The Maximum Squared Correlation, Total Asymptotic Efficiency, and Sum Capacity of Minimum Total-Squared-Correlation Quaternary Signature Sets*

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Abstract

In this paper, we derive closed-form expressions for the maximum squared correlation (MSC), total asymptotic efficiency (TAE), and sum capacity ($C_{\rm sum}$) of minimum total squared correlation (TSC) quaternary signature sets. While TSC, MSC, TAE, and $C_{\rm sum}$ are equivalent optimization metrics over the real/complex field, our developments show that such equivalence does not hold, in general, over the quaternary field. We establish conditions on the number of signatures and signature length under which simultaneous optimization can or cannot be possible.

1. Introduction

In code-division multiplexing (CDM) systems, individual users/signals use distinct signatures (spreading codes) to access a common, in time and frequency, communication channel. In conjunction with channel and receiver design specifics, the overall system performance is determined by the selection of the user signature set. Signature set metrics of interest include the total squared correlation (TSC) [1]-[16], maximum squared correlation (MSC) [1] [2], total asymptotic efficiency (TAE) [21], and sum capacity $C_{\rm sum}$ [22]. We recall the definitions of these metrics below.

If $\mathcal{S} \triangleq [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K]$, $\mathbf{s}_k \in \mathbb{C}^L$, $\|\mathbf{s}_k\| = 1$, $k = 1, 2, \ldots, K$, is an $L \times K$ matrix that represents a set of K normalized (complex, in general) signatures of length (processing gain) L, then

(*i*) TSC of $\mathcal{S}$ is the sum of the squared magnitudes of all inner products between signatures

$$\mathrm{TSC}(\mathcal{S}) \triangleq \sum_{m=1}^{K}\sum_{n=1}^{K} \left|\mathbf{s}_m^H \mathbf{s}_n\right|^2; \tag{1}$$

(*ii*) MSC of $\mathcal{S}$ is the maximum squared magnitude among all inner products between distinct signatures

$$\mathrm{MSC}(\mathcal{S}) = \max_{m \neq n} \left|\mathbf{s}_m^H \mathbf{s}_n\right|^2; \tag{2}$$

(*iii*) TAE of $\mathcal{S}$ is equal to the determinant of the signature cross correlation matrix $\mathcal{S}^H\mathcal{S}$

$$\mathrm{TAE}(\mathcal{S}) \triangleq \left|\mathcal{S}^H\mathcal{S}\right|; \tag{3}$$

(*iv*) the sum capacity of $\mathcal{S}$ is defined as the maximum possible sum of user transmission rates with reliable reception and for a common additive white Gaussian noise (AWGN) channel is given by

$$C_{\rm sum} \triangleq \log_2\left|\mathbf{I}_L + \gamma\mathcal{S}\mathcal{S}^H\right| = \log_2\left|\mathbf{I}_K + \gamma\mathcal{S}^H\mathcal{S}\right| \tag{4}$$

where γ is the received signal-to-noise-ratio (SNR) of each user signal and $\mathbf{I}_L$, $\mathbf{I}_K$ are the size-L and size-K identity matrices.

For real/complex-valued signature sets ($\mathcal{S} \in \mathbb{C}^{L\times K}$or $\mathcal{S} \in \mathbb{R}^{L\times K}$), TSC is bounded from below by [1]-[3]

$$\mathrm{TSC}(\mathcal{S}) \geq \frac{KM}{L} \tag{5}$$

where $M = \max\{K, L\}$. The bound in (5) is called the "Welch bound" and the signature sets that satisfy (5) with equality are called Welch-bound-equality (WBE) sets. While for real/complex-valued signature sets the Welch bound is always achievable [4]-[12], this is not the case in general for finite-alphabet signatures. Tight bounds for the TSC of binary (alphabet $\{\pm 1\}$) signature sets for all lengths L and set sizes K together with optimal set designs for (almost) all K and L values were derived in [13]-[15]. The sum capacity, total asymptotic efficiency, and maximum squared correlation of the minimum-TSC optimal binary sets were found in [16]-[17]. Minimum-TSC and other digital sequence sets were studied in [18]-[20].

Recently, to gain insight into the problem of selecting an appropriate alphabet size for code-division multiplexing sequences, we considered the quaternary (or quadriphase or 4-phase) alphabet $\{\pm 1, \pm j\}$, $j = \sqrt{-1}$. In [23], we derived new bounds on the TSC of any quaternary signature matrix $\mathcal{S}_Q = [\mathbf{s}_1,\, \mathbf{s}_2,\, \ldots,\, \mathbf{s}_K] \in \frac{1}{\sqrt{L}}\{\pm 1, \pm j\}^{L\times K}$, for all possible K and L values, (the subscript "Q" in $\mathcal{S}_Q$ identifies a quaternary signature set). In [23] we also designed minimum-TSC optimal quaternary sets that meet the new bounds for all K and L values. The new bounds for overloaded and underloaded systems are summarized in Table I and Table II, respectively.

In this present work, we focus exclusively on minimum-TSC quaternary sets, i.e. quaternary signature sets that meet the lower bounds in Table I and Table II. For all K and L with $K \leq L$ (underloaded systems), we derive analytic

*This work was supported in part by the Air Force Research Laboratory under Grant FA8750-08-1-0063.
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TABLE I
UNDERLOADED QUATERNARY SEQUENCE SETS ($K \leq L$)

Length	Number of Sequences	Lower Bound on TSC
$L \equiv 0(\text{mod}2)$	Any K	K
$L \equiv 1(\text{mod}2)$	Any K	$K + \frac{K(K-1)}{L^2}$

TABLE II
OVERLOADED QUATERNARY SEQUENCE SETS ($K \geq L$)

Number of Sequences	Length	Lower Bound on TSC
$K \equiv 0(\text{mod}2)$	Any L	$\frac{K^2}{L}$
$K \equiv 1(\text{mod}2)$	Any L	$\frac{K^2}{L} + \frac{L-1}{L}$

expressions for the MSC, C_{sum}, and TAE of minimum-TSC quaternary sets. For all K and L with $K \geq L$ (overloaded systems), we derive analytic expressions for the C_{sum} of minimum-TSC quaternary sets. In particular, we show that minimum-TSC quaternary sets exhibit the following properties: (*i*) if $K \leq L$, $\text{MSC}(\mathcal{S})$ is also minimum; (*ii*) if $K \leq L$, $\text{TAE}(\mathcal{S})$ is single-valued when $L \equiv 0(\text{mod } 2)$ and multi-valued when $L \equiv 1(\text{mod } 2)$; (*iii*) $C_{\text{sum}}(\mathcal{S})$ is single-valued when $\max\{L, K\} \equiv 0(\text{mod } 2)$ and multi-valued when $\max\{L, K\} \equiv 1(\text{mod } 2)$. We derive the exact value of MSC, TAE, and C_{sum} when these metrics are single-valued. When TAE and/or C_{sum} are multi-valued, we establish lower and upper bounds and prove their tightness; the exact value of C_{sum} and/or TAE depends on the particular design of the minimum-TSC signature set. A direct conclusion from this study is that minimum-TSC optimal quaternary sets are not necessarily C_{sum} and/or TAE-optimal, which is also the case for binary antipodal signature sets [16] (we recall that all three metrics are equivalent for real/complex-valued sets [2], [7], [21]).

2. Maximum Squared Correlation (MSC) of Minimum-TSC Quaternary Signature Sets

It can be easily verified that the maximum squared correlation of a quaternary signature matrix $\mathcal{S}_Q$, denoted by $\text{MSC}(\mathcal{S}_Q)$, is lower-bounded as follows:

$$\text{MSC}(\mathcal{S}_Q) \geq \begin{cases} 0, & L \equiv 0(\text{mod } 2) \\ \frac{1}{L^2}, & L \equiv 1(\text{mod } 2). \end{cases} \tag{6}$$

The following two Propositions summarize our findings about the MSC of underloaded minimum-TSC quaternary signature sets. The proofs are omitted due to space limitation.

Proposition 1: Let $\mathcal{S}_Q \in \frac{1}{\sqrt{L}}\{\pm 1, \pm j\}^{L \times K}$, $1 < K \leq L$, be a quaternary signature matrix that achieves the corresponding TSC lower bound in Table I. Then,

(*i*) $\text{MSC}(\mathcal{S}_Q) = 0$, if $L \equiv 0$(mod 2);
(*ii*) $\text{MSC}(\mathcal{S}_Q) = \frac{1}{L^2}$, if $L \equiv 1$(mod 2). ■

Proposition 2: An underloaded quaternary signature set achieves the lower bound on TSC in Table I if only if it also achieves the lower bound on MSC in (6). ■

We conclude that for all K, L with $K \leq L$, the minimum-TSC signature sets are doubly optimal for underloaded systems: they exhibit both minimum TSC and minimum MSC at the same time. It is also interesting to note that while we showed that TSC and MSC minimization are equivalent for quaternary sets for any K, L with $K \leq L$ (subject to the existence of a quaternary Hadamard matrix of size $2\lceil L/2 \rceil$ [23]), this is *not true*, in general, for binary sets[1] [16].

3. Total Asymptotic Efficiency (TAE) of Minimum-TSC Quaternary Signature Sets

The TAE of a complex-valued signature matrix $\mathcal{S} = [\mathbf{s}_1, \ldots, \mathbf{s}_K]$, $\mathbf{s}_k \in \mathbb{C}^L$, $\|\mathbf{s}_k\| = 1$, $k = 1, 2, \ldots, K$, is real-valued and bounded as $0 \leq \text{TAE}(\mathcal{S}) \leq 1$. Since $\mathcal{S}^H\mathcal{S}$ is rank-deficient and $\text{TAE}(\mathcal{S}) = 0$ when $K > L$ (overloaded system), we only consider the underloaded case. $\text{TAE}(\mathcal{S})$ achieves the unit upper bound if $\mathcal{S}$ has orthogonal columns. However, it has been an open question whether tightness is maintained when $\mathcal{S}$ is quaternary, that is $\mathbf{s}_k \in \frac{1}{\sqrt{L}}\{\pm 1, \pm j\}^L$, $k = 1, 2, \ldots, K$. In this section, we obtain closed form expressions for the TAE of minimum-TSC quaternary signature sets for all $K \leq L$. Our developments are based on the proposition that we state below. The proof is given in Appendix A.

Proposition 3: Let $\mathcal{S}_Q \in \frac{1}{\sqrt{L}}\{\pm 1, \pm j\}^{L \times K}$, $K \leq L$, be a quaternary signature matrix that achieves the corresponding TSC lower bound in Table I and $[\mathcal{S}_Q^H\mathcal{S}_Q]_{mn}$ denotes the (m, n)th element of $\mathcal{S}_Q^H\mathcal{S}_Q$, $m = 1, 2, \ldots, K$, $n = 1, 2, \ldots, K$. Then, $\mathcal{S}_Q^H\mathcal{S}_Q$ has the following properties:

(*i*) If $L \equiv 0$(mod 2), $\mathcal{S}_Q^H\mathcal{S}_Q = \mathbf{I}_K$;
(*ii*) if $L \equiv 1$(mod 2), then $[\mathcal{S}_Q^H\mathcal{S}_Q]_{mm} = 1$ and $[\mathcal{S}_Q^H\mathcal{S}_Q]_{mn} \in \frac{1}{L}\{\pm 1, \pm j\}$, $m \neq n$, $m = 1, 2, \ldots, K$, $n = 1, 2, \ldots, K$;
(*iii*) if $L \equiv 1$(mod 2) and there exists a quaternary Hadamard matrix $\mathbf{H}_Q$ of size $L + 1$, we can obtain a minimum-TSC signature set which has $[\mathcal{S}_Q^H\mathcal{S}_Q]_{mn} = -\frac{1}{L}$, $m \neq n$, $m = 1, 2, \ldots, K$, $n = 1, 2, \ldots, K$;
(*iv*) if $L \equiv 1$(mod 2) and there exists a quaternary Hadamard matrix $\mathbf{H}_Q$ of size $L - 1$ and $K \leq L - 1$, we can obtain a minimum-TSC signature set which has $[\mathcal{S}_Q^H\mathcal{S}_Q]_{mn} = \frac{1}{L}$, $m \neq n$, $m = 1, 2, \ldots, K$, $n = 1, 2, \ldots, K$. ■

Based on the above proposition, the TAE of an underloaded minimum-TSC quaternary signature set can be derived and the findings are presented in the form of a proposition given below. The proof is given in Appendix B.

Proposition 4: Let $\mathcal{S}_Q \in \frac{1}{\sqrt{L}}\{\pm 1, \pm j\}^{L \times K}$, $K \leq L$, be a quaternary signature matrix that achieves the corresponding TSC lower bound in Table I. Then,

(*i*) $\text{TAE}(\mathcal{S}_Q) = 1$, if $L \equiv 0$(mod 2);
(*ii*) $\frac{(L+1)^{K-1}(L-K+1)}{L^K} \leq \text{TAE}(\mathcal{S}_Q) \leq \frac{(L-1)^{K-1}(L+K-1)}{L^K}$, if $L \equiv 1$(mod 2). The lower bound is tight if there exists a quaternary Hadamard matrix of size $L + 1$ while the upper bound is tight if $K \leq L - 1$ and there exists a quaternary Hadamard matrix of size $L - 1$. ■

We recall that for real/complex-valued sets TAE maximization and TSC minimization are equivalent problems for all K, L with $K \leq L$. As shown by Proposition 4, however, this property no longer holds true for quaternary signature sets. If

[1] TSC and MSC minimization are equivalent for binary sets for any K, L with $K \leq L$ (subject to the existence of a binary Hadamard matrix of size $4\lfloor \frac{L+2}{4} \rfloor$) except for $L = K \equiv 1(\text{mod } 4)$ or for $L \equiv 2(\text{mod } 4)$.

$L \equiv 1 (\text{mod } 2)$ and $K < L$, then there exist minimum-TSC sets that do not have maximum TAE.

4. Sum Capacity of Minimum-TSC Quaternary Signature Sets

The sum capacity C_{sum} of a multiple-access communication channel is the maximum sum of user transmission rates at which reliable decoding at the receiver end is possible [2], [21], [22]. In a synchronous code-division multiplexing system that employs an $L \times K$ complex-valued signature matrix $\mathcal{S} = [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K]$, $\mathbf{s}_k \in \mathbb{C}^L$, $\|\mathbf{s}_k\| = 1$, $k = 1, 2, \ldots, K$, for transmissions over a common additive white Gaussian noise (AWGN) channel, the received data vector is of the form $\mathbf{r} = \sum_{k=1}^{K} d_k \mathbf{s}_k + \mathbf{n}$ where $d_k \in \mathbb{C}$, $k = 1, 2, \ldots, K$, is the k-th user transmitted symbol (complex in general) and $\mathbf{n}$ is a zero-mean complex Gaussian vector with auto-covariance matrix $N_0 \mathbf{I}_L$. If $E\{|d_k|^2\} = E$, $k = 1, 2, \ldots, K$, it is known [2], [21] that

$$C_{\text{sum}} \triangleq \log_2 \left| \mathbf{I}_L + \gamma \mathcal{S} \mathcal{S}^H \right| = \log_2 \left| \mathbf{I}_K + \gamma \mathcal{S}^H \mathcal{S} \right| \quad (7)$$

where $\gamma \triangleq \frac{E}{N_0}$ is the received signal-to-noise ratio (SNR) of each user signal and $\mathbf{I}_L$, $\mathbf{I}_K$ are the size-L and size-K identity matrices. It is also well known that the sum capacity is bounded as follows [2], [7]

$$0 \le C_{\text{sum}}(\mathcal{S}) \le \begin{cases} K \log_2(1+\gamma), & K \le L \\ L \log_2(1 + \frac{K}{L}\gamma), & K \ge L. \end{cases} \quad (8)$$

While the upper bound in (8) is tight for real/complex-valued signature sets for any K, L, it has been shown in [16] that tightness is *not* always maintained if $\mathcal{S}$ is *binary*. In this section, we consider minimum-TSC quaternary signature sets $\mathcal{S}_Q$ and obtain closed-form expressions for C_{sum} for any K, L. Our developments are presented in the form of a proposition given below. The proof is given in Appendix C.

Proposition 5: Let $\mathcal{S}_Q \in \frac{1}{\sqrt{L}} \{\pm 1, \pm j\}^{L \times K}$ be a quaternary signature matrix that achieves the corresponding TSC lower bound in Table I or Table II. Then,

A) if $K \le L$ (underloaded system)

(*i*) $C_{\text{sum}}(\mathcal{S}_Q) = K \log_2(1+\gamma)$, if $L \equiv 0(\text{mod } 2)$;

(*ii*) $(K-1)\log_2(1 + \frac{L+1}{L}\gamma) + \log_2(1 + \frac{L-K+1}{L}\gamma) \le C_{\text{sum}}(\mathcal{S}_Q) \le (K-1)\log_2(1 + \frac{L-1}{L}\gamma) + \log_2(1 + \frac{L+K-1}{L}\gamma)$, if $L \equiv 1(\text{mod } 2)$. The lower bound is tight if there exists a quaternary Hadamard matrix of size $L+1$, while the upper bound is tight if $K \le L-1$ and there exists a quaternary Hadamard matrix of size $L-1$.

B) If $K \ge L$ (overloaded system)

(*i*) $C_{\text{sum}}(\mathcal{S}_Q) = L \log_2(1 + \frac{K}{L}\gamma)$, if $K \equiv 0(\text{mod } 2)$;

(*ii*) $(L-1)\log_2(1 + \frac{K+1}{L}\gamma) + \log_2(1 + \frac{K-L+1}{L}\gamma) \le C_{\text{sum}}(\mathcal{S}_Q) \le (L-1)\log_2(1 + \frac{K-1}{L}\gamma) + \log_2(1 + \frac{K+L-1}{L}\gamma)$, if $K \equiv 1(\text{mod } 2)$. The lower bound in (*ii*) is tight if there exists a quaternary Hadamard matrix of size $K+1$ while the upper bound is tight if $L \le K-1$ and there exists a quaternary Hadamard matrix of size $K-1$. ■

Comparing Proposition 5 with expression (8) for real/complex-valued sets, we see that minimum-TSC quaternary signature sets meet the upper bound in (8) only if $L \equiv 0(\text{mod } 2)$ for underloaded systems or $K \equiv 0(\text{mod } 2)$ for overloaded systems. In addition, by Proposition 5, when $L \equiv 1(\text{mod } 2)$ for underloaded systems or $K \equiv 1(\text{mod } 2)$ for overloaded systems and $K \ne L$, there exist quaternary minimum-TSC sets that do not exhibit maximum sum capacity. Thus, minimum-TSC and maximum-C_{sum} criteria are not equivalent, in general, for quaternary sets for all K, L.

To visualize the theoretical developments of Proposition 5 on the sum capacity of quaternary signature sets, we consider the relative sum-capacity-loss expression

$$\Delta(\mathcal{S}) \triangleq 1 - \frac{C_{\text{sum}}(\mathcal{S})}{C^*_{\text{sum}}} \quad (9)$$

where C^*_{sum} is the sum capacity of a real/complex-valued Welch-bound-equality (WBE) signature set of the same size as $\mathcal{S}$. In Fig. 1, we plot the sum-capacity loss $\Delta(\mathcal{S})$ of minimum-TSC quaternary sets as a function of K for a common received SNR per user $\gamma = 12$ dB and four different signature length values $L = 31$, 32, 33, and 34. For comparison purposes, we also include the sum-capacity loss of minimum-TSC binary signature sets which was analyzed in [16]. We observe that minimum-TSC quaternary sets exhibit rather negligible sum-capacity-loss for almost all K, L (Fig. 1) in comparison with WBE real/complex-valued sets. In addition, the sum-capacity loss of quaternary minimum-TSC sets is quite less than the sum-capacity loss of binary minimum-TSC sets for almost all values of K. In Fig. 2, we repeat the same study as in Fig. 1 for $L = 63$, 64, 65, and 66. Similar conclusions can be drawn.

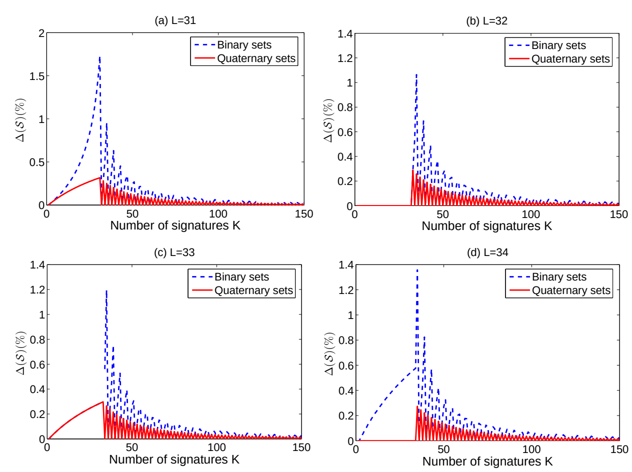

Fig. 1. Sum-capacity loss $\Delta(\mathcal{S})(\%)$ of minimum-TSC binary and quaternary signature sets versus number of signatures K of length (a) $L = 31$, (b) $L = 32$, (c) $L = 33$, and (d) $L = 34$ ($\gamma = 12$ dB).

5. Conclusions

In this paper, we derived closed-form expressions for the MSC, TAE, and sum capacity that minimum-TSC quaternary signature sets achieve for all K, L with $K \le L$ and the sum capacity that minimum-TSC quaternary sets achieve for all

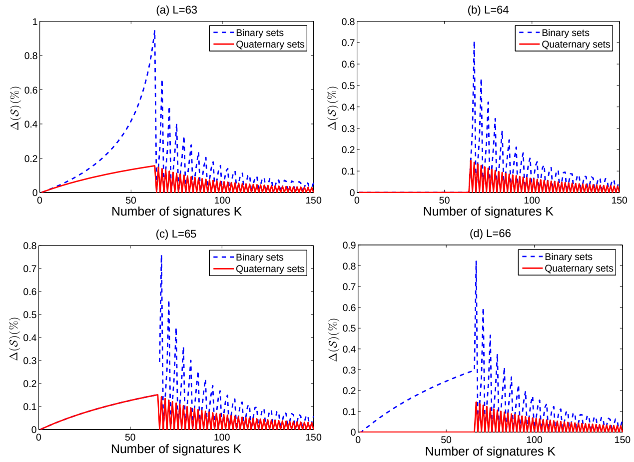

Fig. 2. Sum-capacity loss $\Delta(\mathcal{S})(\%)$ of minimum-TSC binary and quaternary signature sets versus number of signatures K of length (a) $L = 63$, (b) $L = 64$, (c) $L = 65$, and (d) $L = 66$ ($\gamma = 12$ dB).

K, L with $K > L$. We recall that minimum-TSC, minimum-MSC, maximum-TAE, and maximum-sum-capacity are equivalent optimization criteria for real/complex-valued signature sets, i.e. real/complex-valued minimum-TSC signature sets are minimum-MSC and maximum-TAE when the number of signatures K is less than or equal to the signature length L and have maximum sum-capacity for any K, L. Interestingly, for quaternary (and binary [16]) signature sets, there exist K, L values for which different metrics are optimized by different quaternary sets. Our studies showed that the sum-capacity loss of the minimum-TSC quaternary signature sets is negligible in comparison with minimum-TSC real/complex-alphabet (Welch-bound-equality) sets and quite smaller than that exhibited by minimum-TSC binary signature sets.

Appendix A
Proof of Proposition 3

The proof of parts (*i*) and (*ii*) is trivial and is omitted herein. With respect to part (*iii*), we recall that if the rows and columns of a quaternary Hadamard matrix are permuted or any row or column is multiplied by -1 or $\pm j$, the Hadamard orthogonality property is retained. Hence, we can always arrange one row or one column of a quaternary Hadamard matrix to have only $+1$ entries. If there exists a quaternary Hadamard matrix $\mathbf{H}_Q$ of size $L+1$ and $L \equiv 1(\text{mod } 2)$, a minimum-TSC signature set can be obtained by taking K columns from $\mathbf{H}_Q$ and removing one row which contains only $+1$ entries. After normalization, the cross-correlation matrix of the created minimum-TSC signature set is

$$\mathcal{S}_Q^H \mathcal{S}_Q = \frac{L+1}{L}\mathbf{I}_K - \frac{1}{L}\mathbf{1}_K \mathbf{1}_K^T. \tag{10}$$

With respect to part (*iv*), if there exists a quaternary Hadamard matrix $\mathbf{H}_Q$ of size $L-1$ and $K \leq L-1$, a minimum-TSC signature set can be obtained by appending an all-one row $\mathbf{1}_{L-1}^T$ to $\mathbf{H}_Q$ and taking K columns. After normalization, the cross-correlation matrix of the created minimum-TSC signature set is

$$\mathcal{S}_Q^H \mathcal{S}_Q = \frac{L-1}{L}\mathbf{I}_K + \frac{1}{L}\mathbf{1}_K \mathbf{1}_K^T. \tag{11}$$

■

Appendix B
Proof of Proposition 4

(*i*) When $L \equiv 0(\text{mod } 2)$ and $\mathcal{S}_Q$ achieves the TSC lower bound in Table I, by Proposition 3, part (*i*), we obtain $\text{TAE}(\mathcal{S}_Q) = \left|\mathcal{S}_Q^H \mathcal{S}_Q\right| = |\mathbf{I}| = 1$.

(*ii*) By Proposition 3, $[\mathcal{S}_Q^H \mathcal{S}_Q]_{mm} = 1$ and $\left|[\mathcal{S}_Q^H \mathcal{S}_Q]_{mn}\right| = \frac{1}{L}$, $m \neq n$, $m = 1, 2, \ldots, K$, $n = 1, 2, \ldots, K$. Then, by *Lemma 2* of [16], we obtain that

$$\begin{aligned}(1+\frac{1}{L})^{K-1}(1-(K-1)\frac{1}{L}) &\leq \left|\mathcal{S}_Q^H \mathcal{S}_Q\right| \\ &\leq (1-\frac{1}{L})^{K-1}(1+(K-1)\frac{1}{L}).\end{aligned} \tag{12}$$

Expression (12) leads to the bounds on TAE as they appear in Proposition 4. If there exists a quaternary Hadamard matrix $\mathbf{H}_Q$ of size $L+1$, by (10) we can obtain a minimum-TSC quaternary set which has

$$\begin{aligned}\left|\mathcal{S}_Q^H \mathcal{S}_Q\right| &= \left|\frac{L+1}{L}\mathbf{I}_K - \frac{1}{L}\mathbf{1}_K \mathbf{1}_K^T\right| \\ &= \left(\frac{L+1}{L}\right)^K \left(\frac{L-K+1}{L+1}\right)\end{aligned} \tag{13}$$

and this reaches the lower bound in Proposition 4. If there exists a quaternary Hadamard matrix $\mathbf{H}_Q$ of size $L-1$, by (11) we can obtain a minimum-TSC quaternary set with TAE

$$\left|\mathcal{S}_Q^H \mathcal{S}_Q\right| = \left(\frac{L-1}{L}\right)^K \left(\frac{L+K-1}{L-1}\right) \tag{14}$$

and this is the upper bound value in Proposition 4. ■

Appendix C
Proof of Proposition 5

Part A

(*i*) If $L \equiv 0(\text{mod } 2)$ and $\mathcal{S}_Q$ achieves the TSC lower bound in Table I, it has orthogonal columns, i.e $\mathcal{S}_Q^H \mathcal{S}_Q = \mathbf{I}_K$. Therefore,

$$\begin{aligned}C_{\text{sum}}(\mathcal{S}_Q) &= \log_2\left|\mathbf{I}_K + \gamma \mathcal{S}_Q^H \mathcal{S}_Q\right| \\ &= \log_2\left|(1+\gamma)\mathbf{I}_K\right| \\ &= K\log_2(1+\gamma).\end{aligned} \tag{15}$$

(*ii*) By Proposition 3, the minimum-TSC quaternary set $\mathcal{S}_Q$ has following properties: 1) $[\mathbf{I}_K + \gamma\mathcal{S}_Q^H \mathcal{S}_Q]_{mm} = 1+\gamma$, $m = 1, 2, \ldots, K$; 2) $\left|[\mathbf{I}_K + \gamma\mathcal{S}_Q^H \mathcal{S}_Q]_{mn}\right| = \frac{\gamma}{L}$, $m \neq n$, $m = 1, 2, \ldots, K$, $n = 1, 2, \ldots, K$. Then, *Lemma 2* of [16] implies that the determinant of $\mathbf{I}_K + \gamma\mathcal{S}_Q^H \mathcal{S}_Q$ is bounded as follows:

$$\begin{aligned}&(1+\gamma+\tfrac{\gamma}{L})^{(K-1)}(1+\gamma-(K-1)\tfrac{\gamma}{L}) \\ &\quad\leq \left|\mathbf{I}_K + \gamma\mathcal{S}_Q^H \mathcal{S}_Q\right| \\ &\leq (1+\gamma-\tfrac{\gamma}{L})^{(K-1)}(1+\gamma+(K-1)\tfrac{\gamma}{L}).\end{aligned} \tag{16}$$

Therefore, $C_{\rm sum}(\mathcal{S}_Q) = \log_2 \left|\mathbf{I}_K + \gamma \mathcal{S}_Q^H \mathcal{S}_Q\right|$ is bounded as

$$\begin{aligned} &(K-1)\log_2(1+\tfrac{L+1}{L}\gamma) + \log_2(1+\tfrac{L-K+1}{L}\gamma) \\ &\qquad \le C_{\rm sum}(\mathcal{S}_Q) \\ &\le (K-1)\log_2(1+\tfrac{L-1}{L}\gamma) + \log_2(1+\tfrac{L+K-1}{L}\gamma). \quad (17) \end{aligned}$$

If there exists a quaternary Hadamard matrix $\mathbf{H}_Q$ of size $L+1$, by Proposition 3, part (*ii*), we can obtain a minimum-TSC quaternary set that satisfies (11). Therefore,

$$\begin{aligned} C_{\rm sum}(\mathcal{S}_Q) &= \log_2 \left|\mathbf{I}_K + \gamma \mathcal{S}_Q^H \mathcal{S}_Q\right| \\ &= \log_2 \left|\left(1+\frac{L+1}{L}\right)\mathbf{I}_K - \frac{\gamma}{L}\mathbf{1}_K \mathbf{1}_K^T\right| \\ &= (K-1)\log_2\left(1+\frac{L+1}{L}\gamma\right) \\ &\quad + \log_2\left(1+\frac{L-K+1}{L}\gamma\right) \qquad (18) \end{aligned}$$

which is equal to the lower bound in Proposition 5, *Part A(ii)*.

If there exists a quaternary Hadamard matrix $\mathbf{H}_Q$ of size $L-1$ and $K \le L-1$, we can obtain a minimum-TSC quaternary set that satisfies (11) and by similar to (18) derivation we can evaluate $C_{\rm sum}$ as follows:

$$\begin{aligned} C_{\rm sum}(\mathcal{S}_Q) &= \log_2 \left|\mathbf{I}_K + \gamma \mathcal{S}_Q^H \mathcal{S}_Q\right| \\ &= (K-1)\log_2\left(1+\frac{L-1}{L}\gamma\right) \\ &\quad + \log_2\left(1+\frac{L+K-1}{L}\gamma\right) \qquad (19) \end{aligned}$$

which is the upper bound in Proposition 5, *Part A(ii)*.

Part B

Set $\mathbf{D} \triangleq \sqrt{\frac{L}{K}}\mathcal{S}_Q^H$ Then

$$\begin{aligned} C_{\rm sum}(\mathcal{S}_Q) &= \log_2 \left|\mathbf{I}_L + \gamma \mathcal{S}_Q \mathcal{S}_Q^H\right| \\ &= \log_2 \left|\mathbf{I}_L + \gamma \frac{K}{L}\mathbf{D}^H\mathbf{D}\right| \\ &= \log_2 \left|\mathbf{I}_K + \gamma \frac{K}{L}\mathbf{D}\mathbf{D}^H\right|. \qquad (20) \end{aligned}$$

$\mathbf{D} \in \frac{1}{\sqrt{K}}\{\pm 1, \pm j\}^{K\times L}$ can be viewed as a signature matrix with L unit-norm quaternary signatures of length $K \ge L$. Therefore, $C_{\rm sum}(\mathcal{S}_Q)$ at SNR γ equals $C_{\rm sum}(\mathbf{D})$ at SNR $\gamma\frac{K}{L}$ where $\mathcal{S}_Q$ is the overloaded and $\mathbf{D}$ is the corresponding underloaded set. We can show that if TSC($\mathcal{S}_Q$) achieves the TSC lower bound for overloaded sets in Table II, then TSC($\mathbf{D}$) achieves the TSC lower bound for underloaded sets in Table I. Hence, we can apply our results in *Part A* of Proposition 5 to $\mathbf{D}$ and obtain the $C_{\rm sum}(\mathcal{S}_Q)$ expressions in all cases of Proposition 5, *Part B*, directly. ∎

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