POSTERIOR CRAMER-RAO LOWER BOUND FOR MOBILE TRACKING IN MIXED LOS/NLOS CONDITIONS

Chen Liang [1,2], *Wu Lenan* [3], *and Robert Piche* [1]

[1] Institute of Mathematics, Tampere University of Technology, Finland
[2] Department of Testing and Control, Jiangsu University, China
[3] School of Information Science and Engineering, Southeast University, China
Email: cliang@seu.edu.cn; wuln@seu.edu.cn; robert.piche@tut.fi

ABSTRACT

The paper studies the theoretical error lower bound of the mobile tracking problem in mixed Line-of-sight (LOS) and non-line-of-sight (NLOS) conditions. A new method is presented to compute the posterior Cramer-Rao lower bound (CRLB): the mobile state is first estimated by decentralized EKF method, then sigma point set and unscented transformation are applied to calculate Fisher information matrix (FIM). Numerical results show that the error performance of three algorithms is in a good agreement with the theoretical bounds.

1. INTRODUCTION

In dense urban regions, the non-line-of-sight (NLOS) condition is very common, where the direct path from the mobile station (MS) to a base station (BS) is blocked by buildings and other obstacles. Propagation wave may actually travel excess path lengths due to effects of reflection, refraction and scattering in such condition. In terms of range based measurements such as time of arrival (TOA), time difference of arrival (TDOA) and receiver signal strength (RSS), this extra propagation distance imposes positive biases on the true path, which cause large errors on the MS location estimations.

Many methods have been proposed to deal with the NLOS problem. Reference [1] has summarized the methods for static position systems. Several algorithms have been proposed to track the mobile more effectively by exploiting the redundant measurements in time series, including the two-step Kalman filtering technique for smoothing range measurements and mitigating NLOS errors in [2] and a Kalman based interacting multiple model (IMM) smoother [3]. In our previous work, modified EKF banks with data fusion method [4] and the Rao-Blackwellized particle filtering algorithm [5] have improved performance compared with the existing methods.

In this paper, we investigate the theoretical lower bound of the performance error of mobile tracking in mixed LOS/NLOS conditions. For simplicity, we consider the problem under the assumption that the LOS and NLOS transition history is known, which avoids the false detection of sight condition. A new method is presented to calculate the posterior CRLB, which adopts the decentralized EKF method to estimate the mobile state first, then applies sigma point set and unscented transformation to calculate Fisher information matrix (FIM), the inverse of posterior CRLB value.

Although we study the theoretical lower bound in the context of mobile cellular positioning, the methodology is completely general for other platforms, such as ultra wide band (UWB), satellite based position etc. The posterior CRLB we derive could be used as the theoretical basis for developing new algorithms in this kind of problems. It is also useful in predicting the performance for various sampling intervals and sensor accuracy.

The paper is organized as follows: Section II presents the dynamic system models and formulates the problem of mobility tracking in the mixed LOS/NLOS conditions. Sect. III describes the derivation of the performance lower bound. Numerical results and performance comparison are presented and discussed in Sect. IV. Sect. V draws some conclusions.

2. SYSTEM MODEL

2.1 Mobile State Model

In this work, we assume that the MS moves according to a dynamic white noise acceleration model in a 2-D Cartesian coordinate plane [6]:

$$X_k = \Phi X_{k-1} + W_k \tag{1}$$

where Φ models the state kinematics, the mobile state at time k is defined as $X_k = [x_k, y_k, \dot{x}_k, \dot{y}_k]^T$, $[x_k, y_k]$ denotes the mobile position; $[\dot{x}_k, \dot{y}_k]$ denotes the velocity. W_k is a white zero mean Gaussian noise, with covariance matrix Q.

2.2 Measurement Model

method. Suppose $d_{i,k}$ represents the true range between the mobile position $[x_k, y_k]$ and the location of BS_i $[x_i, y_i]$:

$$d_{i,k} \triangleq h_i(X_k) = \sqrt{(x_k - x_i)^2 + (y_k - y_i)^2}, i \in (1, 2 \ldots M) \tag{2}$$

In a LOS environment, the range measurement between MS and the BS_i is only corrupted by the system measurement noise $n_{i,k}$, which can be modeled as an independent and identically distributed (i.i.d.) zero mean white Gaussian noise $N(0, \sigma_m^2)$. In NLOS conditions, the range measurement is corrupted by two sources of errors: the measurement noise $n_{i,k}$ and the NLOS error $b_{i,k}$. Field tests showed that, the mean and standard deviation of the NLOS range error are in the order of 513 m and 436 m respectively [7]. As in ref. [3-5], NLOS error is also modeled as a positively biased Gaussian distribution with $N(m_{NLOS}, \sigma_{NLOS}^2)$ here. Then the range measurement equations are

$$LOS : z_{i,k} = d_{i,k} + n_{i,k} \tag{3}$$

$$NLOS : z_{i,k} = d_{i,k} + n_{i,k} + b_{i,k} \tag{4}$$

We introduce a variable $s_{i,k} \in \mathscr{S} \triangleq \{0,1\}$ to represent LOS/NLOS condition between MS and the BS_i at the time instant k, with $s_{i,k} = 0$ for LOS and $s_{i,k} = 1$ for NLOS. Eq. (3) and (4) can be further transformed to:

$$z_{i,k} = d_{i,k} + m(s_{i,k}) + R(s_{i,k}) \cdot v_{i,k}, \quad s_{i,k} \in \mathscr{S} \tag{5}$$

where $v_{i,k}$ is the normalized i.i.d. zero mean white Gaussian noise and

$$m(s_{i,k}) = \begin{cases} 0, & if \quad s_{i,k} = 0 \\ m_{NLOS}, & if \quad s_{i,k} = 1 \end{cases} \tag{6}$$

$$R(s_{i,k}) = \begin{cases} \sigma_m, & if \quad s_{i,k} = 0 \\ \sqrt{\sigma_m^2 + \sigma_{NLOS}^2}, & if \quad s_{i,k} = 1 \end{cases} \tag{7}$$

Field measurements have shown a dynamic transition between LOS and NLOS conditions in typical cellular communication environments [8]. Thus, the transitions between the two-state sight condition variable $s_{i,k}$ can be further assumed as a first-order Markov chain with initial probability vector π_i and transition probability matrix $A_i : s_{i,k} \sim MC(\pi^i, A^i)$ [3-5]. Note that the M sight conditions are assumed as i.i.d. first-order Markov chains for the independence of the BSs.

Denote the total observation sequence up to time k as $\mathbf{Z}_{1:k}$, where $\mathbf{Z}_k \triangleq [z_{1,k}, z_{2,k}, ..., z_{M,k}]^T$, the corresponding discrete sight condition sequence $\mathbf{S}_{1:k}$, where $\mathbf{S}_k \triangleq [s_{1,k}, s_{2,k}, ..., s_{M,k}]^T$ and the continuous state sequence $X_{1:k}$. The problem of mobile positioning in mixed LOS/NLOS conditions is to infer the current mobile state $X_{1:k}$ from the observation $\mathbf{Z}_{1:k}$, which corresponds to compute the marginal posterior $p(X_{1:k}/\mathbf{Z}_{1:k})$ or the joint posterior $p(X_{1:k}, \mathbf{S}_{1:k}/\mathbf{Z}_{1:k})$.

The optimal Bayesian solution to the problem, unfortunately, cannot be computed analytically, because the measurement equation is nonlinear. Moreover, the required density $p(X_{1:k}/\mathbf{Z}_{1:k})$ is a mixture density with the number of components growing exponentially with time, which involves high-dimensional integrals and is prohibitive to compute. Several suboptimal solutions have been proposed, in which different approximation methods have been applied [3-5]. However, in dealing with approximations, a lower bound of performance should be derived, so as to predict the best achievable performance before running the algorithms and assess the level of approximation introduced by a particular algorithm.

Section III presents the derivation of the performance lower bound in detail. Also a new method is proposed to calculate the posterior CRLB with relatively low computation complexity.

3. POSTERIOR CRLB

Let $\hat{X}_k$ be an unbiased estimator of the state vector X_k, based on the set of measurements $\mathbf{Z}_{1:k}$. Then, the estimate covariance P_k is bounded by the P-CRLB J_k^{-1}:

$$P_k = E\{[\hat{X}_k - X_k][\hat{X}_k - X_k]^T\} \geq J_k^{-1} \tag{8}$$

where J_k is the posterior Fisher information matrix (FIM):

$$J_k = E\{-\nabla_{X_k} \nabla_{X_k}^T \log p(X_k, \mathbf{Z}_k)\} \tag{9}$$

and ∇_{X_k} is the first-order partial derivative operator with respect to X_k.

Tichavsky et al [9] show that the FIM J_k can be recursively calculated as

$$J_{k+1} = D_k^{22} - D_k^{21}(J_k + D_k^{11})^{-1} D_k^{12} \tag{10}$$

where

$$D_k^{11} = E\{-\nabla_{X_k} \nabla_{X_k}^T \log p(X_{k+1} | X_k)\} \tag{11}$$

$$\begin{aligned} D_k^{12} &= [D_k^{21}]^T \\ &= E\{-\nabla_{X_k} \nabla_{X_{k+1}}^T \log p(X_{k+1} | X_k)\} \end{aligned} \tag{12}$$

$$\begin{aligned} D_k^{22} &= E\{-\nabla_{X_{k+1}} \nabla_{X_{k+1}}^T \log p(X_{k+1} | X_k)\} \\ &\quad + E\{-\nabla_{X_{k+1}} \nabla_{X_{k+1}}^T \log p(\mathbf{Z}_{k+1} | X_{k+1})\} \\ &= D_k^{22,a} + D_k^{22,b} \end{aligned} \tag{13}$$

We initialize the recursion of Equ. (10) with

$$J_0 = E\{-\nabla_{X_0} \nabla_{X_0}^T \log p(X_0)\} \tag{14}$$

For the case of the linear dynamic Gaussian white noise acceleration state model in Equ.(1), the Equation (11-13) can be simplified as :

$$D_k^{11} = \Phi^T Q^{-1} \Phi \tag{15}$$

$$D_k^{12} = (D_k^{21})^T = -\Phi^T Q^{-1} \tag{16}$$

$$D_k^{22} = Q^{-1} + D_k^{22,b} \tag{17}$$

Substitute Equ. (15)-(17) into Equ. (10) and apply the matrix inversion lemma, we get:

$$J_{k+1} = (Q + \Phi J_k^{-1} \Phi^T)^{-1} + D_k^{22,b} \tag{18}$$

The expectation $D_k^{22,b}$ relates to nonlinear measurement equation, thus has no analytically closed-form results. Monte Carlo random sampling approach could be used to circumvent the difficulty by converting the above integrals to summations [10]. Considering that the mobile state can be approximately estimated using decentralized EKF method, we propose a new method using sigma point set and unscented transformation, a deterministically sampling method with relatively low computation complexity.

Using linearization approximation, $D_k^{22,b}$ can be further computed as

$$\begin{aligned} D_k^{22,b} = \tfrac{1}{2} E\{\nabla_{X_{k+1}} \nabla_{X_{k+1}}^T \{[\mathbf{Z}_{k+1} - \mathbf{h}(X_{k+1}) - \mathbf{m}(\mathbf{S}_{k+1})]^T \Sigma_{k+1}^{-1} \\ [\mathbf{Z}_{k+1} - \mathbf{h}(X_{k+1}) - \mathbf{m}(\mathbf{S}_{k+1})]\}\} = E\left\{H_{k+1}^T \Sigma_{k+1}^{-1} H_{k+1}\right\} \end{aligned} \tag{19}$$

where $H_{k+1} = \left[H_{1,k+1} ..., H_{M,k+1}\right]^T$ and $H_{i,k+1} = \frac{\partial h_i(X_{k+1})}{\partial X_{k+1}}$. Σ_{k+1} denotes the measurement covariance matrix, and the subscript $k+1$ represents the time varying LOS or NLOS sight conditions.

Let

$$\Lambda_k^{22,b} \triangleq H_{k+1}^T \Sigma_{k+1}^{-1} H_{k+1} \tag{20}$$

then Equ. (19) can be simplified as

$$D_k^{22,b} = E\{\Lambda_k^{22,b}\} = \int \Lambda_k^{22,b} dp(X_{k+1}) \tag{21}$$

Under the assumption that the LOS and NLOS condition between MS and each BS is known during the whole MS trajectory, the density $p(X_k/\mathbf{Z}_k)$ conforms to Gaussian distribution. Since the time varying LOS and NLOS conditions have different mean and variance, we apply the decentralized EKF method to compute the $p(X_k/\mathbf{Z}_k)$ approximately. The mean matrix $\hat{X}_k$ is :

$$\hat{X}_k = \hat{X}_{k/k-1} + \sum_{i=1}^{M} K_{i,k}(z_{i,k} - \hat{z}_{i,k/k-1}) \tag{22}$$

in which,

$$\hat{z}_{i,k/k-1} = h_i(\hat{X}_{k/k-1}) + m(s_{i,k}) \tag{23}$$

the Kalman gain:

$$K_{i,k} = \hat{P}_{i,k} H_{i,k}^T R(s_{i,k})^{-2} \tag{24}$$

$$\hat{P}_{i,k} = \left[\hat{P}_{k/k-1}^{-1} + H_{i,k}^T R(s_{i,k})^{-2} H_{i,k}\right]^{-1} \tag{25}$$

And the covariance matrix $\hat{P}_k$ is :

$$\hat{P}_k = [\hat{P}_{k/k-1}^{-1} + \sum_{i=1}^{M} H_{i,k}^T R(s_{i,k})^{-2} H_{i,k}]^{-1} \tag{26}$$

Assume a n_x dimension motion state variable X_k is estimated with mean $\hat{X}_k$ and covariance $\hat{P}_k$. To calculate the statistics of $D_k^{22,b}$, we use unscented transformation method as follows: First, a set of $2n_x+1$ sigma points $SS_k^{(j)} = \left\{W_k^{(j)}, X_k^{(j)}\right\}$ can be deterministically sampled from the multivariate Gaussian distribution $X_k^{(j)} \sim N(\hat{X}_k, \hat{P}_k)$. A symmetric set of sigma points can be generated according to the following requirements:

- $j = 0$
  $X_k^{(0)} = \hat{X}_k$
  $W_k^{(0)} = \kappa/(n_x+\kappa)$
- $j = 1, ..., n_x$
  $X_k^{(j)} = \hat{X}_k + \left(\sqrt{(n_x+\kappa)\hat{P}_k}\right)_j$ (27)
  $W_k^{(j)} = 1/\{2(n_x+\kappa)\}$
- $j = n_x+1, ..., 2n_x$
  $X_k^{(j)} = \hat{X}_k - \left(\sqrt{(n_x+\kappa)\hat{P}_k}\right)_{j-n_x}$
  $W_k^{(j)} = 1/\{2(n_x+\kappa)\}$

where κ is a scaling parameter and $(\sqrt{(n_x+\kappa)\hat{P}_k})_j$ is the jth column of the matrix square root of $(n_x+\kappa)\hat{P}_k$. $W_k^{(j)}$ is the weight associated with jth point, which satisfies $\sum_{i=0}^{2n_x} W_k^{(j)} = 1$. Substituting $X_k^{(j)}$ into (20), $\Lambda_{k-1}^{22,b(j)}$ can be calculated. Then, the expectation $D_{k-1}^{22,b}$ can be computed as:

$$D_{k-1}^{22,b} = \sum_{j=0}^{2n_x} \Lambda_{k-1}^{22,b(j)} W_k^{(j)}. \tag{28}$$

A detailed scheme of computing the P-CRLB is given below.

Iteratively compute the Posterior CRLB
I. *Initialization*: k=0, Set $\{\hat{X}_0, \hat{P}_0\}$; II. *Recursive estimation*: *for* k=1,2,..., 1) Predict the mean and covariance of mobile state $\hat{X}_{k/k-1} = \Phi\hat{X}_{k-1}$ $\hat{P}_{k/k-1} = \Phi\hat{P}_{k-1}\Phi^T + Q$ 2) For $i = 1, ..., M$, $s_{i,k} = 0$ and 1 2.1) Predict the measurements according to known sight conditions (LOS or NLOS) according to (23) 2.2) Compute Kalman gain according to (24)(25) 3) Update the $\{\hat{X}_k, \hat{P}_k\}$ using decentralized EKF method according to (22)(26) 4) deterministically choose a set of $2n_x+1$ sigma points $SS_k^{(j)} = \left\{W_k^{(j)}, X_k^{(j)}\right\}$ according to (27) 5) Compute the $\Lambda_{k-1}^{22,b(j)} = g(X_k^{(j)})$ according to (20) and estimate the expectation according to (28) 6) Update J_k according to (18) 7) The position MSE bound is then $locationP-CRLB = \sqrt{J_k^{-1}(1,1) + J_k^{-1}(2,2)}$, where, $J_k^{-1}(1,1)$ and $J_k^{-1}(2,2)$ are the bounds on the MSE corresponding to x_k and y_k respectively .

Symmetric sigma point set and unscented transformation method is known to compute the projected mean and covariance to the second order accuracy [11]. Thus, based on the approximately estimation of the mobile state, and the deterministic sampling method using sigma point set and unscented transformation, the PCRLB can be effectively calculated.

4. NUMERICAL RESULTS

The mobile trajectories are generated according to the motion model described in Section II.A. The MS is assumed to receive the signals from only three BSs all the time. Thus, during the mobile tracking, M is known and fixed. The random acceleration variance σ_x^2, σ_y^2 are both chosen to 0.5 m/s^2. The simulated trajectory has $L = 1600$ time samples, and the sample interval $\Delta t = 0.5$ s. The simulated measurement data are generated by adding the measurement noise and the NLOS noise to the true distance from MS to each BS. The measurement noise is assumed to be a white random variable with zero mean and standard deviation $\sigma_m = 150$ m, whereas the NLOS measurement noise is also assumed to be a white random variable but with positive mean $m_{NLOS} = 513$ m and standard deviation $\sigma_{NLOS} = 409$ m [2-5]. The initial estimation of the mode probability are set to $p(s_{i,0}) = p(s_{i,1}) = 0.5$ for $i = 1$ to 3. The mode transition probability is chosen by $p_0 = p_1 = 0.85$. The LOS or NLOS mode between the MS with each BS is generated by sampling from the transition probability of the Markov chain, and is changed every 200 samples in each transition case [2-5].

We compare the posterior CRLB with the performance of the IMM-KF smoother [3], the modified EKF banks [4] and the improved Rao-Blackwellized particle filtering method [5]. The sight condition is assumed to be known when computing the posterior CRLB. Table 1 shows one realization of the actual sight condition used in the simulation with ev-

Table 1: Actual sight condition during the whole trajectory with 200 samples in each transition. ($s_{i,k} = 0$ for LOS and $s_{i,k} = 1$ for NLOS)

T(k)	1	2	3	4	5	6	7	8
$s_{1,k}$	1	0	0	0	0	0	0	0
$s_{2,k}$	0	0	0	0	0	0	0	0
$s_{3,k}$	0	1	1	0	0	0	0	0

ery 200 samples in one time period of T(k). 100 particles are used in the improved Rao-Blackwellized particle filtering method. All the simulation results are obtained based on $MC = 50$ Monte Carlo realizations with the same parameters.

Define the position root mean square error (RMSE) at instant k as: $RMSE_k = \sqrt{\frac{1}{MC}\sum_{mc=1}^{MC}[(\hat{x}_{k,mc} - x_k)^2 + (\hat{y}_{k,mc} - y_k)^2]}$. The comparison of position RMSE among all three algorithms and the Posterior CRLB is presented in Fig. 1. It shows that, among all three tracking algorithms the improved RBPF method is the most accurate, followed by modified EKF bank methods and the IMM-KF method, which has the worst error performance. And the level of the error standard derivation is in general higher than the posterior bound, because: (1) all the algorithms are based on approximations to some extent; and (2) the errors in estimation of the LOS or NLOS condition will also increase the location error. Reason (2) can be further supported by the following observation.

The error estimation of position RMSE in all the three algorithms is relatively larger in the time period 1, 2 and 3. This is because there exits the NLOS sight condition between the MS and a certain BS, which interferes with the estimation of the mobile positions. However, posterior CRLB does not suffer much adverse impact from the NLOS measurement, for the reasons that the sight condition is known, and the NLOS error could be mitigated. Moreover, during the time period from 4 to 8, the sight conditions are all LOS, and Figure 1 shows the estimation errors are reduced, especially for improved RBPF method and modified EKF bank methods, which are in a good agreement with the posterior CRLB.

We further present the cumulative distribution function (CDF) of root square error (RSE) in Fig. 2. RSE is defined as $RSE = \sqrt{((\hat{x}_k - x_k)^2 + (\hat{y}_k - y_k)^2)}$. From Fig. 2, most RSEs of the posterior CRLB are within [30m, 32.9m], while a small portion (less than 5%) is larger than 32.9m. The relative large errors are caused by the initial value settings, as is showed in Fig.1. Statistical results show that, in the CDF of PCRLB, 67% error is within 31.9m, and 95% error is within 32.9m. US Federal Communications Commission (FCC) has mandated that the location accuracy of emergency calls should be 100 meters with 67% possibility and 300 meters with 95% possibilty for the network-based location systems [12]. Obviously, in Fig. 2, it is clear that the location accuracy of the three algorithms all satisfy the FCC requirement. However, when comparing with the posterior CRLB, there is still room for improving the location performance. Whether or not there exits a more efficient tracking method in this application is a topic for future study.

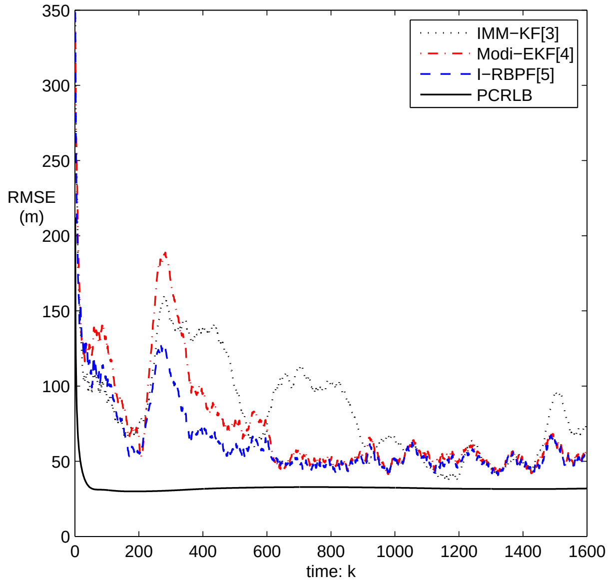

Figure 1: Position RMSE vs. Time instant

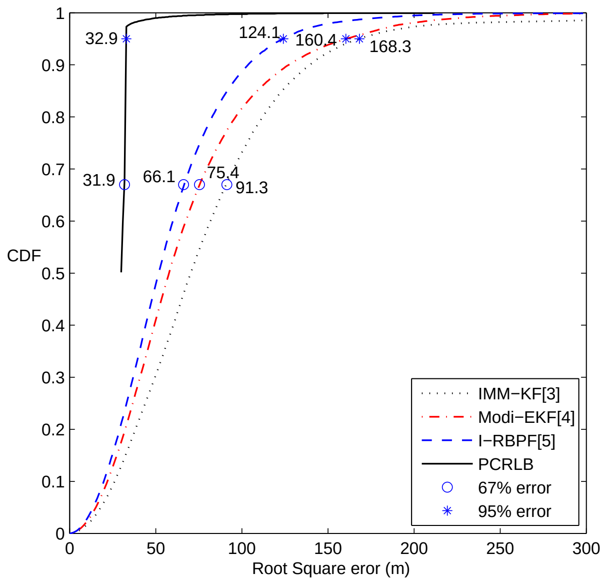

Figure 2: CDF of RSE

5. CONCLUSION

The paper presented the derivation of a posterior Cramer-Rao lower bound of the mobile tracking problem in mixed LOS/NLOS conditions. The theoretical bound was derived under the assumption that the LOS/NLOS sight condition is known during the whole trajectory. Simulation results in the error performance of all three algorithms showed the agreement with the theoretical bounds.

Future work will relax the assumption that the transition sequence of the sight conditions is known and calculate the theoretical bound by referring to the method in ref. [13][14]. In addition, more accurate tracking method is also needed to be further studied.

REFERENCES

[1] Y. -T. Chan, W. -Y. Tsui, H. -C. So, and P. -C. Ching, "Time-of-arrival based localization under NLOS conditions," *IEEE Trans. Veh. Technol.*, vol.55, no.1, pp.17–24, 2006.

[2] B. L. Le, K. Ahmed and H. Tsuji, "Mobile location estimator with NLOS mitigation using Kalman filtering," in *Proc. IEEE Wireless Communications and Networking Conference (WCNC'03)*, New Orleans, USA, vol.3, 2003, pp.1969-1973.

[3] J. F. Liao, and B. -S. Chen, "Robust mobile location estimator with NLOS mitigation using interacting multiple model algorithm," *IEEE Trans. Wirel. Commun.*, vol.5, no.11, pp.3002-3006, 2006.

[4] L. Chen and L. N. Wu, "Mobile positioning in mixed LOS/NLOS conditions using modified EKF banks and data fusion method", *IEICE Transactions on Communications*, vol. EB-92, no. 4, pp.1318-1325, April, 2009 .

[5] L. Chen and L. N. Wu, "Mobile localization with NLOS mitigation using improved Rao-Blackwellized particle filtering algorithm,", in *Proc. IEEE International Symposium on Consumer Electronics (ISCE 2009)*, Kyoto, Japan, May, 2009.

[6] X. R. Li, Y. Bar-Shalom and T. Kirubarajan, *Estimation with Applications to Tracking and Navigation*, John Wiley and Sons, 2001.

[7] M.I. Silventoinen, T. Rantalainen, "Mobile station emergency locating in GSM,", in *IEEE Inter. Conf. Personal Wireless Communications*, India, Feb.,1996.

[8] H. Asplund, A. A. Glazunov, A. F. Molisch, K. I. Pedersen and M. Steinbauser, "The COST 259 directional channel model-part II: macrocells," *IEEE Trans. Wireless Commun.*, vol.5, no.12, pp.3434–3450, 2007.

[9] P. Tichavshy, C.H. Muravchik, and A. Nehorai, "Posterior Cramer-Rao Bounds for discrete-time nonlinear filtering", *IEEE Trans. on Signal Processing*, vol.46, no.5, pp. 1386-1396, 1998.

[10] R. Taylor, B. R. Flanagan, and J. A. Uber, "Computing the recursive posterior Cramero-Rao bound for a nonlinear nonstationary system,", in *ICASSP*,Hong Kong, April 2003, pp.673-676.

[11] S. J. Julier and J. K. Uhlmann, "Unscented filtering and nonlinear estimation", *Proceedings of the IEEE*, vol.92, no.3,pp. 401-422, March 2004.

[12] "Revision of the commissions rules to insure compatibility with enhanced 911 emergency calling systems," Tech. Rep. RM-8143, Fed. Commun. Commission (FCC), Washington, DC, 1996.

[13] M. Hernandez, B. Ristic, A. Farina, A. Sathyan, T. Kirubarajan, "Performance measure for Markovian switching systems using best-fitting Gaussian distributions," *IEEE Trans. Aerospace and Electronic Systems*, vol.44, no.2, pp.724-747, April 2008.

[14] M. Morelande, M. Hernandez, B. Ristic, "A new best fitting Gaussian performance measure for jump Markov systems,", in *Int. Conf. Information Fusion* , Cologne, Germany, June 2008, pp.1-6.