# DIRECT POSITION ESTIMATION APPROACH OUTPERFORMS CONVENTIONAL TWO-STEPS POSITIONING

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## ABSTRACT

GNSS receivers compute its position by a two-steps procedure. First, synchronization parameters are estimated and, then, a geometrical problem is solved to obtain receiver's position. This is the approach typically taken due to its simplicity and modularity. However, recent results pointed out the potential pitfalls of such approach. In that vein, Direct Position Estimation arise as a potential alternative, computing receiver's position directly from the digitized GNSS signal. The latter is performed as a single-step procedure, obtaining the Maximum Likelihood estimate of position. We base on a recent result to show that the variance of the single-step estimator is lower than the variance of the conventional two-steps estimation of position. The result is validated by computer simulations, comparing the performances of both alternatives.

## 1. INTRODUCTION

A Global Navigation Satellite Systems (GNSS) antenna receives measurements which are considered to be a superposition of plane waves corrupted by noise and, possibly, interferences and multipath. An antenna receives $M$ scaled, time-delayed and Doppler-shifted signals with known signal structure. Each signal corresponds to the line-of-sight signal (LOSS) of one of the $M$ visible satellites. The receiving complex baseband signal can be modeled as

$$x(t) = \sum_{i=1}^{M} a_i q_i(t - \tau_i) \exp\{j2\pi f_{d_i} t\} + n(t) \ , \quad (1)$$

where $q_i(t)$ is the transmitted complex baseband low–rate navigation signal spread by the pseudorandom code of the $i$–th satellite, considered known. $a_i$ stands for its complex amplitude, $\tau_i$ is the time-delay, $f_{d_i}$ is the Doppler deviation and $n(t)$ represents zero-mean additive noise and other unmodeled terms.

This work has been partially funded by the Spanish/Catalan Science and Technology Commissions and FEDER funds from the European Commission: TEC2006-06481/TCM, TEC2004-04526, TIC2003-05482, CONSOLIDER CSD2008-00010 (COMONSENS), 2005SGR-00639, CENIT2007-2002 (TIMI), TEC2008-02685/TEC (NARRA), CTP2006-27641-E, and by the European Commission in the framework of the FP7 Network of Excellence in Wireless COMmunications NEWCOM++ (contract n. 216715) and COST Action IC0803 (RFCSET).

There are two approaches to positioning using GNSS receivers: the conventional two-steps approach and Direct Position Estimation (DPE for short). Both approaches are illustrated in Figure 1. The conventional approach (upper diagram) consists in a two-steps procedure:

1. **Estimation of synchronization parameters.** The receiver is equipped with a number of tracking channels, in charge of estimating both time-delays and Doppler-shifts of the acquired satellites. In general, this estimation is performed independently among channels by a bank of Delay/Phase Lock Loops or more sophisticated signal processing techniques. We denote as $\tau_i$ and $f_{d_i}$ the delay and Doppler deviation of the $i$-th satellite, being the $i$-th entries in vectors $\boldsymbol{\tau}$ and $\mathbf{f}_d$ respectively. We form a vector with these parameters, $\boldsymbol{\upsilon} \triangleq \left[\boldsymbol{\tau}^T, \mathbf{f}_d^T\right]^T$.
2. **Position calculation.** The estimates of $\boldsymbol{\upsilon}$ provides a measure of the relative distance between the receiver an each satellite. Then, an estimate of the receiver's position is obtained by solving a geometrical problem, referred to as trilateration. This is typically done relying on Least Squares (LS) or Weighted Least Squares (WLS) algorithms.

The lower diagram in Figure 1 shows the conceptual idea of DPE approach, i.e., merging the two-steps approach into a single estimation problem. DPE, as a GNSS positioning alternative, was proposed in [1]. The underlying idea of DPE is that synchronization from all satellites share a strong constraint: they all depend on the same motion parameters. If we write $\tau_i$ and $f_{d_i}$ as

$$\tau_i = \frac{1}{c} \parallel \mathbf{p}_i - \mathbf{p} \parallel + (\delta t - \delta t_i) \quad (2)$$

$$f_{d_i} = -(\mathbf{v}_i - \mathbf{v})^T \frac{\mathbf{p}_i - \mathbf{p}}{\parallel \mathbf{p}_i - \mathbf{p} \parallel} \frac{f_c}{c} \ , \quad (3)$$

this dependency becomes clear [2]. In these definitions, $c$ is the speed of light, $f_c$ is the carrier frequency of the RF signal constructed with (1), $\mathbf{p} = [x, y, z]^T$ is the user's position coordinates and $\mathbf{v}$ its velocity vector. $\mathbf{p}_i$ and $\mathbf{v}_i$ are the position and velocity vector of the $i$-th satellite, respectively, which can be computed from the navigation message. Finally, $\delta t$ represents the unknown receiver's clock bias and $\delta t_i$ is the known bias of the $i$-th satellite. Therefore, we express the dependency of

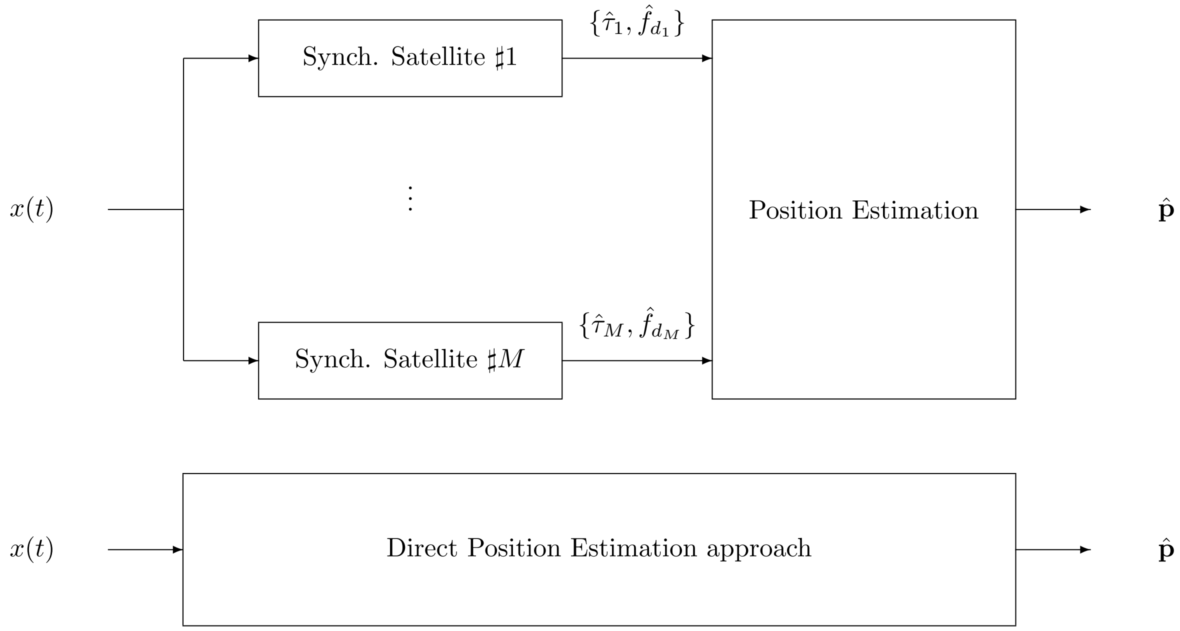


Figure 1: Block diagram comparing the operation of a conventional two-steps GNSS receiver and the Direct Position Estimation approach.

synchronization parameters to a common parameter $\boldsymbol{\gamma}$ by

$$\boldsymbol{\upsilon} \triangleq \boldsymbol{\upsilon}(\boldsymbol{\gamma}) = \begin{pmatrix} \boldsymbol{\tau}(\boldsymbol{\gamma}) \\ \mathbf{f}_d(\boldsymbol{\gamma}) \end{pmatrix} , \tag{4}$$

with $\boldsymbol{\gamma}$ being a vector gathering all motion parameters of the model. The simplest configuration is $\boldsymbol{\gamma} = \left[\mathbf{p}^T, \mathbf{v}^T, \delta t\right]^T$ and $n_\gamma \triangleq \dim\{\boldsymbol{\gamma}\} = 7$. However, DPE is a quite general approach, allowing a plethora of configurations for $\boldsymbol{\gamma}$. Besides, DPE can consider side information such as Inertial Measurement Unit data or atmospheric models [3, 4].

DPE addresses some of the inherent drawbacks of the conventional two-steps approach, at the expenses of an increased computational cost. The dependencies between channels are efficiently exploited, in the sense that signals from visible satellites are jointly processed to obtain user's position (which is the common driving parameter of these signals). Due to the joint processing of satellite signals, Multiple Access Interference (MAI) is also optimally mitigated according to the Maximum Likelihood (ML) principle.

DPE proposes an alternative where the estimation of user's position is performed directly from the received and sampled signal. It avoids intermediate steps and jointly considers signals from all satellites when estimating user's position.

This paper uses the results in [5] to prove that DPE cannot be outperformed by the conventional two-steps positioning approach. Section 2 delves into the proof of such claim and Section 3 provides some simulation results to validate the theory. Simulation results present the performance of both position estimators, following either positioning alternatives. Finally, Section 4 concludes the paper summarizing its main contributions.

## 2. DIRECT POSITION ESTIMATION APPROACH OUTPERFORMS CONVENTIONAL TWO-STEPS POSITIONING

A recent result in [5] provides the means to show that the conventional two-steps approach cannot overcome the performance of DPE. This interesting result is the core of Proposition 2.1.

Proposition 2.1 provides the mathematical justification to the DPE approach. Roughly speaking, the result means that the covariance of the two-steps approach cannot be smaller than the covariance of the one-step estimator. Thus, the estimation performance of the conventional approach can only be equal or worse than the one provided by the DPE approach, in the Mean Squared Error (MSE) sense. This is a strong result that is the basis of DPE framework, as it opens the door to the design of future GNSS receivers with improved performance.

**Proposition 2.1.** *Let $\boldsymbol{\upsilon} \in \Upsilon \subset \mathbb{R}^{n_\upsilon}$ and $\boldsymbol{\gamma} \in \Gamma \subset \mathbb{R}^{n_\gamma}$ be two unknown parameters s.t. there exist an injective function $g(\cdot) : \Gamma \longmapsto \Upsilon$,*

$$\boldsymbol{\upsilon} = g(\boldsymbol{\gamma}) \ , \forall \boldsymbol{\gamma} \in \Gamma \tag{5}$$

*that relates both. Function $g(\cdot)$ has a unique inverse mapping*

$$\boldsymbol{\gamma} = g^{-1}(\boldsymbol{\upsilon}) \ , \forall \boldsymbol{\upsilon} \in \bar{\Upsilon} \tag{6}$$

*under the subset $\bar{\Upsilon} = \{\boldsymbol{\upsilon} \mid \boldsymbol{\upsilon} = g(\boldsymbol{\gamma}), \forall \boldsymbol{\gamma} \in \Gamma\} \subset \Upsilon$.*

*Denote by $\hat{\boldsymbol{\gamma}}_1$ and $\hat{\boldsymbol{\gamma}}_2$ the $K$-samples estimators of $\boldsymbol{\gamma}$ based on single–step and two–steps approaches, respectively. Similarly, $\boldsymbol{\Sigma}(\hat{\boldsymbol{\gamma}}_1)$ and $\boldsymbol{\Sigma}(\hat{\boldsymbol{\gamma}}_2)$ represent the covariance matrix of each estimator.*

*Then,*

$$\mathbf{C} \triangleq \lim_{K \to \infty} \left(\boldsymbol{\Sigma}(\hat{\boldsymbol{\gamma}}_2) - \boldsymbol{\Sigma}(\hat{\boldsymbol{\gamma}}_1)\right) \tag{7}$$

*is a positive semidefinite matrix.*

*Proof.* Denote the log-likelihood function of measurements $\mathbf{x}$ given $\boldsymbol{\gamma}$ by $\mathcal{L}(\mathbf{x}|g(\boldsymbol{\gamma})) \triangleq \ln p(\mathbf{x}|g(\boldsymbol{\gamma}))$. Then, the Maximum likelihood Estimator (MLE) of $\boldsymbol{\gamma}$ is

$$\hat{\boldsymbol{\gamma}}_{\mathrm{ML}} = \arg\max_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}} \left\{\mathcal{L}(\mathbf{x}|g(\boldsymbol{\gamma}))\right\} \ , \tag{8}$$

which equals $\hat{\boldsymbol{\gamma}}_1$ by definition. Notice that the optimization search in (8) shall be performed only over the space of possible positions $\boldsymbol{\Gamma}$.

Conversely, the two-steps approach consists in first estimating the MLE of $\boldsymbol{\upsilon}$ as the solution to

$$\hat{\boldsymbol{\upsilon}}_{\mathrm{ML}} = \arg\max_{\boldsymbol{\upsilon}\in\boldsymbol{\Upsilon}} \left\{\mathcal{L}(\mathbf{x}|\boldsymbol{\upsilon})\right\} \ , \tag{9}$$

and, then, use this estimate to obtain $\hat{\boldsymbol{\gamma}}_2$ following its relation with $\boldsymbol{\upsilon}$. For instance, the second estimation step can be done by a WLS, which is the common choice in conventional GNSS receivers:

$$\begin{aligned}\hat{\boldsymbol{\gamma}}_2 &= \arg\min_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}} \left\{\Lambda(\boldsymbol{\gamma})\right\} \qquad (10)\\ &= \arg\min_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}} \left\{\left(\hat{\boldsymbol{\upsilon}}_{\mathrm{ML}} - g(\boldsymbol{\gamma})\right)^T \mathbf{W} \left(\hat{\boldsymbol{\upsilon}}_{\mathrm{ML}} - g(\boldsymbol{\gamma})\right)\right\} \ ,\end{aligned}$$

where $\mathbf{W}$ is a real, positive definite and symmetric weighting matrix.

To prove (7), we first obtain the asymptotical expressions of the covariance matrices of the estimators, $\boldsymbol{\Sigma}(\hat{\boldsymbol{\gamma}}_1)$ and $\boldsymbol{\Sigma}(\hat{\boldsymbol{\gamma}}_2)$ respectively.

Since the one-step estimator is the MLE, it is well-known that it is asymptotically efficient under regularity conditions [6]. This means that its asymptotical covariance equals the inverse of the Fisher Information Matrix (FIM), defined as $\mathbf{J}_F(\boldsymbol{\gamma}) \in \mathbb{R}^{n_\gamma \times n_\gamma}$. Thus,

$$\begin{aligned}\lim_{K\to\infty} \boldsymbol{\Sigma}(\hat{\boldsymbol{\gamma}}_1) &= \left(\mathbb{E}\left\{\frac{\partial\mathcal{L}(\mathbf{x}|g(\boldsymbol{\gamma}))}{\partial\boldsymbol{\gamma}} \left(\frac{\partial\mathcal{L}(\mathbf{x}|g(\boldsymbol{\gamma}))}{\partial\boldsymbol{\gamma}}\right)^T\right\}\right)^{-1}\\ &\triangleq \mathbf{J}_F^{-1}(\boldsymbol{\gamma}) \ . \qquad (11)\end{aligned}$$

Using the chain rule we can extend the derivative as

$$\frac{\partial\mathcal{L}(\mathbf{x}|g(\boldsymbol{\gamma}))}{\partial\boldsymbol{\gamma}} = \frac{\partial\mathcal{L}(\mathbf{x}|\boldsymbol{\upsilon})}{\partial\boldsymbol{\gamma}} = \frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\frac{\partial\mathcal{L}(\mathbf{x}|\boldsymbol{\upsilon})}{\partial\boldsymbol{\upsilon}} \ , \tag{12}$$

which substituted in (11) results in

$$\begin{aligned}&\lim_{K\to\infty} \boldsymbol{\Sigma}(\hat{\boldsymbol{\gamma}}_1)\\ &= \left(\mathbb{E}\left\{\left(\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\frac{\partial\mathcal{L}(\mathbf{x}|\boldsymbol{\upsilon})}{\partial\boldsymbol{\upsilon}}\right)\left(\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\frac{\partial\mathcal{L}(\mathbf{x}|\boldsymbol{\upsilon})}{\partial\boldsymbol{\upsilon}}\right)^T\right\}\right)^{-1}\\ &= \left(\mathbb{E}\left\{\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\frac{\partial\mathcal{L}(\mathbf{x}|\boldsymbol{\upsilon})}{\partial\boldsymbol{\upsilon}}\frac{\partial\mathcal{L}^T(\mathbf{x}|\boldsymbol{\upsilon})}{\partial\boldsymbol{\upsilon}}\frac{\partial\boldsymbol{\upsilon}}{\partial\boldsymbol{\gamma}}\right\}\right)^{-1}\\ &= \left(\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\mathbb{E}\left\{\frac{\partial\mathcal{L}(\mathbf{x}|\boldsymbol{\upsilon})}{\partial\boldsymbol{\upsilon}}\frac{\partial\mathcal{L}^T(\mathbf{x}|\boldsymbol{\upsilon})}{\partial\boldsymbol{\upsilon}}\right\}\frac{\partial\boldsymbol{\upsilon}}{\partial\boldsymbol{\gamma}}\right)^{-1}\\ &= \left(\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\ \mathbf{J}_F(\boldsymbol{\upsilon})\ \frac{\partial\boldsymbol{\upsilon}}{\partial\boldsymbol{\gamma}}\right)^{-1} \ , \qquad (13)\end{aligned}$$

where $\mathbf{J}_F(\boldsymbol{\upsilon}) \in \mathbb{R}^{n_\upsilon \times n_\upsilon}$ is the FIM of $\boldsymbol{\upsilon}$.

Now we focus on the covariance matrix of $\hat{\boldsymbol{\gamma}}_2$. According to [7], for large data sets,

$$\begin{aligned}\boldsymbol{\Sigma}(\hat{\boldsymbol{\gamma}}_2) \simeq &\left(\mathbb{E}\left\{\frac{\partial^2\Lambda(\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}^2}\right\}\right)^{-1}\\ &\mathbb{E}\left\{\frac{\partial\Lambda(\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}}\frac{\partial\Lambda^T(\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}}\right\}\left(\mathbb{E}\left\{\frac{\partial^2\Lambda(\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}^2}\right\}\right)^{-1} \qquad (14)\end{aligned}$$

can be used as an approximation. We know that:

$$\begin{aligned}\Lambda(\boldsymbol{\gamma}) &\triangleq \left(\hat{\boldsymbol{\upsilon}}_{\mathrm{ML}} - g(\boldsymbol{\gamma})\right)^T \mathbf{W} \left(\hat{\boldsymbol{\upsilon}}_{\mathrm{ML}} - g(\boldsymbol{\gamma})\right)\\ \frac{\partial\Lambda(\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}} &= -2\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\mathbf{W}\left(\hat{\boldsymbol{\upsilon}}_{\mathrm{ML}} - g(\boldsymbol{\gamma})\right)\\ \frac{\partial^2\Lambda(\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}^2} &= 2\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\mathbf{W}\frac{\partial\boldsymbol{\upsilon}}{\partial\boldsymbol{\gamma}} \ , \qquad (15)\end{aligned}$$

which, when substituted in (14), yields to

$$\begin{aligned}\boldsymbol{\Sigma}(\hat{\boldsymbol{\gamma}}_2) \simeq &\left(\mathbb{E}\left\{2\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\mathbf{W}\frac{\partial\boldsymbol{\upsilon}}{\partial\boldsymbol{\gamma}}\right\}\right)^{-1}\\ &\mathbb{E}\left\{2\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\mathbf{W}\left(\hat{\boldsymbol{\upsilon}}_{\mathrm{ML}} - g(\boldsymbol{\gamma})\right)\left(2\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\mathbf{W}\left(\hat{\boldsymbol{\upsilon}}_{\mathrm{ML}} - g(\boldsymbol{\gamma})\right)\right)^T\right\}\\ &\left(\mathbb{E}\left\{2\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\mathbf{W}\frac{\partial\boldsymbol{\upsilon}}{\partial\boldsymbol{\gamma}}\right\}\right)^{-1} \ . \qquad (16)\end{aligned}$$

The first and last expectations in (16) do not contain terms with $\hat{\boldsymbol{\upsilon}}$, thus we can neglect the expectation operator there,

$$\begin{aligned}\boldsymbol{\Sigma}(\hat{\boldsymbol{\gamma}}_2) \simeq &\left(\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\mathbf{W}\frac{\partial\boldsymbol{\upsilon}}{\partial\boldsymbol{\gamma}}\right)^{-1}\\ &\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\mathbf{W}\mathbb{E}\left\{\left(\hat{\boldsymbol{\upsilon}}_{\mathrm{ML}} - g(\boldsymbol{\gamma})\right)\left(\hat{\boldsymbol{\upsilon}}_{\mathrm{ML}} - g(\boldsymbol{\gamma})\right)^T\right\}\mathbf{W}\frac{\partial\boldsymbol{\upsilon}}{\partial\boldsymbol{\gamma}}\\ &\left(\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\mathbf{W}\frac{\partial\boldsymbol{\upsilon}}{\partial\boldsymbol{\gamma}}\right)^{-1} \ . \qquad (17)\end{aligned}$$

We now recall that the covariance matrix of $\hat{\boldsymbol{\upsilon}}_{\mathrm{ML}}$ tends to the inverse FIM under regularity conditions, then

$$\begin{aligned}\lim_{K\to\infty} \boldsymbol{\Sigma}(\hat{\boldsymbol{\gamma}}_2) = &\left(\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\mathbf{W}\frac{\partial\boldsymbol{\upsilon}}{\partial\boldsymbol{\gamma}}\right)^{-1}\\ &\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\mathbf{W}\mathbf{J}_F(\boldsymbol{\upsilon})\mathbf{W}\frac{\partial\boldsymbol{\upsilon}}{\partial\boldsymbol{\gamma}}\left(\frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\mathbf{W}\frac{\partial\boldsymbol{\upsilon}}{\partial\boldsymbol{\gamma}}\right)^{-1} \ . \qquad (18)\end{aligned}$$

Defining

$$\begin{aligned}\mathbf{V}_1 &= \frac{\partial\boldsymbol{\upsilon}^T}{\partial\boldsymbol{\gamma}}\\ \mathbf{V}_2 &= \left(\mathbf{V}_1\mathbf{W}\mathbf{V}_1^T\right)^{-1}\mathbf{V}_1\mathbf{W} \ , \qquad (19)\end{aligned}$$

for the sake of clarity, we have that the asymptotical variances can be expressed as

$$\lim_{K\to\infty} \boldsymbol{\Sigma}(\hat{\boldsymbol{\gamma}}_1) = \left(\mathbf{V}_1\mathbf{J}_F(\boldsymbol{\upsilon})\mathbf{V}_1^T\right)^{-1} \tag{20}$$

and

$$\lim_{K\to\infty} \boldsymbol{\Sigma}(\hat{\gamma}_2) = \mathbf{V}_2 \mathbf{J}_F(\boldsymbol{\upsilon}) \mathbf{V}_2^T \ . \tag{21}$$

Substituting (20) and (21) into (7),

$$\mathbf{C} \triangleq \mathbf{V}_2 \mathbf{J}_F(\boldsymbol{\upsilon}) \mathbf{V}_2^T - \left(\mathbf{V}_1 \mathbf{J}_F(\boldsymbol{\upsilon}) \mathbf{V}_1^T\right)^{-1} \ , \tag{22}$$

we are ready to verify its positive semidefiniteness. That is to say, for any real vector $\mathbf{u} \neq \mathbf{0}$, we have that

$$\mathbf{u}^T \mathbf{C} \mathbf{u} \geq 0 \ . \tag{23}$$

Define the vectors

$$\begin{aligned} \mathbf{a} &= \mathbf{J}_F^{-1/2}(\boldsymbol{\upsilon}) \mathbf{V}_2^T \mathbf{u} \\ \mathbf{b} &= \mathbf{J}_F^{1/2}(\boldsymbol{\upsilon}) \mathbf{V}_1^T \left(\mathbf{V}_1 \mathbf{J}_F(\boldsymbol{\upsilon}) \mathbf{V}_1^T\right)^{-1} \mathbf{u} \end{aligned} \tag{24}$$

and, by the Cauchy-Schwartz inequality[1], we have

$$\begin{aligned} \left(\mathbf{u}^T \mathbf{V}_2 \mathbf{J}_F(\boldsymbol{\upsilon}) \mathbf{V}_2^T \mathbf{u}\right) \left(\mathbf{u}^T \left(\mathbf{V}_1 \mathbf{J}_F(\boldsymbol{\upsilon}) \mathbf{V}_1^T\right)^{-1} \mathbf{u}\right) \\ - \left(\mathbf{u}^T \left(\mathbf{V}_1 \mathbf{J}_F(\boldsymbol{\upsilon}) \mathbf{V}_1^T\right)^{-1} \mathbf{u}\right)^2 \geq 0 \ . \end{aligned} \tag{25}$$

Recalling that $\left(\mathbf{V}_1 \mathbf{J}_F(\boldsymbol{\upsilon}) \mathbf{V}_1^T\right)^{-1}$ is a non-negative definite matrix, since it represents a covariance matrix, we can write (25) as

$$\mathbf{u}^T \left(\mathbf{V}_2 \mathbf{J}_F(\boldsymbol{\upsilon}) \mathbf{V}_2^T - \left(\mathbf{V}_1 \mathbf{J}_F(\boldsymbol{\upsilon}) \mathbf{V}_1^T\right)^{-1}\right) \mathbf{u} \geq 0 \ , \tag{26}$$

proving (23). □

Notice that the inequality in (26) becomes equality when we choose the weighting matrix such that $\mathbf{W} = \mathbf{J}_F(\boldsymbol{\upsilon})$. However, the true value of $\boldsymbol{\upsilon}$ is not available, which reinforces the idea that the one-step estimation cannot be outperformed by the two-steps approach.

## 3. SIMULATION RESULTS

This section aims at comparing the variances of the estimators of position in either cases: two-steps and DPE approaches. The former was computed by first computing ML estimates of synchronization parameters [8] and transforming them by the WLS procedure, as given in [2]. The latter was obtained by solving the ML estimator of position, reported in [1]. Both ML estimators ($\boldsymbol{\upsilon}_{\text{ML}}$ and $\boldsymbol{\gamma}_{\text{ML}}$) required the optimization of a non-convex cost function, which was performed by the Accelerated Random Search (ARS) algorithm [9].

When using WLS to compute user's position from synchronization parameter estimates, the construction of weighting matrix $\mathbf{W}$ is not unique. We considered that the diagonal entries in $\mathbf{W}$ are the carrier-to-noise density ratios ($C/N_0$) of the corresponding satellites, normalized to the highest $C/N_0$ value.

The recreated scenario consisted of $M = 7$ satellites, in a realistic geometry. The simulated constellation is described by the following azimuth and elevation angles (in degrees)

$$\begin{aligned} \boldsymbol{\theta} &= [288.9, 215.2, 87.9, 295.4, 123.5, 46.1, 130.6]^T \\ \boldsymbol{\phi} &= [46.9, 24.5, 29.1, 32.1, 71.5, 24.4, 60.7]^T \ , \end{aligned} \tag{27}$$

[1]Cauchy-Schwartz inequality: $\| \mathbf{a} \|^2 \| \mathbf{b} \|^2 - (\mathbf{a}^T \mathbf{b})^2 \geq 0$

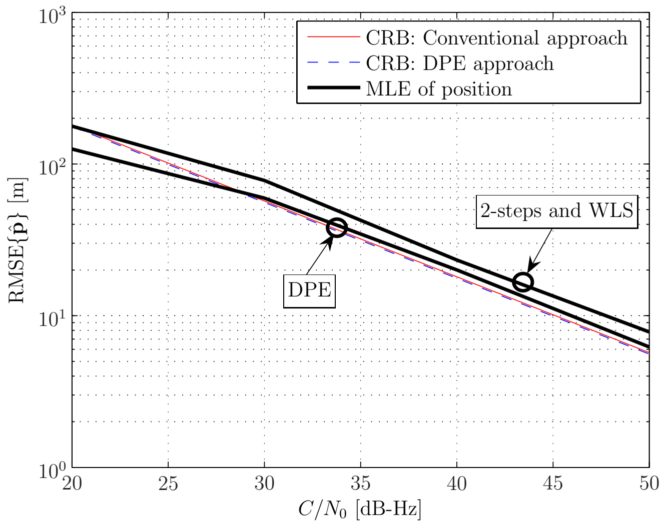


Figure 2: MSE performance versus $C/N_0$ of the satellites.

respectively. These satellites transmit civilian GPS-like signals at a carrier frequency of $f_c = 1575.42$ MHz. We considered a single antenna based receiver, with a 1.1 MHz pre-correlation filter and a sampling frequency of $f_s = 5.714$ MHz.

With this setup, the Root-MSE (RMSE) performance of either position estimators is plotted and compared to their respective theoretical lower bounds, provided by the Cramér-Rao Bound (CRB) and reported in [10]. In Figure 2 the curves are plotted against the $C/N_0$ of the satellites, assumed all equal in this simulation. It arises that both estimators have similar asymptotical bounds, although a close look will reveal that the CRB derived under DPE's framework is lower than the bound of a conventional 2-steps approach. In addition, we see that the MSE performance of DPE is smaller than the one achieve by the WLS processing of ML estimates of time-delays and Doppler-shifts, in accordance to Proposition 2.1.

Since the strength of DPE approach comes due to its ability to jointly process signals coming from independent channels, a slightly different scenario was tested. Figure 3 shows the RMSE performance of the estimators when the $C/N_0$ of one satellite is swept, while the rest remain at 45 dB-Hz. On the one hand, we can see how the CRBs differ for low-SNR conditions and that the DPE's CRB is always below conventional's positioning CRB. On the other hand, the performance of the conventional position estimator is severely degraded when accounts for low-powered signals, in contrast to DPE. When the swept $C/N_0$ exceeds the value of 45 dB-Hz, this satellite introduces a MAI to the rest of visible satellite. The latter is seen as a degradation of the position solution in the MSE sense. Since the estimation of synchronization parameters is performed independently, a conventional receiver is not immune to MAI. Conversely, DPE provides an optimal approach to jointly process all signals, following the ML principle. Thus, effectively mitigating the effect.

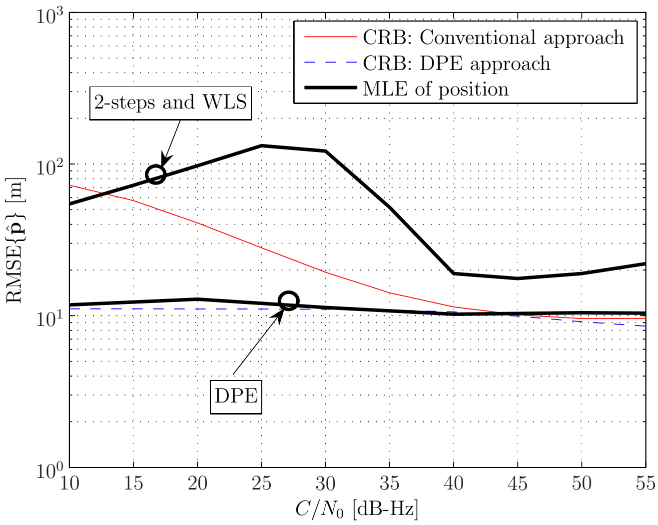


Figure 3: MSE performance versus $C/N_0$ of one satellite, the rest are assumed all equal to 45 dB-Hz.

Simulation results also point out that the bound provided by the CRB might not be valid for low $C/N_0$ values. This can be seen in Figure 2 for either positioning approaches and in Figure 3 for the conventional two-steps procedure, where the estimators are likely to be biased for low $C/N_0$ conditions. The reason is that the CRB falls in the category of *small-error bounds*, meaning that its validity is conditional on having *small* estimation errors. Thus, other bounds could be explored to have more accurate benchmarks under that regime, see for instance [11].

## 4. CONCLUSIONS

Although the conventional approach to positioning using GNSS receivers consists in a two-steps procedure (estimate synchronization parameters and solve a geometrical problem), this paper shows that this solution is not ML-optimal. In particular, this scheme does not provide in general ML estimates of receiver's position. These optimal estimates, in the ML sense, are given by DPE's approach. DPE proposes to estimate position directly from digitized signal, thus, merging the two-steps into a single process. The paper delved into the proof of Proposition 2.1, showing that DPE provides better MSE performances than a two-steps approach. In particular, the result states that the performance of the two-steps position estimator is, at most, equal to that given by DPE. This result reinforces the interest on DPE's philosophy to design GNSS receivers. A number of computer simulations were performed and discussed to validate the statement under realistic signal conditions.

DPE was seen to provide enhance positioning capabilities in scenarios where satellite links are affected by independent degradation effects. In the simulation results we discussed the case of a weak satellite signal, however, other cases are of interest. For instance, the problem of multipath mitigation can be mitigated by DPE, relying on the joint processing of signals affected by independent propagation channels [1].

The contribution of the paper is twofold. First, to provide a mathematical justification for DPE's approach, establishing a solid basis for its further investigation. Secondly, studying the performances of both position estimators and commenting on some of their issues, such as the Multiple Access Interference mitigation of DPE's approach.

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