Improved Correlation of Generalized DFT with Nonlinear Phase for OFDM and CDMA Communications

Ali N. Akansu and Handan Agirman-Tosun
New Jersey Institute of Technology
Department of Electrical & Computer Engineering
University Heights, Newark, NJ 07102 USA
Email: Akansu@NJIT.edu

ABSTRACT

Recently, Generalized Discrete Fourier Transform (GDFT) has been forwarded as the theoretical framework to design a variety of constant modulus orthogonal complex transforms with non-linear phase. In this paper, we evaluate the auto- and cross-correlation properties of several GDFT solutions and compare them with the popular code families like Gold, Walsh and DFT. It is shown that the GDFT brings significant correlation performance improvements over known code families. We predict that GDFT based OFDM and CDMA solutions will offer performance improvements in multicarrier communications systems of the future.

***Index Terms*— Generalized Discrete Fourier Transform, Correlation Performance, OFDM, DMT, CDMA.**

I. INTRODUCTION

Among various binary spreading families, Gold codes have been successfully used for asynchronous communications in DS/CDMA systems due to their lower cross-correlation features [1]. Walsh, Gold and several other real spreading code sets are designed to optimize even correlation functions [2-6]. However, the odd correlations are also important as much as even correlations. Therefore, Fukumasa, Kohno and Hideki proposed a new set of complex PN sequences, called Equal Odd and Even (EOE) sequences, with good odd and even correlations [7]. EOE sequences are generated by using one of the real code sets, e.g. Gold and Walsh.

Spreading codes with non-binary real chip values were also proposed in the literature in order to improve their auto- and cross-correlations. More recently, research has refocused on constant amplitude spreading codes due to the efficiency concerns of non-linear RF power amplifiers employed in wireless transceivers. Hence, complex roots of unity were proposed as complex spreading codes by several authors in the literature. All codes of such a set are placed on the unit circle of the complex plane. Frank-Zadoff, Chu and Oppermann introduced a variety of complex spreading codes [8-11]. Moreover, Oppermann has shown that Frank-Zadoff and Chu Sequences are the special cases of his family of spreading sequences. The dimension of an Oppermann set with the code length of N is determined by Euler's totient function. For the case where N is a prime number, the size of the Opperrmann code set is equal to N-1. This is one of the basic limitations of Oppermann Codes [8,12].

More recently, Generalized Discrete Fourier Transform (GDFT) was introduced and it provides a theoretical framework where many popular constant modulus orthogonal function sets including DFT and others shown to be the special solutions [11]. In contrast to linear phase DFT, GDFT family explores the phase space in its entirety in order to improve correlation properties of constant power orthogonal spreading codes. We present GDFT and its correlation properties with respect to the well know correlation metrics along with popular families like Gold and Walsh in the following sections of the paper.

II. GENERALIZED DFT

An N^{th} root of unity is a complex number satisfying the equation [11]

$$z^N = 1 \quad N = 0,1,2,... \tag{1}$$

If z holds Eq. (1) but $z^m \neq 1$; $0 < m < N-1$, then z is defined as a *primitive N^{th} root of unity*. The complex number $z_0 = e^{j(2\pi/N)}$ is the primitive N^{th} root of unity with the smallest positive argument. The other primitive N^{th} roots of unity are expressed as

$$z_k = e^{j(2\pi/N)k} \quad k = 1,2,3, \ \dots\ , N-1 \tag{2}$$

where *k* and *N* are *co-prime*. All *primitive N^{th} roots of unity* satisfy the unique summation property of a geometric series expressed as follows

$$\sum_{n=0}^{N-1} z_k^n = \frac{(z_k)^N - 1}{z_k - 1} = \begin{cases} 1, N = 1 \\ 0, N > 1 \end{cases} \quad \forall k = 0,1,2,..,N-1 \tag{3}$$

Now, we define a periodic, with the period of N, constant modulus, complex discrete-time sequence $e_r(n)$ as

$$e_r(n) \triangleq (z_r)^n = e^{j(2\pi/N)rn} \quad r,n = 0,1,2,\ldots, N-1 \qquad (4)$$

This complex sequence over a finite discrete-time interval in a geometric series is expressed according to Eq. (3) as follows

$$\frac{1}{N}\sum_{n=0}^{N-1} e_r(n) = \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)rn} = \begin{cases} 1, & r = mN \\ 0, & r \neq mN \\ & m,n = \text{integer} \end{cases} \qquad (5)$$

This mathematical property is utilized with the factorization into two orthogonal exponential functions where one defines the discrete Fourier transform (DFT) set $\{e_k(n)\}$ satisfying

$$\frac{1}{N}\sum_{n=0}^{N-1} e_k(n)e_l^*(n) = \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)(k-l)n} = \begin{cases} 1, & k-l = r = mN \\ 0, & k-l = r \neq mN \\ & m,n = \text{integer} \end{cases} \qquad (6)$$

The notation (*) represents the complex conjugate function of a function. Note that $\omega_0 = 2\pi/N$ is the n[th] root of unity on the unit circle and also called the fundamental frequency defined in the unit of radians per cycle. We are going to expand the linear phase functions of Eq. (6) in the definition of GDFT.

Let's generalize Eq. (5) by introducing a product function in the phase defined as $\varphi(n) = \varphi_k(n) - \varphi_l(n)$ and expressing a constant amplitude orthogonal set as follows,

$$\frac{1}{N}\sum_{n=0}^{N-1} e_k(n)e_l^*(n) = \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)rn}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)\varphi(n)n} = \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)[\varphi_k(n)-\varphi_l(n)]n}$$

$$= \begin{cases} 1, & \varphi(n) = \varphi_k(n) - \varphi_l(n) = r = mN \\ 0, & \varphi(n) = \varphi_k(n) - \varphi_l(n) = r \neq mN \\ & m,n = \text{integer} \end{cases} \qquad (7)$$

Hence, the basis functions of the new orthogonal set are defined as

$$e_k(n) \triangleq e^{j(2\pi/N)\varphi_k(n)n} \quad k \,\&\, n = 0,1,\ldots, N-1 \qquad (8)$$

The new orthogonal function set in Eq. (8) is called the *Generalized Discrete Fourier Transform* (GDFT) [11]. It is noted that there are infinitely many function sets with constant power are available.

As an example, one might define the discrete time rational function $\varphi_k(n)$ in Eq. (8) as the ratio of two polynomials,

$$\varphi_k(n) = \frac{N(n)}{D(n)} = \frac{\sum_{j=1}^{N} a_j n^{b_j}}{\sum_{j=1}^{M} c_j n^{d_j}} \quad N \leq M \qquad (9)$$

Let's assume that the denominator polynomial $D(n)$ is equal to one and *the order N* numerator polynomial in *n* is defined as follows

$$\varphi_k(n) = \sum_{j=1}^{N} a_j n^{b_j} = a_1 n^{b_1} + a_2 n^{b_2} + a_3 n^{b3} + \ldots + a_N n^{b_N} \qquad (10)$$

In general, $\{a_j, b_j\}$ coefficients are real numbers. Now, we will define several correlation metrics to compare various code sets. These correlation types are known to dictate the performance of a multicarrier communications system.

III. CORRELATION METRICS

Since there are infinitely many possible GDFT's with nonlinear and linear phase in the solutions space, we define a few metrics to compare the performance of various code sets. These metrics basically depend on the auto- and cross-correlation properties of the orthogonal sets. On the other hand, the *multi-user interference* and *synchronization amiability* of code sets are theoretically shown to depend on their auto- and cross-correlations properties. There are several different types of correlation functions defined in the literature to characterize code sets of different families. In this study, we mainly focus on the *aperiodic* correlation function (ACF). ACF for two complex sequences $\{e_k(n), e_l(n)\}$ of an *N-dimensional* set is defined as [4]

$$d_{k,l}(m) = \begin{Bmatrix} \frac{1}{N}\sum_{n=0}^{N-1-m} e_k(n)e_l^*(n+m), \; 0 < m \le N-1 \\ \frac{1}{N}\sum_{n=0}^{N-1+m} e_k(n-m)e_l^*(n), \; 1-N < m \le 0 \\ 0 \qquad , \qquad |m| \ge N \end{Bmatrix} \quad (11)$$

The following metrics employed in the brute force search process to design various GDFT examples presented in this paper.

III.A Maximum Value of Auto- and Cross-Correlation Sequences: The maximum correlation value $d_{\max}$ of a set of sequences $\{e_k(n)\}; k,n = 0,1,2,...,N-1$ is calculated as

$$d_{\max} = \max\{d_{am}, d_{cm}\} \quad (12)$$

where d_{am} is the maximum value of N autocorrelation sequences for the entire set obtained from Eq. (11) when $\{e_k(n) = e_l(n)\}; k = 0,1,2,...,N-1$ as given in the following equation,

$$d_{am} = \max_{\substack{0 \le k < M \\ 1 \le m < M}} \{|d_k(m)|\} \quad (13)$$

Similarly, d_{cm} is the maximum value of all possible cross-correlation sequences in a code set also calculated from Eq. (8) and is expressed as

$$d_{cm} = \max_{\substack{0 \le k,l < M \;\; k \ne l \\ 0 \le m < M}} \{|d_{k,l}(m)|\} \quad (14)$$

In Eq. (13) and Eq. (14), M is the size of the code set and N is the code length in the set.

Sarwate showed the relationship between the maximum out-of-phase auto-correlation d_{am} and the maximum cross-correlation d_{cm} as follows [13],

$$d_{cm}^2 + \frac{(N-1)}{N(M-1)} d_{am}^2 \ge 1 \quad (15)$$

leading to the Welch bound for complex spreading sequences expressed as [14],

$$d_{\max} = \max\{d_{am}, d_{cm}\} = \sqrt{\frac{M-1}{NM-1}} \quad (16)$$

In Table 1, we display the achievable Welch bounds for constant modulus complex spreading codes for various lengths.

Table 1: Achievable Welch Bounds for Various Spreading Code Lengths.

(M=N)	$d_{\max}$
8	0.333
16	0.243
32	0.174
64	0.124

III.B Mean Square Value of Auto- and Cross-Correlation Sequences: The quantitative measures given above are important to highlight the worst case scenarios. In contrast, the average performance counts more in some applications. Therefore, we take into account the mean square value of cross-correlation sequences as another performance metric. Furthermore, the average of mean square auto-correlation sequences for each code in the set, R_{AC}, and the average of mean square cross-correlation sequences for all code pairs in the set, R_{CC}, are introduced as follows [4],

$$R_{AC} = \frac{1}{M}\sum_{k=1}^{M}\sum_{\substack{m=1-N \\ m \ne 0}}^{N-1} |d_{k,k}(m)|^2 \quad (17)$$

$$R_{CC} = \frac{1}{M(M-1)}\sum_{k=1}^{M}\sum_{\substack{l=1 \\ l \ne k}}^{M}\sum_{m=1-N}^{N-1} |d_{k,l}(m)|^2 \quad (18)$$

III.C The Merit Factor (F_k): Code synchronization is crucial for the performance of CDMA systems and it is strongly related to the auto-correlation properties of codes. In order to incorporate this requirement in code evaluation another metric called the merit factor (F_k) was introduced in [6]. The merit factor for the k^{th} code is the ratio of the energy in the main lobe of the autocorrelation function over the energy in the side lobes and it is mathematically expressed as

$$F_k = \frac{d_k(0)}{2\sum_{m=1}^{N-1} |d_k(m)|^2} \quad (19)$$

In CDMA communications systems, merit factor is desired to be as large as possible in order to improve the code synchronization and amiability. Next, we are going to search for solutions of a G matrix type with brute force search technique with respect to the correlation performance metrics defined above in order to be able to design A_{GDFT} matrices defined in Eq. (22) below.

IV. GDFT DESIGN WITH DIAGONAL G MATRIX

In this design example we used a non-constant $\varphi_k(n)$ function of Eq. (10) in the phase for each function of the set expressed as

$$\begin{aligned} \varphi_k(n) &= a_1 n^{b_1} + a_2 n^{b_2} \\ a_1 &= k \\ b_1 &= 0 \\ \varphi_k(n) &= k + a_2 n^{b_2} \end{aligned} \tag{20}$$

Therefore, the basis functions of the set are defined according to Eq. (8) as

$$e_k(n) = e^{j(2\pi/N)\varphi_k(n)n} = e^{j(2\pi/N)(kn)} e^{j(2\pi/N)(a_2 n^{b_2+1})}$$
$$k, n = 0, 1, ..., N-1 \tag{21}$$

Note that the first exponential term of the last equation is merely the DFT kernel with linear phase while the second exponential term defines the G matrix and $\{e_k(n)\}$ are the row sequences of A_{GDFT} matrix that is defined as follows

$$A_{GDFT} = A_{DFT} G \tag{22}$$

In this form, by changing the values of real a_2 and b_2 coefficients, one might obtain many different transform sets with desirable auto- and cross-correlation properties and nonlinear phase functions.

In the previous section, we defined several metrics for the evaluation of various spreading code sets. Now, we display the values of these metrics for optimal A_{GDFT} matrices obtained in the solutions space utilizing a brute-force search where the search resolution is defined by the binary valued a_2 and b_2 coefficients with the corresponding number of bits per coefficient as defined above. Table 2 tabulates the optimal values of the metric $d_{\max}$ along with other performance metrics for various search resolutions defined as $\Delta_{a_2,b_2} = 5/2^b$ where b is the search resolution bits per coefficient and $0 < a_2, b_2 \le 5$ for the code length of $N = 8$.

Table 2: Values of Various Metrics when Optimal Design is Based on the Performance Metric $d_{\max}$ for the Code Length of $N = 8$.

***b* (bits/c)**	d_{am}	d_{cm}	$d_{\max}$ ***(OPT)***	R_{AC}	R_{CC}	F
4	0.301	0.442	0.442	0.526	0.925	1.900
6	0.376	0.409	0.409	0.854	0.878	1.171
8	0.341	0.388	0.388	0.576	0.918	1.738
9	0.377	0.388	0.388	1.096	0.844	0.913

Similarly, Table 3 displays the correlation metrics for various known codes along with the optimal A_{GDFT} matrix obtained through a search based on the design metric $d_{\max}$ for the code length of *N=8*.

Table 3: Correlation Performance Metrics for Various Popular Spreading Code Families with the Code length of N=8.

Code	d_{am}	d_{cm}	$d_{\max}$	R_{AC}	R_{CC}	**F**
Walsh [8x8]	0.88	0.88	0.88	2.38	0.66	0.42
Walsh-like [8x8], [3]	0.63	0.63	0.63	0.88	0.88	1.14
DFT [8x8]	0.88	0.33	0.88	4.38	0.38	0.22
7/8 Gold	0.71	0.71	0.71	0.86	0.88	1.17
Oppermann Set, [8,12] (*opt* $d_{\max}$) (m=1, p=1, n=2.98, N=7)	0.47	0.42	0.47	1.28	0.59	0.78
A_{GDFT} [8x8] (*opt* $d_{\max}$)	0.38	0.39	0.39	1.1	0.84	0.91

In Figures 1.a and 1.b, we display the inter-dependences of the auto- and cross-correlation metrics R_{AC} and R_{CC}, respectively, on both of the design parameters a_2 and b_2. For a multi-carrier communications application, depending on the system either OFDM based or CDMA based, one can choose optimum values of a_2 and b_2 for the desired values of auto- and cross-correlation metrics, R_{AC} and R_{CC}. In OFDM systems, frequency localization is more important and the optimization is performed on R_{CC} parameters whereas in a DS-CDMA system, R_{AC} and R_{CC} both are equally important. The low values of R_{AC} is desired for synchronization purposes of the system. Similarly, the low values of R_{CC} are required to minimize multi-user interference (MUI) that dictates the system performance.

4. CONCLUSIONS

Multi-carrier interference and synchronization amiability of orthogonal sets in an OFDM or CDMA based communications system are theoretically shown to depend on their auto- and cross-correlations properties. GDFT with nonlinear phase offers a unified framework to design constant modulus orthogonal sets. We propose in this paper a correlation-based optimal orthogonal set design method providing performance improvements over the known families. It is expected that GDFT based orthogonal multiplexers with improved performance and design flexibilities will allow us to build better OFDM and CDMA based multicarrier communications systems in the future.

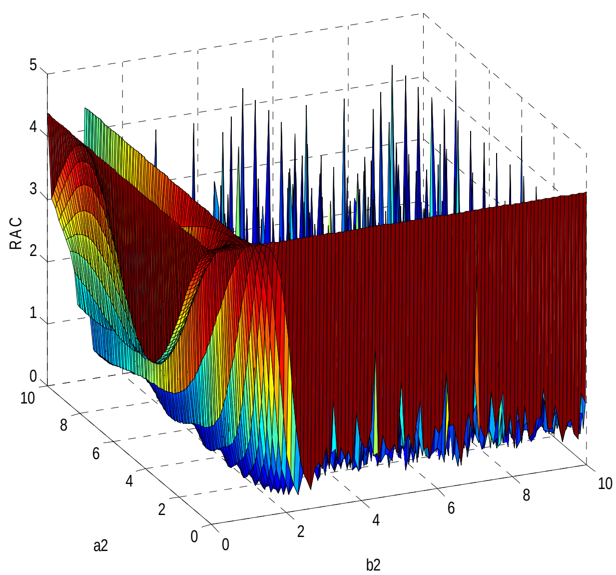

Figure 1.a: Variation of the auto-correlation metric R_{AC} as a function of the design parameters a_2 and b_2.

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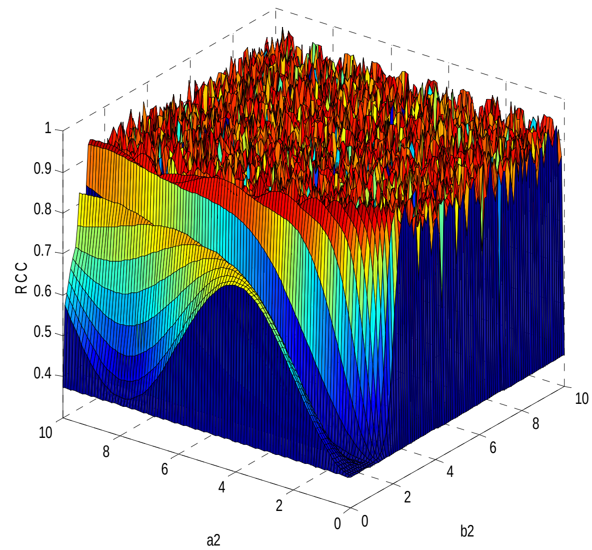

Figure 1.b: Variation of the cross-correlation metric R_{CC} as a function of the design parameters a_2 and b_2.

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