CRAMER-RAO BOUND FOR TIME-DELAY ESTIMATION IN THE FREQUENCY DOMAIN

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ABSTRACT

In this paper a measure of performance of time delay estimators arising from a frequency model of the received signal is presented. The aim is to derive theoretical analysis of the Cramer-Rao bound (CRB) for time delay estimation under realistic wireless channel propagation conditions. The derivation of the CRB is considered from the frequency domain signal model yielding a closed-form expression without imposing any constraints on the different propagation paths.

1. INTRODUCTION

Time delay estimation is of paramount importance in communications systems. Range finding in RADAR, synchronization, channel estimation or geolocation are just some of the possible applications. Unfortunately, characteristic phenomena of wireless channels such as multipath propagation impair the quality of the estimation. Therefore, it is desirable to possess a tool for the comparison of the performance of the designed estimator with some optimality criterion. This tool is provided by the CRB, which is a lower bound on the variance of any unbiased estimator. The derivation of this bound relies on the model assumed for the received signal. Moreover, it depends on the problem definition, that is, on the parameters to estimate and the nuisance parameters. The simplest model consists of a delayed version of a known deterministic signal corrupted by an additive white Gaussian noise. The performance in such a model is well understood, see for instance [1], the higher the SNR (Signal to Noise Ratio) or the larger the signal bandwidth, the lower the CRB.

In [2] a model of the received signal, in the temporal domain, consisting of multiple overlapping echoes of a known waveform was considered. Given that model, a CRB for the time delays was presented assuming that the amplitudes of the different channel paths were known. That work showed that the bound for the variance of the *i*-th time delay estimate depends on the relative delays of the other paths. More complex models involving several time delays related to different propagation paths have been considered in the context of synchronization in DS-CDMA in [3]. Therein, the CRB for the vector of time delays was derived taking into account a temporal model for the received signal and considering the noise variance and the path gains as nuisance parameters.

More recently a plethora of works have considered a space-time signal model for the problem of joint estimation of time-delays, bearings and other multipath parameters. For instance, in [4] the CRB for the time-delays, the directions of arrival (DOA) and the path amplitudes is derived in the context of asynchronous DS-CDMA considering the noise variance as a nuisance parameter. In [5], the CRB is obtained for the same parameters, but in a more general context. Therein, the assumption is that multiple replicas of a known narrowband signal impinge on an array of sensors.
The problem addressed in this paper is the performance of time delay estimation when multiple replicas of a known signal are received. However, unlike [2] herein the amplitudes related to each channel path are considered to be unknown, they are nuisance parameters. Moreover, in contrast to the references stated above we consider a signal model in the frequency domain which led the authors to derive a set of time delay estimators [6-8].

The aim of this paper is twofold. First, to present an alternative approach to obtain the CRB of the time delays based on the assumption of a signal model in the frequency domain. Second, to show that this approach leads to a more compact and simple expression of the CRB than the one using a time domain model.

The paper is organized as follows. In section 2 the signal model and the problem statement are defined. The CRB for that model is obtained in section 3. Then, simulation results

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are presented in section 4. Finally, section 5 contains some concluding remarks.

2. SIGNAL MODEL AND PROBLEM STATEMENT

In this section a signal model in the frequency domain will be presented for the problem of time delay estimation. The analogy between this model and the problems of spatial and spectral analysis will be stressed as well. Afterwards, given that signal model the problem statement will be introduced.

2.1 Signal model

The channel model considered is given by the general expression for the multipath fading propagation channel as follows,

$$h(t) = \sum_{i=0}^{L-1} h_i \delta(t - \tau_i)$$

with no loss of generality we assume $\tau_0 < \tau_1 < ... < \tau_{L-1}$, where L is the number of propagation paths.
Then, the n-th observation of the received signal consists of multiple superimposed delayed and attenuated replicas of a known waveform $g(t)$ embedded in noise:

$$y(t;n) = \sum_{i=0}^{L-1} h_i(n) g(t - \tau_i) + v(t;n), \quad n = 1, \ldots, N \tag{1}$$

We assume the received pulse from each i-th path exhibits the same waveform but experiences a different fading coefficient, $h_i(n)$ and time delay τ_i. The amplitudes h_i are considered deterministic but unknown. The additive noise $v(t;n) \sim N\left(0, \sigma^2\right)$ is modelled as Gaussian circularly symmetric. Finally, the number of observations of the received signal is given by N.

The received signal in the frequency domain is transformed into a weighted sum of complex exponentials:

$$Y(w;n) = \sum_{i=0}^{L-1} h_i(n) G\left(w\right) e^{-jw\tau_i} + V(w;n), \quad n = 1, \ldots, N \tag{2}$$

where $Y(w;n)$ is the noisy frequency domain observed signal, $G\left(w\right)$ is the transmitted signal in the frequency domain and $V(w;n)$ is the additive noise.
Sampling the signal in (2) at $w_m = w_0 m$ for $m = 0, 1, ..., M-1$ and $w_0 = 2\pi/M$ and rearranging the frequency domain samples $Y\left[m, n\right]$ into the vector $\mathbf{Y}_n \in \mathbb{C}^{M\times 1}$:

$$\mathbf{Y}_n = \mathbf{G}\mathbf{E}_\tau \mathbf{h}(n) + \mathbf{V}_n \tag{3}$$

where the matrix $\mathbf{G} \in \mathbb{C}^{\mathbf{M\times M}}$ is a diagonal matrix which components are the frequency samples of $G(w)$ and the matrix $\mathbf{E}_\tau \in \mathbb{C}^{\mathbf{M\times L}}$ contains the delay-signature vectors (harmonic components) associated to each arriving delayed signal,

$$\mathbf{E}_\tau = \left[\mathbf{e}_{\tau_0} \quad \mathbf{e}_{\tau_1} \quad \cdots \quad \mathbf{e}_{\tau_{L-1}}\right]$$
$$\mathbf{e}_{\tau_i} = \left[1 \quad e^{-j\omega_0\tau_i} \quad \cdots \quad e^{-j(M-1)\omega_0\tau_i}\right]^T$$

The channel fading coefficients are arranged in the vector $\mathbf{h}(n) = \left[h_0(n) \quad h_1(n) \quad \ldots \quad h_{L-1}(n)\right]^T \in \mathbb{C}^{L\times 1}$, and the noise samples in vector $\mathbf{V}_n \in \mathbb{C}^{M\times 1}$.

2.2 Problem statement

At this point it is interesting to stress the analogy between the model in (3), obtained in the framework of time delay estimation, and a more general model that arises in the context of spatial and spectral analysis. The relation between both models paves the way to obtain the CRB for the time delays in (3). According to [9], the model for the problem of finding the directions of arrival of multiple plain waves impinging on an array of sensors, and the problem of finding the frequencies of multiple superimposed exponentials in noise, in spectral analysis, can be expressed as:

$$\mathbf{y}(n) = \mathbf{A}(\boldsymbol{\theta})\mathbf{x}(n) + \mathbf{e}(n), \quad n = 1, 2, \ldots, N \tag{5}$$

being $\mathbf{y}(n) \in \mathbb{C}^{M x1}$ the received signal and $\mathbf{x}(n) \in \mathbb{C}^{n x1}$ a vector of complex amplitudes. Besides, N is the number of snapshots or realizations of the experiment and $\mathbf{e}(n)$ is a complex Gaussian additive noise whose statistical properties will be examined in section 3. With regard to $\mathbf{A}(\boldsymbol{\theta})$, it is a matrix with the following structure:

$$\begin{aligned} &\mathbf{A}(\boldsymbol{\theta}) = \left[\mathbf{a}(\omega_o) \quad \mathbf{a}(\omega_2) \quad \ldots \quad \mathbf{a}(\omega_{L-1})\right] \\ &\boldsymbol{\theta} = \left[\omega_0 \quad \omega_1 \quad \ldots \quad \omega_{L-1}\right]^T \end{aligned} \tag{6}$$

In the case of spectral analysis $\mathbf{a}(\omega_i)$ denotes a complex sine wave with angular frequency ω_i whose dimension depends on the number of samples. In an array signal processing context $\mathbf{a}(\omega_i)$ is the spatial signature and its dimension depends on the number of sensors. In this case, ω_i denotes the so called spatial frequency, which is a function of the direction of arrival of the wavefront impinging on the array. Clearly, that function depends on the specific geometry of the problem, Note that when the problem addressed in the expression (5) is finding the angles of arrival, the standard narrowband assumption is made. That is, it is assumed that the time required for the signal wavefronts to propagate across the array is much smaller than the inverse of the signals' bandwidth.

Next, in expression (7), the equivalences between the signal model assumed in this paper, (3), and the model in (5) are described. These relations will be useful in the next section in order to obtain the CRB for the time delays.

$$\begin{cases} \mathbf{A}(\boldsymbol{\theta}) \equiv \mathbf{G}\mathbf{E}_\tau \\ \mathbf{x}(n) \equiv \mathbf{h}(n) \\ \mathbf{e}(n) \equiv \mathbf{V_n} \\ \boldsymbol{\theta} \equiv \left[\tau_0 \quad \ldots \quad \tau_{L-1}\right]^T \end{cases} \tag{7}$$

In the design of an estimator of the time delays several questions arise. Specifically how close it is to the true parameter value and if there exists a better estimator. That is we are interested in assessing the performance.

Let us assume that the design is based on the classical estimation theory, that is, no prior statistical information about the parameters is available and the aim is to design an unbiased estimator. In this case, a measure of the performance can be based on the CRB.

Therefore the problem can be stated as follows. Given the signal model (3), obtain the CRB for the vector of parameters $\boldsymbol{\theta}$ defined in (7).

3. CRAMER RAO LOWER BOUND

As it is well known, the CRB is a lower bound on the covariance of any unbiased estimator, [1]. Specifically, in the problem at hand this can be summarized as follows:

$$\operatorname{cov}\left(\hat{\boldsymbol{\theta}}\right) \geq CRB\left(\boldsymbol{\theta}\right) \tag{8}$$

Where $\operatorname{cov}(\cdot)$ denotes the covariance operator and $\hat{\boldsymbol{\theta}}$ is an unbiased estimate of $\boldsymbol{\theta}$.

At this point it is interesting to mention several remarks about the expression (8). First, note that the signal model has the presence of nuisance parameters $\boldsymbol{\eta}$ that are not of interest but influence $CRB(\boldsymbol{\theta})$:

$$\begin{aligned} &\boldsymbol{\psi} = \begin{bmatrix} \boldsymbol{\theta}^T & \boldsymbol{\eta}^T \end{bmatrix}^T \\ &\left[CRB(\boldsymbol{\theta})\right]_{ij} = \left[\mathbf{I}^{-1}(\boldsymbol{\psi})\right]_{ij} \quad , i \in \{1,2,\ldots,L\} \;\; j \in \{1,2,\ldots,L\} \end{aligned} \tag{9}$$

Where $\mathbf{I}(\boldsymbol{\psi})$ denotes the Fisher information matrix of vector $\boldsymbol{\psi}$.

The second remark is that depending on the application $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ can be slightly modified. In the most general case we could formulate the estimation problem related to the signal model in (3) as:

$$\begin{aligned} &\boldsymbol{\theta} \equiv \begin{bmatrix} \tau_0 & \cdots & \tau_p \end{bmatrix}^T \\ &\boldsymbol{\eta} \equiv \begin{bmatrix} \tau_{p+1} & \cdots & \tau_{L-1} & \mathbf{h}^T(n) & \sigma^2 \end{bmatrix}^T, 0 \leq p \leq L-1 \\ &\left[CRB(\boldsymbol{\theta})\right]_{ij} = \left[\mathbf{I}^{-1}(\boldsymbol{\psi})\right]_{ij}, i \in \{1,2,\ldots,p\} \;\; j \in \{1,2,\ldots,p\} \end{aligned} \tag{10}$$

Depending on the application (10) is particularized. As an example, in wireless location the aim is to obtain an estimate of the first time delay and as a consequence $p = 0$. In other applications, such as channel estimation, the designer is interested in all the time delays, therefore $p = L-1$.

Next, a closed-form expression for $CRB(\boldsymbol{\theta})$ will be obtained. First, the next assumptions are made:

1) $M > L$, that is, the number of samples in the frequency domain is greater than the number of channel propagation paths. Moreover, the set of exponential vectors $\left\{\mathbf{e}_{\tau_i}\right\}_{i=0}^{L-1}$ containing the time delays in its argument are linearly independent. This is a valid assumption since $\mathbf{E}_\tau$ is a Vandermonde matrix.

2) The noise vector $\mathbf{V}_n$ in (3) is assumed to be zero mean $E\left[\mathbf{V}_n\right] = 0$, and Gaussian distributed $E\left[\mathbf{V}_n\mathbf{V}_n^H\right] = \sigma^2\mathbf{I}$.

3) The noise is uncorrelated between observations $E\left[\mathbf{V}_n\mathbf{V}_l^H\right] = 0, \quad \forall n \neq l$.

where $E(\cdot)$ denotes the expectation operator.

Taking into account these assumptions, using the theorem 4.1 of [9] and bearing in mind the relations described in (7), the CRB for time delay estimation in the frequency domain $CRB(\boldsymbol{\theta})$ is:

$$CRB(\boldsymbol{\theta}) = \frac{\sigma^2}{2}\left\{\sum_{n=1}^{N} \mathbb{R}\left\{\mathbf{H}^H\mathbf{D}_{\mathrm{E}}^H\mathbf{G}^H\left\{\mathbf{I} - \boldsymbol{\Omega}\left(\boldsymbol{\Omega}^H\boldsymbol{\Omega}\right)^{-1}\boldsymbol{\Omega}^H\right\}\mathbf{G}\mathbf{D}_{\mathrm{E}}\mathbf{H}\right\}\right\}^{-1} \tag{12}$$

Where $\boldsymbol{\Omega} = \mathbf{G}\mathbf{E}_\tau$, $\mathbf{H}$ is a diagonal matrix containing the unknown channel fading coefficients,

$$\mathbf{H} = \begin{bmatrix} h_0(n) & & 0 \\ & \ddots & \\ 0 & & h_{L-1}(n) \end{bmatrix}$$

$\mathbb{R}\{\cdot\}$ denotes the real part operator and

$$\mathbf{D}_{\mathrm{E}} = \begin{bmatrix} \dfrac{\partial \mathbf{e}_0}{\partial \tau_0} & \cdots & \dfrac{\partial \mathbf{e}_{\tau_{L-1}}}{\partial \tau_{L-1}} \end{bmatrix} = \boldsymbol{\Phi}\mathbf{E}_\tau, \; \left[\boldsymbol{\Phi}\right]_{k,l} = \begin{cases} -j\dfrac{2\pi(k-1)}{M} & k = l \\ 0 & otherwise \end{cases}$$

Regarding expression (12) several remarks are of interest. On the one hand, note that the dependence of the CRB on the noise variance is linear. On the other hand, according to theorem 4.2 of [9], increasing the number of snapshots or the number of frequency samples, the CRB decreases, which seems an intuitive result. Furthermore, the amplitude related to each channel path and the shape of the known waveform seem to have a definite influence on the CRB related to each time delay. However, the most important remark is that the expression obtained in (12) is more compact and easier than the one obtained when considering a temporal signal model. The reason is that using the latter approach, the CRB depends on the derivative of the pulse shape $g(\tau)$ respect to time, see expressions (5)-(7) of reference [2] or (17)-(23) of [10]. On the other hand, in the frequency domain approach presented herein the derivative affects the matrix of exponentials $\mathbf{E}_\tau$ which yields to an easy and compact closed-form expression.

Even more interesting is the result obtained when considering a sufficiently large number of snapshots because expression (12) is simplified. Thus, using the theorem 4.3 of [9] and the relations described in (7), the CRB for the time delays takes the form:

$$CRB(\boldsymbol{\theta}, N\uparrow) = \frac{\sigma^2}{2N}\cdot\left\{\mathbb{R}\left[\left\{\mathbf{D}_{\mathrm{E}}^H\mathbf{G}^H\left\{\mathbf{I} - \boldsymbol{\Omega}\left(\boldsymbol{\Omega}^H\boldsymbol{\Omega}\right)^{-1}\boldsymbol{\Omega}^H\right\}\mathbf{G}\mathbf{D}_{\mathrm{E}}\right\}\odot\mathbf{P}^H\right]\right\}^{-1} \tag{13}$$

Where $N\uparrow$ denotes a large number of snapshots or available realizations of the stochastic process that describes our signal model in (3), $\odot$ denotes the Hadamard or element-wise product and $\mathbf{P} = \dfrac{1}{N}\sum_{n=1}^{N}\mathbf{h}(n)\mathbf{h}(n)^H$.

4. SIMULATION RESULTS

In this section numerical results that study the asymptotic CRB of the time delays obtained in (13) will be presented. The application under consideration is time delay estimation in the context of wireless location for cellular telephony systems based on DS-CDMA. Therefore, the parameter of interest is now the first time delay of the model (3). Hereafter, the simulation environment will be described.

It is assumed that a mobile station traces the trajectory described in Figure 1 and that the received signal model at the base station can be described properly by equation (3). The known waveform $g(t)$ in (1) will be assumed to be a raised cosine with a roll-off factor of 0.22. At every trajectory point the number of rays is generated according to a Poisson variable,

$$\text{Prob}(L) = \bar{L}^{L}\,\frac{e^{-\bar{L}}}{L!}$$

The power delay profile (the power associated to each ray) is generated according to [11] using a one side exponential decaying function:

$$\text{PDP}(\tau) = e^{-\frac{\tau}{\sigma_D}}u(\tau) \qquad (14)$$

being σ_D the delay spread and $u(\tau)$ the Heaviside step function. The probability density funciton of the time delays related to each channel path is also an exponential decaying distribution, according to [11]:

$$f(\tau) = e^{-\frac{\tau}{\tilde{\sigma}_D}}u(\tau),\ \ \tilde{\sigma}_D = \alpha\sigma_D \qquad (15)$$

The specific value of α depends on the base station antenna position, for instance for a high antenna position $\alpha = 1.17$. Moreover, for each point of the trajectory described in figure 1, realistic values of the delay spread were generated according to the statistical models described in [12]. Their range of variation was between 0.131 μs and 0.509 μs with a mean of 0.255 μs and a variance of $4.1{\cdot}10^{-15}$ s. Furthermore note that the nearer the trajectory point the lower was the delay spread.

The CRB for the time delay of the first arrival, given by expression (13), has been studied in figures 2 and 3. More specifically, its dependence on the SNR, the mean number of rays and the number of samples has been assessed. Actually, the CRB in those figures is evaluated in terms of the range which is directly related with the time delay.

For each value of SNR the curves of figures 2 and 3 were obtained by averaging the CRB obtained at each point of the trajectory described in figure 1. In figure 2 we can observe the dependence of the square root of the CRB on the SNR and the sampling rate when the mean number of rays is 5, whereas in figure 3 the same result is presented when the mean number of rays is 10. Observing figures 2 and 3 we can conclude that given a high enough sampling rate, or number of frequency samples, the performance in time delay estimation do not improve. This behaviour seems logical, due to the discrete nature of the model in (3), increasing the sampling rate we reduce the implicit error in the time delay estimate. However, the specific shape of the waveform limits the attainable accuracy in the estimation. We can also infer that the mean number of rays is important in the low to medium SNR range.

5. CONCLUSIONS

In this paper the performance, in terms of CRB, of any unbiased estimator of the time delays associated to multiple echoes of a known waveform has been studied. With this aim, first a model of the received signal in the frequency domain was considered for the time delay estimation problem. Then, an analogy between that model and a more general model, arising in the context of spectral analysis or spatial analysis, was shown. Finally, that analogy was the key issue to obtain a closed-form expression for the CRB of the time delays.

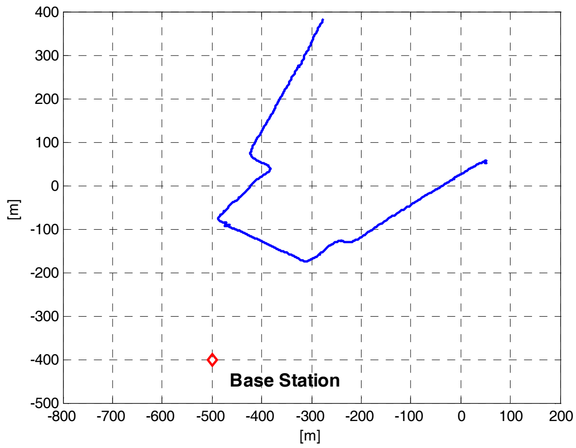

Figure 1. Trajectory considered in the simulations.

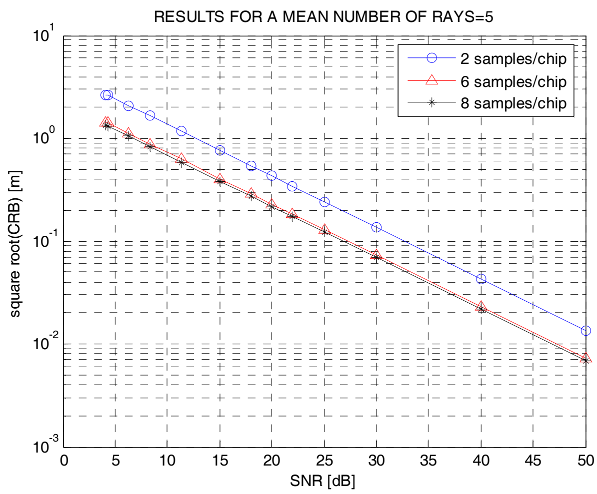

Figure 2. Dependence of the square root of the CRB of the first time delay on the SNR and the sampling rate when the mean number of rays is 5.

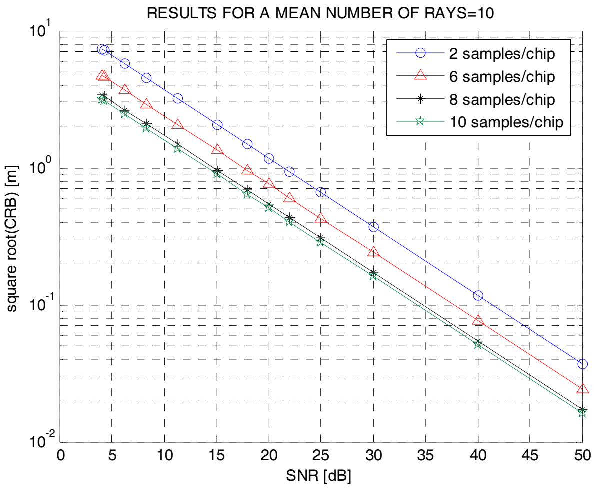

Figure 3. Dependence of the square root of the CRB of the first time delay on the SNR and the sampling rate when the mean number of rays is 10.

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