ON SPECTRAL ESTIMATION AND FOURIER TRANSFORM APPROXIMATION FROM SAMPLED DATA

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ABSTRACT

A method for spectral estimation of a continuous-domain signal, given by its sampled version only, is introduced. Unlike the discrete Fourier transform (DFT), the proposed approach reduces aliasing effects. The proposed approach relies on finite-duration Sobolev functions, for which the ideal sampling process is characterized by means of an inner product operation. The point-wise evaluation of the Fourier transform is based on a Sobolev type inner product too, allowing for a minimax approximation approach to be derived and utilized. Experimental results show that the proposed approach is a preferred alternative over the DFT in cases where spectral analysis of sampled signals is required.

1. INTRODUCTION

The discrete Fourier transform (DFT) is widely used in spectral analysis of digital signals [1–4]. It describes a finite duration sequence by means of discrete-domain complex exponentials, where the DFT values themselves serve as the representation coefficients. An additional interpretation of the DFT was recently suggested in [5], for which every single DFT value was considered to be the average of the input sequence having been modulated beforehand. Considering an ideal sampling procedure, there are cases in which the DFT corresponds to the Fourier transform of the continuous-domain signal; one such case is periodic band-limited functions. It is a well known fact, however, that in the general case of finite energy signals, the DFT of a sample sequence is related to the Fourier transform through aliasing. It is also known that as the sampling step becomes shorter, does the aliasing effect become negligible, as is often assumed in practical situations. Within this context (and albeit the aliasing effect), the DFT of a sampled signal may be viewed as an approximation procedure for the Fourier transform of the original continuous-domain signal itself.

Spectral analysis can also be carried out by Gabor functions [6–9]. Gabor analysis of discrete-domain signals is closely related to the DFT, providing a time-frequency representation of the input sequence. As is the case for the DFT, Gabor analysis of sampled signals does not necessarily correspond to the Gabor representation of the underlying continuous-domain signal, and the former may be viewed as an approximation procedure of the latter, too.

The question raised in this work is whether there exists an approximation procedure that minimizes the aliasing effects that are inherent to the DFT. More precisely, can one suggest a procedure for approximating the Fourier transform of a continuous-domain signal given by its ideal samples only, while minimizing aliasing? Anti-aliasing filtering is commonly used by signal processing practitioners. Nevertheless, these filters do not necessarily comply with an ideal low pass operation and the DFT may still introduce aliasing. Further, sampling the output of a low pass filter is equivalent to orthogonally projecting the signal onto a shift-invariant space; the generating function of this space is the mirrored version of the filter impulse response. In such cases, the DFT still introduces aliasing and one should utilize the shift-invariant model to approximate Fourier transform values instead. Notwithstanding the shift-invariant model, orthogonal projections introduce information loss and by avoiding the low pass filtering operation one may better cope with tasks such as Fourier transform approximation.

A possible approach to this problem relies on a Sobolev model of the original signal. Sobolev spaces consist of smooth functions and they serve as the underlying continuous-domain model in several signal processing tasks [10–14]. More importantly, Sobolev functions are dense in L_2, implying that they can approximate any finite-energy function, e.g. the step function, while attaining arbitrarily small approximation errors. Furthermore, this set of functions includes exponential functions, trigonometric functions, Gaussian-type functions and Polynomials, describing a large class of signal models that fit into this framework. The ideal sampling process of finite duration Sobolev signals will be characterized by means of an inner product operation and the point-wise evaluation of their Fourier transform will be characterized by means of a Sobolev type inner product, too. Such an interpretation would be utilized then for deriving a minimax approach for the approximation task at hand.

2. APPROXIMATION OF LINEAR FUNCTIONALS

Let H_2^p be the Sobolev space of order p. This space consists of all one-dimensional finite-energy functions defined on a finite-support domain $\Omega \in \mathbb{R}$ for which their first p derivatives are of finite energy as well. The cor-

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responding inner product is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle_{H_2^p} = \sum_{n=0}^{p} \lambda_n \cdot \left\langle \mathbf{x}^{(n)}, \mathbf{y}^{(n)} \right\rangle_{L_2}, \tag{1}$$

where

$$\langle \mathbf{x}, \mathbf{y} \rangle_{L_2} = \int_{\Omega} \mathbf{x}(t) \cdot \overline{\mathbf{y}(t)}\, dt, \tag{2}$$

and where the set of weights $\{\lambda_n\}$ provides a positive measure for $\langle \mathbf{x}, \mathbf{x} \rangle_{H_2^p}$. Considering the Fourier domain, this restriction implies that

$$\lambda_0 + \lambda_1 \omega^2 + \cdots + \lambda_p \omega^{2p} \geq 0 \tag{3}$$

would hold for all frequencies $\omega \in \mathbb{R}$. The RKHS property of the Sobolev space implies that there is a unique function, the reproducing kernel $\varphi(t)$, for which $\mathbf{x}(\tau) = \langle \mathbf{x}(t), \varphi(t-\tau) \rangle_{H_2^p}$ holds for every $\mathbf{x} \in H_2^p$. Let $\Omega = (-\pi, \pi)$ and let $\mathbf{x}$ be an arbitrary Sobolev function. Such a function can be expressed as

$$\mathbf{x}(t) = \sum_n a[n] \cdot e^{jnt}, \tag{4}$$

where equality holds point-wise. The sample value $\mathbf{x}(t=0)$ is a linear bounded functional and by Riesz representation theorem can be expressed by means of an inner product operation

$$\mathbf{x}(0) = \langle \mathbf{x}(t), \varphi(t) \rangle_{H_2}. \tag{5}$$

The function $\varphi(t)$ can be expressed by means of its Fourier coefficients, too, giving rise to the following derivation

$$\begin{aligned} \mathbf{x}(0) &= \sum_n a[n] \cdot e^{jn\cdot 0} \qquad (6) \\ &= \left\langle \sum_n a[n] \cdot e^{jnt}, \sum_m b[m] \cdot e^{jmt} \right\rangle_{H_2^p} \\ &= \sum_{n,m} a[n]\overline{b}[m](\lambda_0 + \cdots + \lambda_p n^p m^p) \int_{\Omega} e^{j(n-m)t}\, dt \\ &= 2\pi \sum_n a[n]\, \overline{b}[n]\, (\lambda_0 + \cdots + \lambda_p n^{2p}). \end{aligned}$$

Now, $\mathbf{x}$ is arbitrary yielding $b[n] = [2\pi(\lambda_0 + \cdots + \lambda_p n^{2p})]^{-1}$ and

$$\varphi(t-\tau) = \frac{1}{2\pi} \sum_n \frac{e^{jn(t-\tau)}}{\lambda_0 + \cdots + \lambda_p n^{2p}}. \tag{7}$$

This reproducing kernel is Ω-periodic and several such kernels are depicted in Figure 1.

Adopting the reproducing kernel framework, the sampled version of a signal can be described by a set of inner product operations involving $\{\varphi(t-t_n)\}_n$. This set of functions constitutes a Riesz basis for the sampling space given by

$$\mathcal{S} = \overline{Span}\, \{\varphi(t-t_n)\}_n\,, \tag{8}$$

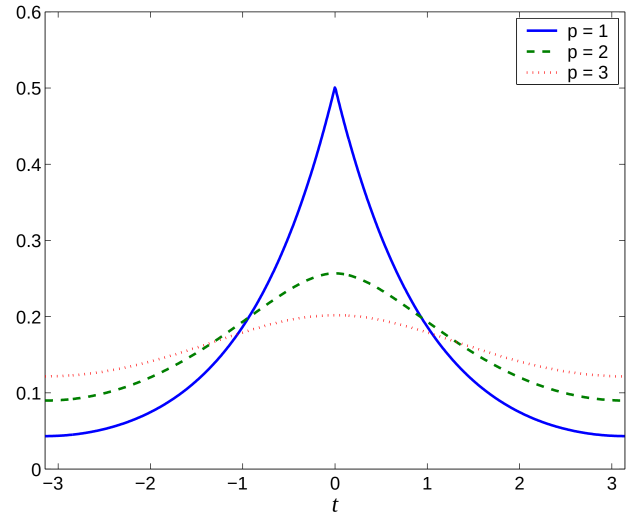

Figure 1: *Reproducing kernels of Sobolev spaces of several orders p. The weights are $\lambda_n = \binom{p}{n}$ and $\Omega = (-\pi, \pi)$.*

and the corresponding Gram matrix is given by

$$G(m,n) = \varphi(t_m - t_n). \tag{9}$$

The orthogonal projection of $\mathbf{x}$ onto the sampling space is then given by

$$P_{\mathcal{S}}\mathbf{x} = \sum_n a_n \cdot \varphi(\cdot, t_n), \tag{10}$$

where $a = G^{-1}c$ and c denotes the ideal samples of $\mathbf{x}$; that is, $c[n] = \mathbf{x}(t_n)$. The sampled versions of both $P_{\mathcal{S}}\mathbf{x}$ and of $\mathbf{x}$ are identical, while the former minimizes the Sobolev norm among all possible functions that interpolate into c. It also holds that the only information available from the sampled version of a Sobolev signal corresponds to $P_{\mathcal{S}}\mathbf{x}$. When considering shift-invariant cases as in [15], the reproducing kernels are shown to be exponential functions. Further, these functions describe autoregressive stochastic processes, which in turn allow one to determine proper Sobolev weights $\{\lambda\}$. Given the sampled version of a signal, it is suggested here to use a least square estimation method for extracting its autoregressive parameters, i.e. the Z-domain roots of the autocorrelation sequence. It is then suggested to determine the s-domain roots of an autocorrelation function by $s = \ln Z$ and set the Sobolev weights to be the coefficient of the corresponding polynomial in s.

Following Riesz representation theorem, every linear and bounded functional in a Hilbert space can be described by means of an inner product operation that involves the input signal $\mathbf{x}$ and a unique function, denoted by $\mathbf{w}$. Given the sampled version of a signal $\mathbf{x}$, it was shown in [12] that the minimax approximation of $d = \langle \mathbf{x}, \mathbf{w} \rangle$ is given by $\hat{d} = \langle P_{\mathcal{S}}\mathbf{x}, P_{\mathcal{S}}\mathbf{w} \rangle$. That is, the latter term minimizes the maximum possible approximation error $|d - \hat{d}|$ among all possible Sobolev functions that interpolate into c. This approximation can be carried out by discrete domain means involving the sampled versions of both the original signal, the analysis function and the Gram matrix of the sampling functions given by (9). In particular,

$$\langle P_{\mathcal{S}}\mathbf{x}, P_{\mathcal{S}}\mathbf{w}\rangle = c^T \cdot G^{-1} \cdot b, \tag{11}$$

where $b[n] = w(t_n)$. The approximation error is then given by

$$\begin{aligned} \left|d - \hat{d}\right|^2 &= \left|\langle \mathbf{x}, w\rangle - \langle P_{\mathcal{S}}\mathbf{x}, P_{\mathcal{S}}\mathbf{w}\rangle\right|^2 \qquad (12) \\ &= \left|\langle \mathbf{x} - P_{\mathcal{S}}\mathbf{x}, \mathbf{w} - P_{\mathcal{S}}\mathbf{w}\rangle\right|^2 \\ &\leq \left(\left\|\mathbf{x}\right\|^2 - c^T G^{-1} c\right) \cdot \left\|\mathbf{w} - P_{\mathcal{S}}\mathbf{w}\right\|^2, \end{aligned}$$

providing a tight upper bound on the approximation error. The analysis function $\mathbf{w}$ is analytically known giving rise to

$$\begin{aligned} \left\|\mathbf{w} - P_{\mathcal{S}}\mathbf{w}\right\|^2 &= \|\mathbf{w}\|^2 - b^T G^{-1} b \\ &= \sum_n \left|a[n]\right|^2 - b^T G^{-1} b, \qquad (13) \end{aligned}$$

where a corresponds to the Fourier coefficients of $\mathbf{w}$ and $b[n] = w(t_n)$ as in (11).

3. FOURIER TRANSFORM APPROXIMATION

The Fourier transform can be described by means of functionals. In particular, the point-wise evaluation of $\mathbf{X}(\omega)$ can be identified by a proper analysis function $\mathbf{w}_\omega(t)$ for which $\mathbf{X}(\omega) = \langle \mathbf{x}, \mathbf{w}_\omega\rangle_{H_2}$. Such an interpretation would not hold for signals of infinite support; the point-wise evaluation of $\mathbf{X}(\omega)$ is not a bounded functional in such a case. One may consider scaled and normalized versions of the $sinc(\cdot)$ function and observe that the DC component $\mathbf{X}(\omega = 0)$ may be arbitrarily large although the L_2 norm is maintained fixed. Nevertheless, finite support signals do fit into this interpretation and the Fourier transform of $x \in L_2(\Omega)$ in such a case is given by

$$\mathbf{X}(\omega) = \left\langle \mathbf{x}(t), e^{j\omega t}\right\rangle_{L_2}.$$

Assuming $\mathbf{x}$ is a Sobolev function, this L_2 inner product operation can be described by means of a Sobolev inner product involving $\mathbf{x}$ and the analysis function $\mathbf{w}_\omega$ which is derived next. Let $\mathbf{x}(t) \in H_2^p$ be given by (4) and let the complex exponential function be given by

$$e^{j\omega t} = \sum_n b[n] \cdot e^{jnt} \tag{15}$$

where $b[n] = sin\left[\pi(\omega - n)\right]/\pi(\omega - n)$. The Fourier transform of $\mathbf{x}$ is given by

$$\left\langle \mathbf{x}, e^{j\omega t}\right\rangle_{L_2} = 2\pi \cdot \sum_n a[n] \cdot \overline{b[n]}. \tag{16}$$

Now, let $\mathbf{w}_\omega(t) = \sum_n c[n] \cdot e^{jnt}$. This function should satisfy $\left\langle \mathbf{x}, e^{j\omega t}\right\rangle_{L_2} = \langle \mathbf{x}, w_\omega\rangle_{H_2}$. It then follows that

$$\sum_n a[n] \cdot \overline{b[n]} = \sum_n \left(\lambda_0 + \cdots + \lambda_p n^{2p}\right) \cdot a[n] \cdot \overline{c[n]} \tag{17}$$

holds for every Sobolev function $\mathbf{x}$. Therefore, $c[n] = b[n]/\left(\lambda_0 + \cdots + \lambda_p n^{2p}\right)$ and

$$w_\omega(t) = \sum_n \frac{1}{\lambda_0 + \cdots + \lambda_p n^{2p}} \cdot \frac{sin\left[\pi(\omega - n)\right]}{\pi(\omega - n)} \cdot e^{jnt}. \tag{18}$$

This analysis function reduces to a single complex exponential for integer values of ω.

Having the uniformly sampled version of a signal, one may apply (11) for approximating the Fourier transform of the original signal at the required frequencies. An alternative method for such an approximation is the DFT given by,

$$\mathbf{X}^D[k] = \sum_{n=0}^{N-1} c[n] \cdot e^{-j\frac{2\pi nk}{N}}. \tag{19}$$

Each DFT value can be interpreted as a Riemann type sum approximation of the Fourier transform at equally spaced frequencies as will be shown next. Considering a sampling step T,

$$\begin{aligned} \mathbf{X}(\omega) &= \int_\Omega \mathbf{x}(t) \cdot e^{-j\omega t}\, dt \qquad (20) \\ &\cong T \cdot \sum_{|n| < \lfloor \frac{\pi}{T} \rfloor} \mathbf{x}(nT) \cdot e^{-j\omega nT} \\ &\cong T \cdot e^{j\omega t_0} \cdot \sum_{n=0}^{N-1} c[n] \cdot e^{-j\omega nT}. \end{aligned}$$

where t_0 is the first sampling coordinate and where N denotes the number of known samples. It then follows that

$$\mathbf{X}\left(\frac{2\pi k}{NT}\right) \cong T \cdot e^{\frac{2\pi j k t_0}{NT}} \cdot \mathbf{X}^D[k] \tag{21}$$

where $k = 0 \ldots N-1$. There are cases in which this relation holds with equality [2]; if a periodic band limited signal is sampled an integer number of times N over one period, such that N is at least $2M+1$ (where M is the index of the highest non-zero harmonic), the DFT coefficients of the sampled sequence are equal, up to a constant factor N, to the corresponding Fourier coefficients of the signal harmonics. However, this approximation is not optimal for Sobolev functions and the minimax approach may provide an alternative procedure for this task.

4. EXPERIMENTAL RESULTS

Fourier transform approximation was carried out by both the proposed approach and by the DFT while considering $\mathbf{x}$ to be the hat function $\mathbf{x}_1(t) = 1 - |t|$, $|t| < 1$ with its Fourier transform $\mathbf{X}_1(\omega) = \left[\sin(\omega/2)/\omega/2\right]^2$. The approximation error of both methods is shown in Figure 2. The frequency range in this figure is $\left[0, \frac{\pi}{T}\right]$ which corresponds to half of the sampling rate. The complementary range $\left[\frac{\pi}{T}, \frac{2\pi}{T}\right]$ is not shown due to the symmetry property of the DFT sequence around $\frac{\pi}{T}$, which prevents the DFT from properly approximating

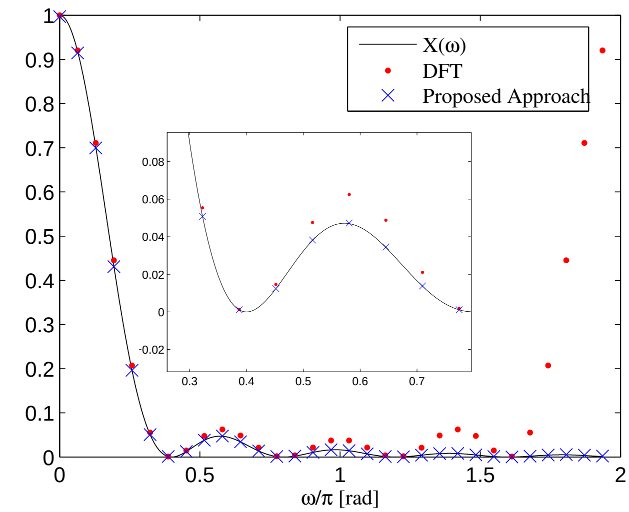

Figure 2: *Spectral estimation comparison. Shown is magnitude of the of the Fourier transform (solid), of the DFT values (dots) and of the proposed approach (x-marks) for the hat function. The frequency range is $\left[0, \frac{2\pi}{T}\right]$. The sampling interval is $T = 0.2$ and the Sobolev order is $p = 1$. The DFT symmetry property within the frequency range $\left[\frac{\pi}{T}, \frac{2\pi}{T}\right]$ is not shared by the proposed approach. Also the proposed approach outperforms the DFT method in $\left[0, \frac{\pi}{T}\right]$ too.*

$\mathbf{X}(\omega)$ there. It is noted that this symmetry property is not shared by the proposed approach, making it suitable for approximating spectral content of high frequencies, too. Figure 3 further depicts SNR values as a function of the sampling interval; the proposed approach outperfoms the DFT in this case too. SNR values were calculated based on the frequency range $\left[0, \frac{\pi}{T}\right]$,

$$\mathrm{SNR} = 10 \log \frac{\sum_{n=0}^{\lfloor \pi/T \rfloor} \left| \mathbf{X}\left(\frac{2\pi k}{NT}\right) \right|^2}{\sum_{n=0}^{\lfloor \pi/T \rfloor} \left| \mathbf{X}\left(\frac{2\pi k}{NT}\right) - \widehat{\mathbf{X}}[n] \right|^2}, \tag{22}$$

where $\widehat{\mathbf{X}}$ is either the DFT sequence or the sequence that stems from the proposed approach. DFT values with an index larger than $N/2$ correspond to negative frequencies and this is the reason for excluding the frequency range $\left[\frac{\pi}{T}, \frac{2\pi}{T}\right]$ from the SNR calculations. To further demonstrate the performance of the proposed approach within the low frequency band, another experiment was carried out involving input signals that are band-limited to $\frac{\pi}{T}$. These signals are given by

$$\mathbf{x}_2(t) = \sum_{n=0}^{\lfloor \pi/T \rfloor} a[n] \cdot \cos(nt) + b[n] \cdot \sin(nt), \tag{23}$$

where the coefficients $\{a[n], b[n]\}_{n=0}^{\lfloor \pi/T \rfloor}$ are randomly chosen from a uniform distribution occupying the interval $[0, 1]$. Restricting $\mathbf{x}_2(t)$ to $\Omega = [-\pi, \pi]$, its Fourier transform is given by

$$\begin{aligned} \mathbf{X}_2(\omega) \quad = \quad & \sum_{n=0}^{\lfloor \pi/T \rfloor} \Big(a[n] + b[n]\Big) \frac{sin\left[\pi(\omega - n)\right]}{\omega - n} + \\ & + \Big(a[n] - b[n]\Big) \frac{sin\left[\pi(\omega + n)\right]}{\omega + n}. \end{aligned} \tag{24}$$

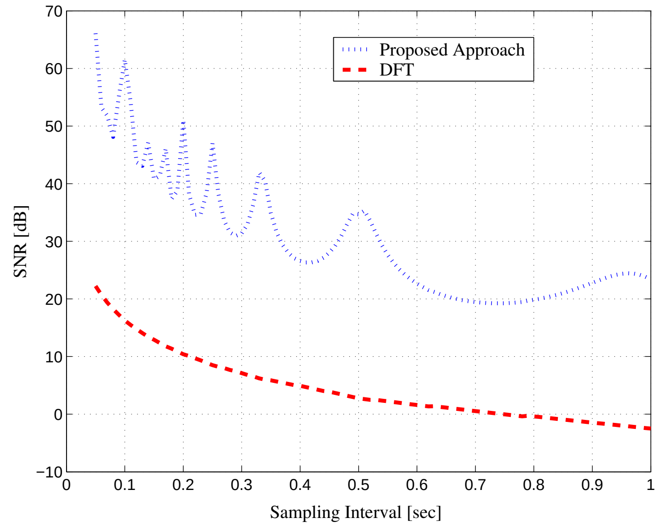

Figure 3: *Spectral estimation comparison. Considering the hat function, Shown are SNR values as a function of the sampling interval . These values were calculated based on frequencies lower than π/T, which is half of the sampling rate. The Sobolev order is $p = 1$.*

The experiment involved 100 realizations of $\mathbf{x}_2(t)$ and an SNR value was calculated for every realization by applying (22). The results are shown in Figure 4, for which the proposed approach achieves higher SNR values in most of the cases. Figure 5 depicts an experiment that involves polynomial B-spline modeling. In this experiment, a continuous-domain model was derived from the known samples and the Fourier transform values were calculated accordingly. The Sobolev order was $p = 2$ and the B-spline order was $L = 4$ (cubic).

Figure 6 depicts upper bounds on the approximation error of the proposed approach. Following (12), the term $\|P_{\mathcal{S}}\mathbf{x}\|^2 = c^T G^{-1} c$ converges to $\|x\|^2$ as the sampling step becomes shorter, while the term $\|\mathbf{w} - P_{\mathcal{S}}\mathbf{w}\|^2$ converges to zero. Also, the rate of decay is proportional to the Sobolev order p as shown in the Figure, too.

5. CONCLUSIONS

This work has introduced a new method for spectral estimation of continuous-domain signals given by their sampled version only. Based on Sobolev spaces, the proposed approach reduces the aliasing effects that are inherent to the DFT. Sobolev spaces were utilized for describing the ideal sampling process by means of an inner product operation and a similar interpretation was used for point-wise evaluation of the contiguous domain Fourier transform. This, in turn, allowed for a minimax approximation approach to be derived. Experimental results show that the proposed approach may provide a preferred alternative over the DFT and over polynomial modeling in cases where spectral analysis of sampled signals is required. Since Sobolev functions are dense in L_2, our conclusion is that the proposed approach could be instrumental in most spectral analysis tasks.

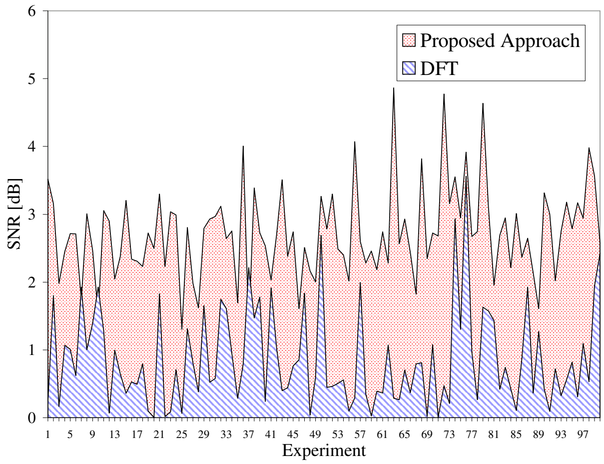

Figure 4: *Comparison of spectral estimation. Shown are 100 realizations of bandlimited signals given by (23). Every realization has undergone spectral analysis by means of the DFT and by means of the proposed approach. SNR values correspond to frequencies lower than π/T which is half of the sampling rate. The sampling interval is $T = 0.2$ and the Sobolev order is $p = 1$.*

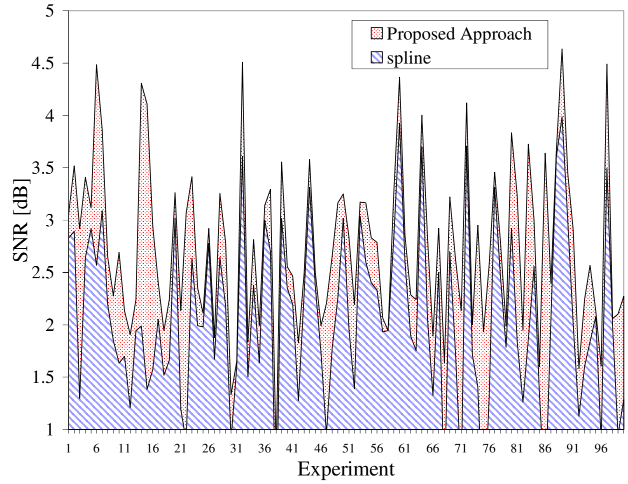

Figure 5: *Similar to Figure 4 while considering a polynomial B-spline model. The Sobolev order is $p = 2$ and the B-spline order is $L = 4$ (cubic). The proposed approach outperforms the polynomial model in all 100 realizations.*

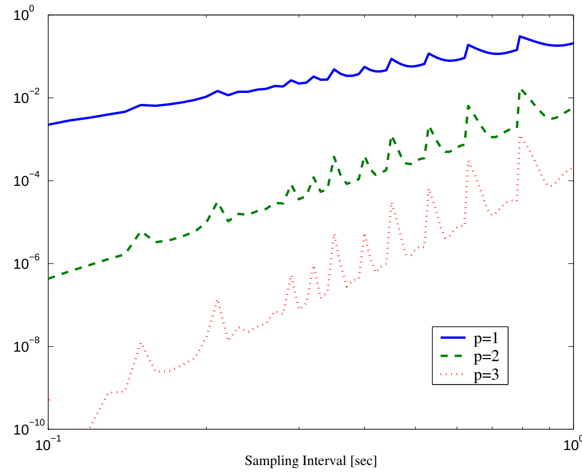

Figure 6: *Upper bound on the approximation error. Shown here is $\|\mathbf{w}_\omega - P_{\mathcal{S}}\mathbf{w}_\omega\|^2$ given by (12) for $\omega = 1\,[rad/sec]$.*

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