THE DYNAMICS OF IMAGE PROCESSING VIEWED AS DAMPED ELASTIC DEFORMATION

Vadim. Ratner, Yehoshua Y. Zeevi

Technion – Israel Institute of Technology,
Technion City, Haifa, 32000, Israel
vad@tx.technion.ac.il, zeevi@ee.technion.ac.il

ABSTRACT

Diffusion-type algorithms have been integrated in recent years successfully into the toolbox of image processing. We introduce a new more flexible and powerful family of parabolic-hyperbolic partial differential equations (PDEs) that somewhat resembles the structure of the parabolic diffusion equation, but incorporates the second order derivative in time. It is instructive to consider intuitively in this context the dynamics of image processing as the deformation of an 'elastic sheet'. Indeed, our parabolic-hyperbolic PDE models elastic deformation. This analogy between a well-known physical system and process on one hand, and the dynamics of an image processing scheme on the other hand, contributes interesting and important insight about images and their processing. We explore and demonstrate the capabilities and advantages afforded by the application of the proposed family of equations in image enhancement. The problem of computational complexity is addressed, and efficient numeric schemes are also presented.

Index Terms— Parabolic-hyperbolic equations, PDE image processing, image enhancement, denoising, image edge analysis

1. INTRODUCTION

Relations between pixels in an image can be considered as analogous to interaction between particles in a physical system. Correlation, for example, can be considered to be a measure of the strength of interaction, and one may elaborate a Hook's law of the force produced by the attraction of pixels. Likewise, operations encountered in image processing, such as sharpening or denoising, have counterparts among physical processes. Finding an adequate representation of an image as a physical entity may, therefore, yield new image processing techniques, that have already been thoroughly explored and understood in the context of physics. Several such physics-based models lend themselves to the application of partial differential equations (PDE) in image processing. One advantage of this approach is the possibility to implement image processing as a short-time evolutionary process by means of the dynamics of PDEs. Deep insight into the behavior of such processing schemes is then facilitated by research results obtained in both applied and pure mathematics.

One such approach that has led to very productive research and powerful algorithms, considers an image as a particle density map. Image processing schemes then become analogous to random particle motion, or diffusion. Diffusion-type PDEs have been successfully applied in recent years in enhancement and segmentation of images (see, for example, [1]-[4]).

The purpose of the present study is to further improve the capability of the PDE-based image processing, by using the Telegraph-Diffusion operator, introduced in this context in [11].

2. BACKGROUND

2.1. The Problem

Consider an image degraded by random Gaussian additive blurring noise. Improving image quality insofar as one image attribute is concerned results in a compromise regarding other image attributes. Most filtering-based denoising schemes yield simultaneous noise reduction and image blurring. Likewise, sharpening or deblurring of an image entails an undesired side effect of noise enhancement.

An important goal of image enhancing algorithms is to further improve the processes of deblurring and denoising. Solutions that achieve this goal are based on adaptive local control of the restoration process in accordance with image structure.

Approaches presented in this paper are inspired by processes that favor smooth surfaces over peaks, striving to achieve minimal surface area. Such phenomena are often encountered in nature (diffusion, elastic motion). While these approaches benefit from a profound mathematical basis, they are also highly intuitive since we deal with them in everyday life. Further, the physics of such phenomena has been deeply investigated and is well understood.

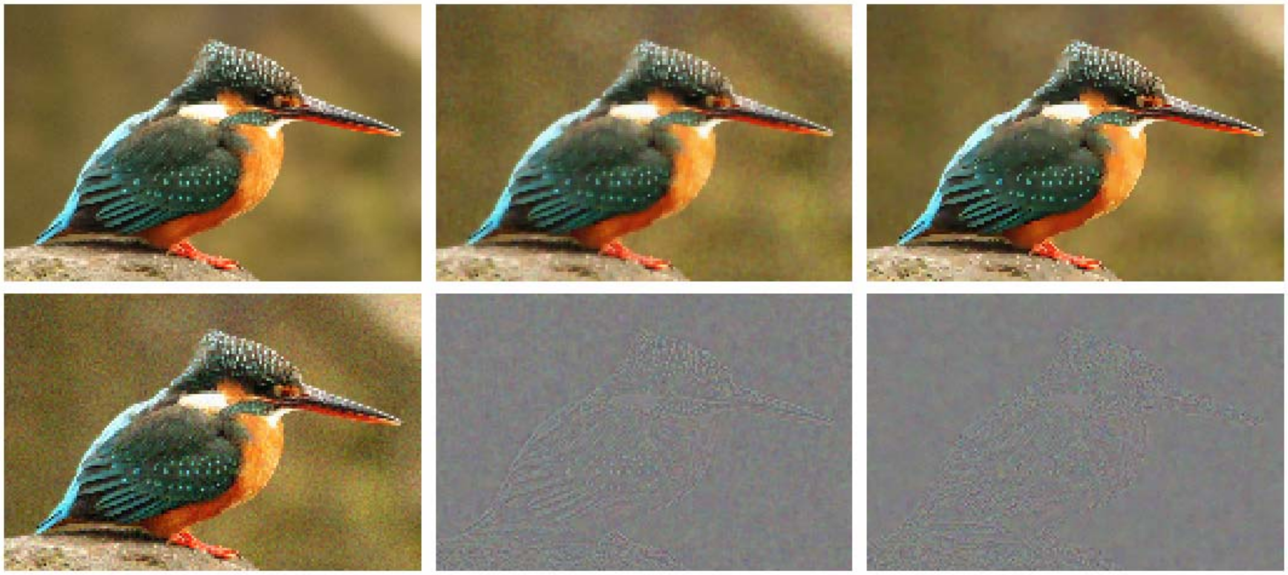

Figure 1: Leftmost column: original image, noisy image (PSNR 28.0dB). Middle column: PM-TeD denoising result and error (PSNR 30.0dB). Rightmost column: FAB-TeD denoising result and error (PSNR 30.7dB). The error image is the difference between the noiseless and the denoised images added to a uniform gray image (gray level 0.5) to improve visibility. FAB-denoising error image contains much less details than PM denoising error, in which the bird is still clearly visible.

2.2. Diffusion Processes

The non-linear diffusion denoising process, defined by (2.1), was introduced by Perona and Malik (PM) in [1].

$$-\nabla\cdot\left(k\nabla u\right)+u_t=0, \tag{2.1}$$

where u is the processed image and u_t, its first derivative in time. Setting the diffusion coefficient k to be large over smooth areas and small around edges allows anisotropic, content-dependant smoothing. The linear smoothing process can be described as a convolution with a Gaussian smoothing kernel.

Gilboa *et al.* have generalized the process, allowing negative and complex diffusion coefficient k (or time). This resulted in Forward-And-Backward (FAB) diffusion [3], which enables local sharpening, as well as smoothing. Complex diffusion, [4], further generalizes the operator by incorporating the imaginary part which approximates a second spatial derivative of an image. It also lends itself to the application of the Schroedinger potential which is instrumental in enhancement of texture [14].

These properties of the diffusion operator are useful in image denoising and enhancement, since they allow both smoothing (denoising) and sharpening of an image by adjusting the diffusion coefficient.

2.3.Damped Elastic Deformation Processes

Damped elastic deformation (DED) processes are also endowed with smoothing properties. These processes are represented by a damped wave equation (also known as the telegraphers' equation):

$$u_{tt}-\nabla\cdot\left(k\left(\nabla u\right)\nabla u\right)+cu_t=0, \tag{2.2}$$

where u_t and u_{tt} are respectively the first and second derivatives in time of u, k is the elasticity coefficient and c is the damping coefficient. The dynamics of (2.2) depicts the deformation of a thin elastic sheet, placed in a (liquid) damping environment. Elastic nature of the process encourages reduction of surface area, and therefore facilitates smoothing of singular structural anomalies, such as noise peaks. A discrete representation of an elastic sheet is a grid of particles connected by springs. Forces between the particles may be viewed as analogous to correlation between pixels. It is therefore possible to control the inter-pixel correlation by adjusting k. Non-zero damping, ensures energy loss and, therefore, convergence. Damping can also control the speed of the process – weaker damping results in more rapid evolution, as well as its nature – over-damped process resembles diffusion, while under-damped process exhibits wave-like characteristics.

This parabolic-hyperbolic equation is often encountered in various fields, such as description of random motion of particles ([10]), transmission of signals over telegraph wires (hence the terminology) and wave transmission to name a few. It has also been thoroughly investigated mathematically ([7], [9]). Equation 2.2 was first introduced in the context of image processing by Ratner and Zeevi in [11].

It is interesting to note that (2.2) converges to the diffusion eq. (2.1) after very long time [6], [7]. Given positive and bounded coefficients, DED (TeD) converges to a unique bounded solution ([9]). The interesting behavior in the context of image processing is, however, short-time

Noise stdev	Noise PSNR	DED PSNR	Diff. PSNR	DED iterations	Diff. iterations
0.2	27.6	51.2	45.4	6	8
0.3	21.1	46	41	15	20
0.4	17	37.3	35	50	70
0.5	13.5	25.3	24.7	250	350

Table 1: Denoising results for image consisting of 2 regions separated by an edge, with different levels of white Gaussian noise. The PSNR was computed near the edge and not over an entire image. Columns (left to right): noise STD, noisy image PSNR, DED denoised image PSNR, diffusion result PSNR, number of iterations of DED simulation, iterations of diffusion simulation. Maximal stable time steps were used (DED: 0.4, diffusion: 0.1).

evolution, i.e. the process that is stopped after time period long enough to reduce noise and produce other desirable effects, yet short enough to preserve meaningful information. It is also important to note that although (2.2) is a wave equation, in most cases addressed in the context of image processing, the wave-like nature is suppressed by over-damping (i.e. $4k < c^2$), to avoid unwanted artifacts. Allowing k to become negative around edges results in edge enhancement. This algorithm, which advances DED into negative time regime, is called Forward-and-Backward (FAB) DED. It is based on FAB-diffusion, proposed by Gilboa *et al.* in [7]. Although it does not necessarily converge to a steady-state solution, in practice it achieves significant improvement over the basic DED denoising scheme (Fig. 2).

Similarly to diffusion, linear 1D elastic deformation can be represented as convolution with kernel h_{td} (described in detail in [12]). An important advantage of h_{td} over Gaussian kernel is that its characteristics can be better controlled by the two coefficients k and c. It also better approximates an ideal ("box") lowpass, due to the higher-order nature (in time) of the equation. This should improve its performance in presence of high-frequency details (such as edges).

3. DED-BASED ALGORITHMS

Due to similarity between the diffusion and DED equations, various diffusion-based research results are applicable to DED-based image processing. Several such applications are presented in this section.

By defining spatially (and temporally) varying elasticity and damping coefficients, it is possible, as in diffusion, to locally control the degree of smoothing. In this paper we address primarily the case of varying elasticity coefficient, although initial experiments have shown that variation of damping coefficient can further improve the performance. To introduce our approach, we assume an image to be an assembly of smooth regions separated by edges. Intra-regional pixels are highly correlated, while inter-region correlation is weak. When represented as an elastic sheet, the process is applied to the noisy input image u_0. To remove noise and preserve edges, the sheet is made more elastic in smooth areas and more rigid around edges. The same k used in diffusion ([1], [3]) is applicable here:

$$u_{tt} - \nabla \cdot \left(\frac{\kappa}{\varepsilon^2 + |\nabla u|^2} \nabla u \right) + c u_t = 0 , \qquad (3.1)$$

where ε is a small constant and κ (positive constant) is the elasticity parameter. Thus, the intra-regional correlation is further increased, while inter-regional correlation remains low. An image that results from the application of the basic DED operator consists mostly of flat surfaces separated by sharp edges (Fig. 2); a cartoon-like representation of the original image. Variations on the theme are presented in [11].

Computational complexity of the algorithm is as in diffusion $O(nN)$, N being the number of input elements and n - the number of iterations. The strength of DED is in denoising fine details and edges (due to the fact that it is closer to an ideal lowpass). Experiments with images indicate that DED performs better than diffusion at lower noise levels (Table 1). This should be expected, since high noise levels degrade an image so badly that it becomes impossible to restore the finer details.

4. EFFICIENT NUMERIC SCHEMES

Discretization schemes and their convergence received little attention so far. Weickert et. al., in [8], presented one such scheme, which differs from the basic approach in that it improves stability. Further development of the scheme and its application to the DED method are presented below.

To simplify matters, we first discuss a one-dimensional case, where $u^{i,j}$ is the i-th element (spatial coordinate $x=i*h$) of vector $\underline{u}^j$ (input at time $t=j*\tau$). Basic discrete representation of (2.2) is as follows (τ and h being the temporal and spatial steps):

$$\frac{u^{x,t+1} - 2u^{x,t} + u^{x,t-1}}{\tau^2} + c^{x,t}\,\frac{u^{x,t+1} - u^{x,t}}{\tau} - \\ - \frac{k^{x,t}\left[u^{x+1,t} - u^{x,t}\right] - k^{x-1,t}\left[u^{x,t} - u^{x-1,t}\right]}{h^2} = 0 . \qquad (4.1)$$

It defines the following iterative, explicit (Forward Euler) update scheme of u:

$$\left(1+c^{x,t}\tau\right)u^{x,t+1}=\left(2+c\tau\right)u^{x,t}-u^{x,t-1}+ \\ +\tau^2\frac{k^{x,t}\left[u^{x+1,t}-u^{x,t}\right]-k^{x-1,t}\left[u^{x,t}-u^{x-1,t}\right]}{h^2}. \tag{4.2}$$

An advantage of Forward Euler scheme is that it is straightforward, and can follow the continuous process with any given accuracy by using small enough time steps. This is also the major drawback of the explicit scheme - it requires small time steps to converge to a stable solution, and is therefore very demanding computationally. The most stable, implicit solution (that uses vectors of time step *t+1* in the right-hand side of the equation) is immediate (i.e. single iteration). However, it requires the solution of N nonlinear equations, where N is the number of elements in u. An intermediate scheme closely related to Backward Euler (BE) discretization, was proposed in the context of image processing by Weickert et. al. in [8], resulting in the following discrete equation:

$$\frac{u^{x,t+1}-2u^{x,t}+u^{x,t-1}}{\tau^2}+c^{x,t}\frac{u^{x,t+1}-u^{x,t}}{\tau}- \\ -\frac{k^{x,t}\left[u^{x+1,t+1}-u^{x,t+1}\right]-k^{x-1,t}\left[u^{x,t+1}-u^{x-1,t+1}\right]}{h^2}=0. \tag{4.3}$$

This leads to the semi-implicit update scheme (for pixel dimension $h = 1$, unity matrix I):

$$\left(\left(1+c\tau\right)I-\tau^2A\right)\underline{u}^{j+1}= \\ =\left(\left(2+c\tau\right)I+\tau^2A\right)\underline{u}^{j}-\underline{u}^{j-1}, \tag{4.4}$$

where $\underline{u}^j$ is the input vector at time $j*\tau$, and A is a tri-diagonal matrix, which allows efficient inversion of $\left[\left(1+c\tau\right)I+\tau^2A\right]$ by using Thomas algorithm (TDMA). Although it requires additional calculations, the scheme retains linear complexity, and is much more stable then the standard approach, allowing larger time step and fewer iterations (reduction by order of magnitude).

Other time discretization schemes are also applicable, e.g Crank-Nicolson (CN), or Richtmeyer-Morton (RM). Experimental comparison shows more stable performance of CN then that of BE scheme for large time-steps and fewer iterations (Fig. 2, 3). It is interesting to note that the improvement of CN over BE is more significant in diffusion then in DED, which may imply that DED is inherently more stable then diffusion. This is further supported by the fact that a numeric stability requirement for linear discrete diffusion equation is $\tau < h^2/2k$, while that of linear discrete DED is $\tau \le h/\sqrt{k}$ [13] (k being the diffusivity in diffusion and elasticity in DED), i.e. DED allows larger time step.

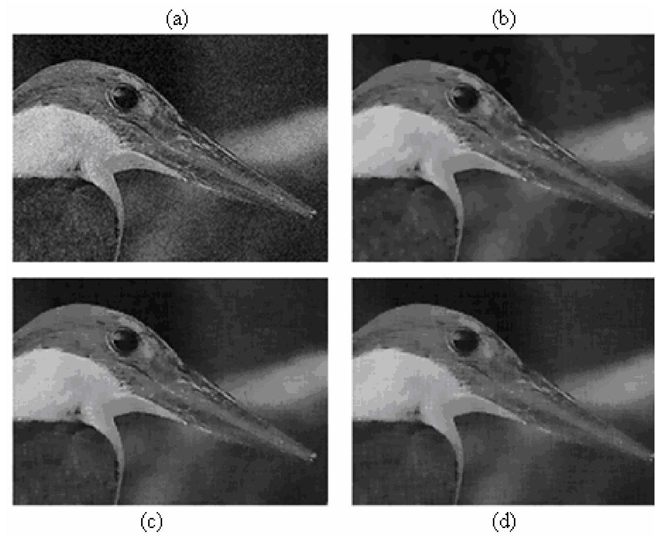

Figure 2: DED denoising. (a) – noisy image (PSNR 28dB). (b) – explicit DED denoising, 20 iterations (PSNR 32.3). (c) – semi-implicit DED denoising, 2 iterations (PSNR 32.2). (d) – semi-implicit DED, 1 iteration (PSNR 30.6)

Generalization of the algorithm to higher dimensions is not trivial. The naïve approach of writing the Backward Euler scheme for 2D results in non-zero values outside the main three diagonals of A, requiring $O(n^2)$ operations. Solution proposed by Weickert *et al.* [8] performs the update separately along each axis, resulting in a good approximation of the original scheme. The 2D input image is raster-scanned twice per iteration – along x and y axes, and the two resulting vectors are updated.

5. CONCLUSIONS

Experimental results and their analysis indicate that the proposed DED operator achieves better results than those obtained previously by means of diffusion-type operators. In addition, the new operator offers greater flexibility by incorporating a controllable coefficient.

Different numeric solution schemes are the subject of ongoing research. These include semi-implicit schemes (Backward Euler, Crank-Nicolson, etc.), and multigrid methods. Initial results show that semi-implicit schemes are applicable to DED, and that further investigation in the context of DED image processing contributes also to enhancement of diffusion-based schemes. Parallelization of the algorithm seems to be a natural step towards on-line implementation in video processing and enhancement.

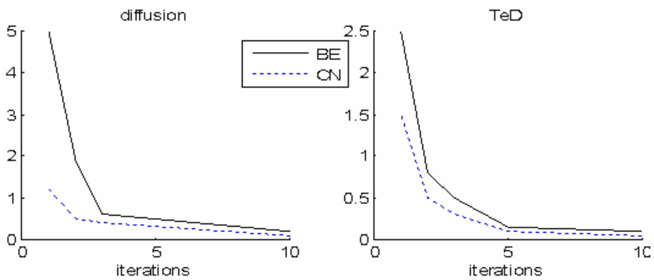

Figure 3: Performance of Backward Euler (BE) and Crank Nicolson (CN) discretization schemes, measured as the difference between PSNR achieved by explicit scheme (60 iterations, close approximation of the continuous process) and the respective semi-implicit scheme. The measurements were taken for several runs of the denoising algorithm, each with a different number of iterations (less iterations – larger time steps). PSNR of the noisy image was 28.0dB, PSNR of explicit diffusion denoising was 30.7dB and of explicit DED denoising 31.0dB.

Since PSNR is not always useful as a measure of an image quality (e.g. in image enhancement applications), some adequate quality metrics remains to be determined. Based on these, it will become possible to develop an automatic parameter adjustment scheme. The proposed operator can be adjusted and implemented in processing of higher dimensional signals such as video or 3D models.

Improved texture preservation schemes, incorporating external force into the equation (similar to [14]) are currently explored.

6. ACKNOWLEDGEMENT

Research supported in part by the Ollendorff Minerva Center of the Technion, and by Philips Consumer Lifestyle.

7. REFERENCES

1. Perona, P., Malik, J.: Scale-Space and Edge Detection Using Anisotropic Diffusion. IEEE Trans. Pattern Analysis and Machine Intelligence, vol. 12, no. 7, pp. 629-639, July 1990.

2. Alvarez, L., Lions, P.L., Morel, J.M.: Image Selective Smoothing and Edge Detection by Nonlinear Diffusion. SIAM Journal on Num. Analysis, vol. 29, no. 3, pp. 845-866, June 1992.

3. Gilboa, G., Sochen, N., Zeevi, Y.Y.: Forward-and-Backward Diffusion Processes for Adaptive Image Enhancement and Denoising. IEEE Trans. on Image Proc., vol. 11, no. 7 , July 2002.

4. Gilboa, G., Sochen, N., Zeevi, Y.Y.: Image Enhancement and Denoising by Complex Diffusion Process. IEEE Trans. Pattern Analysis and Machine Intelligence, vol. 26, no. 8, pp. 1020-1036, August 2004.

5. Sochen, N., Zeevi, Y.Y.:, Images as Manifolds Embedded in a Spatial-Feature Non-Euclidian Space. EE-Technion report No. 1181, November 1998.

6. Zauderer, E., Partial Differential Equations of Applied Mathematics. Wiley, New York 1998.

7. Gallay, Th., Raugel, G.: Scaling Variables and Asymptotic Expansions in Damped Wave Equations. Journal of differential equations, vol. 150, no. 1, pp. 42-97, 1998.

8. Weickert, J., ter Haar Romeny, B. M., Viergever, M. A.: Efficient and Reliable Schemes for Nonlinear Diffusion Filtering. IEEE Trans on Image Processing, vol. 7, No. 3, March 1998.

9. Nakao, M.: Decay and global existence for nonlinear wave equations with localized dissipations in general exterior domains. Operator Theory, Advances and Applications, Vol.159, pp.213-299, 2007.

10. Dunbar, S. R.: A Branching Random Evolution and a Nonlinear Hyperbolic Equation. SIAM Journal on Applied Mathematics, Vol. 48, No. 6. (Dec., 1988), pp. 1510-1526.

11. Ratner, V., Zeevi, Y. Y.: Telegraph-Diffusion Operator for Image Enhancement. ICIP 2007 Proceedings, September 2007.

12. Ratner, V., Zeevi, Y. Y.: Image representation and enhancement on elastic manifolds. PRIME 2008 Proceedings, June 2008.

13. Ames, W. F., Numerical Methods for Partial Differential Equations, 2nd ed. Academic Press, Orlando, 1977.

14. Honigman, O., Zeevi, Y. Y. : Enhancement of Textured Images Using Complex Diffusion Incorporating Schroedinger's Potential. ICASSP 2006 Proceedings, June 2006.