in a scenario with a single target the elements d_{jk} of the *D* matrix can be written as $a\exp(2\pi i f t_j + 2\pi i f x_k \sin\theta)$ where a, f and θ are the target amplitude, frequency, and angle; t_j ranges from 0 to t_{max} (the duration of the time window) in units of Δt (the sampling period); and x_k ranges from 0 (the position of the first antenna) to x_{max} (the position of the last antenna) in units of Δx (the separation of individual antennas). The values of Δt and t_{max} set the values of a frequency grid for which the Fourier transform in time of *D* is sparse. For a given frequency, the values of Δx and x_{max} determine the angular grid for which the array dimension Fourier transform of *D* is sparse.

Our calculations proceed as follows. First, we define the time, position, frequency and sinθ grids. Then we calculate the *D* matrix. Next we mix the *D* matrix from the left and right with wide and skinny matrices whose elements are randomly chosen from +/-1+/-i to calculate the matrix *C*, which is the basic matrix to be measured. We have used a *Mathematica* package for minimizing an ℓ_1-penalized functional called L1Packv2 developed by Loris [11] to recover *S* from *C* and the *G* matrix discussed above. With no noise, recovery is perfect, provided that *m* and *n* satisfy the usual compressive sensing relations relative to *M* and *N* and the number of targets, which determines the number of non-zero elements in *S*.

To insure that the discrete 2D Fourier transformation can be inverted, the dimension of the frequency grid equals the number of time steps while the dimension of the angle grid equals the number of antenna elements (the number of antenna elements need not equal the number of time steps). If the target frequency and/or angle do not lie on the grids, the matrix *S* defined above is not sparse. But sliding the Fourier transforms by the correct offsets as given in eq. (6), transforms a target with an arbitrary angle or frequency on to the grid and makes *S* sparse:

$$\exp[2\pi i(j-1)(k-1)/N] \rightarrow \exp[2\pi i(j-1-\text{offset})(k-1)/N]. \qquad (6)$$

Since the Fourier transforms are used only in the recovery process, the correct offsets can be found by minimizing the penalized norm as a function of offset to find the true target angle and frequency. The data taking process, the random matrices, and the measurement matrix *C* are unchanged.

5. RESULTS

In test calculations the matrices *D* and *S* are known and one can calculate the error in the compressive sensing estimate of the coefficients of *S* compared to the true value. Fig. 2 shows this error as a function of the small dimension of the mixing matrices for several values of σ, the standard deviation of the Gaussian pseudo-random noise added to the real and imaginary parts of each element of *D* in the simulation. In an actual application, where the input angle and frequency are unknown and off the grid, the error shown in Fig. 2 cannot be used to determine the angle and one must evaluate the penalized norm given in eq. (4) as a function of offset for each realization of the random noise. Illustrations of such calculations are shown in Fig. 3. Note that the true offset is 0.8 and in the large noise case (σ = 0.3), particularly, the offset inferred from the minimum in the penalized norm has large errors. We emphasize that for targets off the grid, the value of the offset at the minimum of the penalized norm is the only knowledge of the unknown target angle.

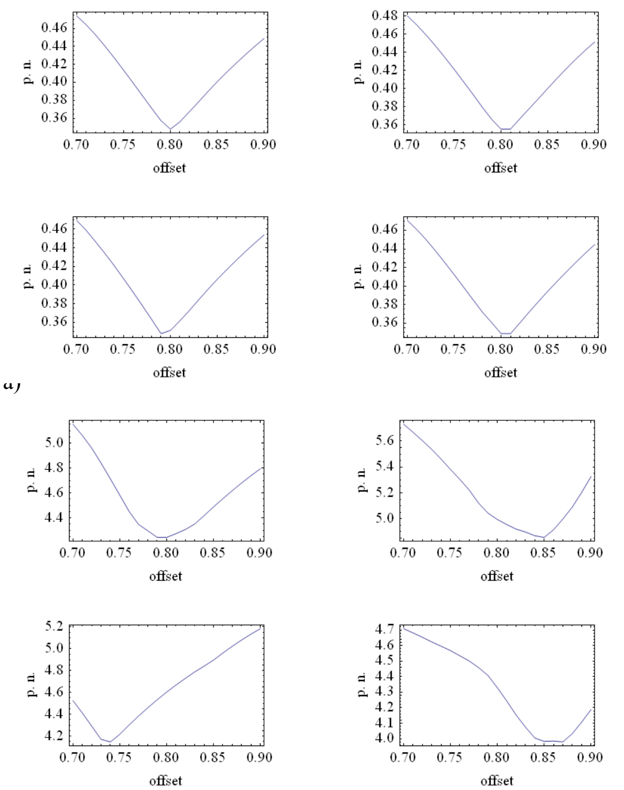

Figure 3 – Penalized norm (p.n.) as a function of Fourier transform offset for 4 realizations of the pseudo-random noise added to each measurement. a) $\sigma = 0.03$, b) $\sigma = 0.3$.

To find the cumulative probability distribution, the angle (or frequency) must be determined from the minimum of the curves shown in Fig. 3 for a large number of independent realizations. Fig. 4 shows the results for a 32 element array and a single time point. We performed 1024 calculations for each curve. The red curves are generated using an identity mixing matrix, that is, unmixed, and using the penalized ℓ_1-norm as the angle estimator. We have shown in separate work that using the ℓ_1-norm as the angle estimator with unmixed signals gives the same results as using the MUSIC algorithm. The leftmost curve is for an unmixed 32-element array while the right hand curve is for an unmixied 16 element array. The median angular error [$\Delta\theta$ for $P(\Delta\theta) = 0.5$] for the 16-element array is about $2^{3/2}$ larger compared to the median angular error for the 32-element array, in agreement with simple arguments for conventional arrays and processign discussed in the next section. The four green curves correspond to compressive sensing with the small dimension of the mixing matrix equal to 4, 8, 16 and 32. Note that the performance of the compressive sensing algorithm with a 4x32 mixing matrix is about the same as the unmixed 16 element array. The 32x32 mixing matrix result is slightly inferior to the identity mixing matrix result as expected since the non-unitary random mixing matrix distorts the distribution of the noise. The median angle error scales by $2^{1/2}$ in going from the unmixed 32 element result to the 16x32 CS result. This reflects the loss in SNR in the 16x32 CS system compared to the 32 element system without CS as discussed in Section 6. In all calculations shown in Figs. 2-4 we set the penalty parameter λ equal to the noise standard deviation σ. Results are not sensitive to this assumption as shown by the probability distributions given in Fig. 5 for $\sigma = 0.3$ and $\lambda = 0.1, 0.3, 0.6$. We emphasize that the choice of λ is a system design issue and determined by *a priori* knowledge of noise levels.

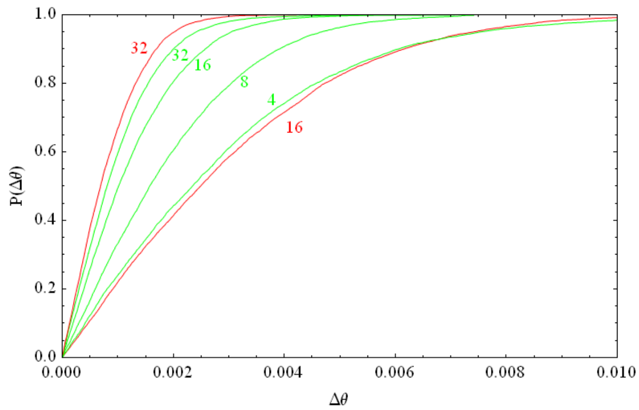

Figure 4 – Cumulative probability $P(\Delta\theta)$ as a function of $\Delta\theta$ for 32 and 16 element arrays mixed with an identity matrix, which yields the same results as conventional MUSIC alogorithm,(red curves) and for a 32 element array randomly mixed to 32, 16, 8 and 4 elements.

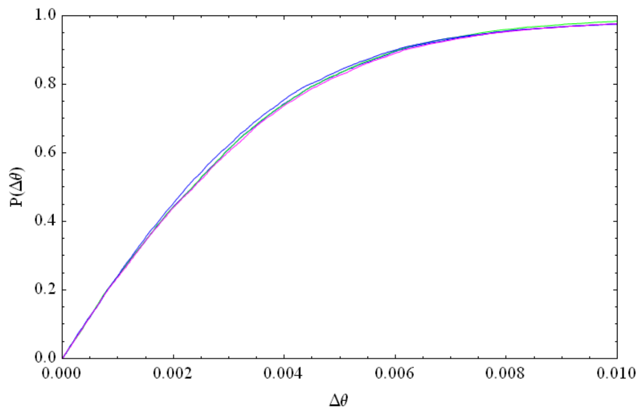

Figure 5 – Cumulative probability $P(\Delta\theta)$ as a function of $\Delta\theta$ for a 32-element array randomly mixed down to 4 measurements with $\sigma = 0.3$ and $\lambda = 0.01$ (magenta), 0.3 (green), 2.0 (blue)

6. SIGNAL-TO-NOISE RATIO DEPENDENCE

Define the signal-to-noise ratio (SNR) of a signal s in the presence of noise to be $\|s\|^2 / (2\sigma^2)$, where σ^2 is the variance of the noise in the real and imaginary part of a single sample in the received signal. On average, compressive measurements reduce the SNR of a signal by the compression ratio

[12], which will limit the utility of large compression ratios in practical applications. Even under the modest compression, say by reducing the number of measurements by half, the SNR will drop by 3 dB and this loss of SNR affects AOA accuracy.

AOA accuracy is proportional to 1/($SNR^{1/2} x_{max}$) [13] where x_{max} is the width of the linear array. Alternatively, the number of measurements can be reduced by simply measuring fewer elements of the array. We compare two reduced measurement approaches to standard array measurements using the full array: compressive sensing and standard sensing on a smaller array. The accuracy standard is the uncompressed, full sized array $\sigma_{AOA} \sim 1/(SNR^{1/2} x_{max})$. The AOA uncertainty for the compressively sensed full sized array for the same signal is $\sigma_{CSAOA} \sim 1/ [(SNR/2)^{1/2} x_{max}] = 2^{1/2}/(SNR^{1/2} x_{max})$. Finally, the uncertainty for the half sized array with standard processing is $\sigma_{HAOA} \sim 1/ [(SNR/2)^{1/2} x_{max}/2] = 2^{3/2} / (SNR^{1/2} x_{max})$. The constant of proportionality is the same in all three cases.

7. CONCLUSIONS

We formulate AOA estimation in the conventional format used in compressive sensing and calculate AOA by finding the minimum in the penalized ℓ_1-norm of a sparse target vector as a function of the offset in the Fourier transform necessary to sparsify the input signal. We calculate cumulative probability distributions for the AOA error for varying noise levels and degrees of compression. Relations between median angle errors for different cases are in good agreement with simple scaling laws. We recognize that the CS recovery algorithms developed in this paper may be computationally complex and/or slow but there is extensive work going on world-wide to develop optimized CS recovery algorithms.

8. APPENDIX

In order to apply standard minimization packages [11], to the matrix formulation of the AOA problem given in equation (4), the problem needs to be restated in terms of ℓ_1-minimization of vectors. This is easily accomplished since the mapping of $C=LDR$ is linear in the elements of D.

Let G be the nm x NM complex matrix with columns G_k; k is in the set {1…NM} defined by the flattened outer product of the i^{th} column of L with the j^{th} row of R where $k = Nj + i$. Then $c=Gd$, where d is the vector formed by flattening D and c is the vector formed by flattening C. In other words, if Col(G, k)=Flatten(Outer(Col(L, i), Row(R, j))) for each k, $1 \leq k \leq NM$, then Flatten(C) = G Flatten(D) = Flatten(LDR).

This flattened formulation can be derived by considering the image of the standard basis $\{E_{ij}\}$ of the space of NxM matrices under the linear transformation given by $C=LDR$, where E_{ij} is a matrix with a one in the i^{th}, j^{th} position and zeros elsewhere. The same technique may be used to derive the matrix formulation of the discrete 2D Fourier transformation F_2^{-1} from F_t^{-1} and F_a^{-1}.

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