EXPLOITING GEOMETRIC TRANSLATIONS IN TLS BASED ROBOT LOCALIZATION FROM LANDMARK BEARINGS

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ABSTRACT

In a previous study we showed that the total least squares (TLS) estimation performance for bearings-only localization is dependent on coordinate shifts. In this paper we extend this study to TLS-based robot localization from landmark bearings. We show that unlike nonlinear least squares estimators the TLS estimation performance is sensitive to where the local Cartesian coordinates are placed in the two-dimensional plane. Thus for a given set of landmark bearing measurements collected by a robot, the TLS performance is influenced by where the local coordinates are placed. The dependence of the TLS estimation bias on local coordinate translations is due to the dependence of TLS perturbations on local coordinate translations. We observe that the TLS estimate performs almost as well as the maximum likelihood estimate as the origin of the local coordinates is placed sufficiently away from the landmarks.

1. INTRODUCTION

In mobile robotics applications accurate determination of the location and orientation of a robot based on information that can be gathered from landmarks is an important research problem. A particularly attractive localization technique is landmark bearings localization in which the robot measures the angles of signals received from multiple landmarks at known locations. The noise in angle measurements necessitates the use of an estimation algorithm. Mobile robot localization from landmark bearings has been an active research area (see e.g. [1] and the references therein). A popular localization method is the so-called geometric solution that is predicated on circle intersection using subtended angles [2]. A similar method was also employed in scan-based emitter localization for scanning radars [3] . For more than three landmarks, the geometric solution loses its appeal due to the nonlinear nature of the estimation problem and performance penalties associated with its linearization.

A computationally attractive approach to robot localization was proposed in [1], which formulates the estimation problem as a linear homogeneous equation with well-defined constraints to find a unique solution. This algorithm is very similar to the well-known pseudolinear estimator (PLE) that was proposed in [4] for bearings-only tracking. The main difference is the unknown orientation angle that needs to be estimated in the case of robot localization. This also makes the estimation problem a homogenous linear matrix equation that can be solved by reduced-rank matrix approximation based on total least squares (TLS) as was done in [1].

In a previous study we applied TLS to the bearings-only localization problem and observed that the TLS localization performance is dependent on coordinate system translations [5]. A key finding of this work was that unlike the PLE, the TLS estimator attempts to correct the errors in both the measurement matrix and the data vector, which can lead to improved bias performance if the origin of the coordinate system is chosen appropriately. A formal proof of the dependence of the TLS estimation bias and mean-squared error (MSE) on local coordinate translations for a given geometry was also provided based on preliminary observations made in [6], [7]. In this paper we extend this study to TLS-based robot localization from landmark bearings. We show that unlike the maximum likelihood estimator which is asymptotically unbiased and efficient, the TLS estimation performance is highly sensitive to where the local Cartesian coordinates are placed in the two-dimensional plane. For a given set of landmark bearing measurements collected by a robot, the TLS performance is influenced by where the local coordinates are placed, i.e., how the landmark locations are recorded. The dependence of the TLS estimation bias on local coordinate translations is due to the dependence of TLS perturbations on local coordinate translations.

2. ROBOT LOCALIZATION FROM LANDMARK BEARINGS

In robot localization from landmark bearings the robot measures the bearing angles of signals it receives from N landmarks. The locations of the landmarks in the global coordinate system, $\boldsymbol{r}_i$, $i = 1, \ldots, N$, are known to the robot. Based on the bearing angle measurements and prior knowledge of the landmark locations, the robot estimates its own location $\boldsymbol{p}$ and orientation ϕ with respect to the global coordinate system. The robot localization problem considered in this paper is shown in Fig. 1.

The bearing angle measurements are modelled as

$$\hat{\theta}_i = \theta_i + n_i \tag{1}$$

where the additive bearing noise n_i is assumed to be i.i.d. zero-mean Gaussian with variance σ^2.

The distribution of the bearing noise is governed by the bearing angle estimation method employed. The Gaussian bearing noise is an approximation of the actual bearing noise which is of necessity a random vari-

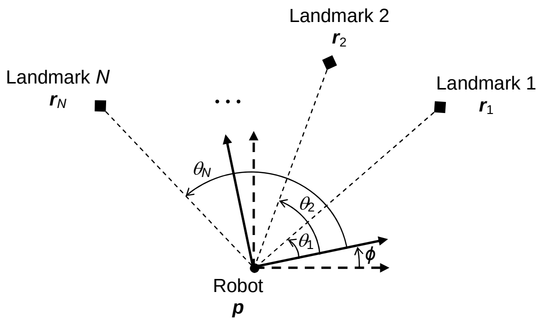


Figure 1: Robot localization from landmark bearings. The angle ϕ is the orientation of robot with respect to global coordinates.

able with finite support unlike a Gaussian random variable [5]. The finite support requirement can be approximated by a sufficiently small noise variance.

3. OVERVIEW OF LOCALIZATION ESTIMATES

3.1 Maximum Likelihood Estimate

Under the Gaussian noise assumption it can be shown that the maximum likelihood estimate (MLE) for the robot location and orientation is simply given by the solution of the nonlinear least-squares problem:

$$\begin{bmatrix} \hat{\boldsymbol{p}}_{ML} \\ \hat{\phi}_{ML} \end{bmatrix} = \arg\min_{\boldsymbol{p},\ \phi} J_{ML}(\boldsymbol{p}, \phi) \tag{2}$$

where $J_{ML}(\boldsymbol{p}, \phi)$ is the so-called maximum likelihood cost function:

$$J_{ML}(\boldsymbol{p}, \phi) = \sum_{i=1}^{N} \left(\hat{\theta}_i + \phi - \angle(\boldsymbol{r}_i - \boldsymbol{p})\right)^2. \tag{3}$$

Here $\angle \boldsymbol{x}$ denotes the bearing angle of the two-dimensional vector $\boldsymbol{x}$. In general the MLE does not have a closed-form solution. Grid search or numerical iterative minimization techniques such as the Gauss-Newton, Newton-Raphson or the Nelder-Mead simplex method can be employed to find the MLE. However a common problem with these search techniques is susceptibility to divergence and high computational complexity. A good initial guess is often a must in order to avoid divergence and instability.

3.2 TLS Estimate

For the bearing line emanating from landmark i we can write

$$\begin{aligned} \tan(\theta_i + \phi) &= \frac{\sin(\theta_i + \phi)}{\cos(\theta_i + \phi_i)} \\ &= \frac{r_{iy} - p_y}{r_{ix} - p_x} \end{aligned} \tag{4}$$

where $\boldsymbol{r}_i = [r_{ix}, r_{iy}]^T$ and $\boldsymbol{p} = [p_x, p_y]^T$. Rearranging the above equation we have

$$[\sin(\theta_i+\phi), -\cos(\theta_i+\phi)]\boldsymbol{p} = [\sin(\theta_i+\phi), -\cos(\theta_i+\phi)]\boldsymbol{r}_i. \tag{5}$$

Define the clockwise rotation matrix

$$\boldsymbol{R}_\phi = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \tag{6}$$

and

$$\boldsymbol{\nu}_i = \begin{bmatrix} \sin\theta_i \\ -\cos\theta_i \end{bmatrix}. \tag{7}$$

Using $\boldsymbol{R}_\phi$ and $\boldsymbol{\nu}_i$, (5) can be rewritten as

$$\begin{aligned} \boldsymbol{\nu}_i^T \boldsymbol{R}_\phi \boldsymbol{p} &= \boldsymbol{\nu}_i^T \boldsymbol{R}_\phi \boldsymbol{r}_i \\ &= [\boldsymbol{\nu}_i^T \boldsymbol{r}_i, \boldsymbol{\gamma}_i^T \boldsymbol{r}_i] \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} \end{aligned} \tag{8}$$

where

$$\boldsymbol{\gamma}_i = \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix}. \tag{9}$$

Collecting all the unknowns in one vector we get

$$[\boldsymbol{\nu}_i^T, \boldsymbol{\nu}_i^T \boldsymbol{r}_i, \boldsymbol{\gamma}_i^T \boldsymbol{r}_i] \underbrace{\begin{bmatrix} \boldsymbol{p}_\phi \\ \cos\phi \\ \sin\phi \end{bmatrix}}_{\boldsymbol{w}} = 0 \tag{10}$$

where $\boldsymbol{p}_\phi = -\boldsymbol{R}_\phi \boldsymbol{p}$. Stacking the above equation for $i = 1, \ldots, N$ we obtain the following homogeneous matrix equation

$$\begin{bmatrix} \boldsymbol{\nu}_1^T & \boldsymbol{\nu}_1^T \boldsymbol{r}_1 & \boldsymbol{\gamma}_1^T \boldsymbol{r}_1 \\ \boldsymbol{\nu}_2^T & \boldsymbol{\nu}_2^T \boldsymbol{r}_2 & \boldsymbol{\gamma}_2^T \boldsymbol{r}_2 \\ & \vdots & \\ \boldsymbol{\nu}_N^T & \boldsymbol{\nu}_N^T \boldsymbol{r}_N & \boldsymbol{\gamma}_N^T \boldsymbol{r}_N \end{bmatrix} \boldsymbol{w} = \boldsymbol{0}. \tag{11}$$

Replacing the true bearing angles with their noisy measurements, we obtain

$$\underbrace{\begin{bmatrix} \hat{\boldsymbol{\nu}}_1^T & \hat{\boldsymbol{\nu}}_1^T \boldsymbol{r}_1 & \hat{\boldsymbol{\gamma}}_1^T \boldsymbol{r}_1 \\ \hat{\boldsymbol{\nu}}_2^T & \hat{\boldsymbol{\nu}}_2^T \boldsymbol{r}_2 & \hat{\boldsymbol{\gamma}}_2^T \boldsymbol{r}_2 \\ & \vdots & \\ \hat{\boldsymbol{\nu}}_N^T & \hat{\boldsymbol{\nu}}_N^T \boldsymbol{r}_N & \hat{\boldsymbol{\gamma}}_N^T \boldsymbol{r}_N \end{bmatrix}}_{\boldsymbol{A}} \boldsymbol{w} \approx \boldsymbol{0} \tag{12}$$

where $\hat{\boldsymbol{\nu}}_i = [\sin\hat{\theta}_i, -\cos\hat{\theta}_i]^T$ and $\hat{\boldsymbol{\gamma}}_i = [\cos\hat{\theta}_i, \sin\hat{\theta}_i]^T$. The total least-squares (TLS) estimate of $\boldsymbol{w}$ is defined by

$$\min \|\boldsymbol{\Delta}\|_F^2 \tag{13a}$$

such that

$$(\boldsymbol{A} + \boldsymbol{\Delta})\hat{\boldsymbol{w}} = \boldsymbol{0} \quad \text{and} \quad \|\hat{\boldsymbol{w}}(3:4)\|^2 = 1 \tag{13b}$$

where $\|\cdot\|_F$ denotes the Frobenius matrix norm. The TLS estimate is easily obtained by using the singular value decomposition (SVD) of $\boldsymbol{A}$:

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{U\Sigma V}^T \\ &= \sum_{i=1}^{4} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T \end{aligned} \tag{14}$$

where $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \sigma_4$ are the ordered singular values, and $\boldsymbol{U}$ and $\boldsymbol{V}$ are unitary matrices. For noisy bearings we have $\sigma_4 > 0$, i.e., $\boldsymbol{A}$ is full-rank. A reduced rank estimate of $\boldsymbol{A}$ with minimum Frobenius-norm perturbation $\boldsymbol{\Delta}$ is obtained from [8]

$$\boldsymbol{A} + \boldsymbol{\Delta} = \sum_{i=1}^{3} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T \tag{15}$$

where

$$\boldsymbol{\Delta} = -\sigma_4 \boldsymbol{u}_4 \boldsymbol{v}_4^T. \tag{16}$$

Since $\boldsymbol{V}$ is a unitary matrix, the TLS estimate $\hat{\boldsymbol{w}}$ is given by

$$\hat{\boldsymbol{w}} = \frac{\boldsymbol{v}_4}{\|\boldsymbol{v}_4(3:4)\|} \tag{17}$$

which also implements the norm constraint in (13b). The TLS estimates for robot orientation and location are

$$\hat{\phi} = \angle \hat{\boldsymbol{w}}(3:4) \tag{18a}$$

$$\hat{\boldsymbol{p}} = -\boldsymbol{R}_{\hat{\phi}}^T \hat{\boldsymbol{w}}(1:2). \tag{18b}$$

The TLS estimate has no sign ambiguity since any sign change in $\hat{\boldsymbol{w}}$ is undone by sing reversal in the rotation matrix.

4. EFFECT OF COORDINATE TRANSLATIONS

All entries of $\boldsymbol{A}$ contain noise originating from noisy landmark bearing measurements. The noise in the last two columns of $\boldsymbol{A}$ is scaled by the landmark locations whereas the noise in the first two columns of $\boldsymbol{A}$ are independent of $\boldsymbol{r}_i$ as long as bearings remain the same. As the norm of landmark locations increases, the noise in the last two columns also increases, affecting the TLS estimate. This observation motivates us to study the effect of geometric shifts on the TLS estimate.

We are free to choose any reasonable location in the plane as the origin of the global coordinate system. We will use geometric translations to move the coordinate origin. Shifting the coordinate system by a 2-D vector $\boldsymbol{\psi} = [\psi_x, \psi_y]^T$ results in robot and landmark locations to be shifted:

$$\begin{aligned} \boldsymbol{p}_\psi &= \boldsymbol{p} + \boldsymbol{\psi} \\ \boldsymbol{r}_{\psi,i} &= \boldsymbol{r}_i + \boldsymbol{\psi}. \end{aligned} \tag{19}$$

Coordinate shifts do not affect landmark bearings. After a shift by $\boldsymbol{\psi}$, (12) becomes

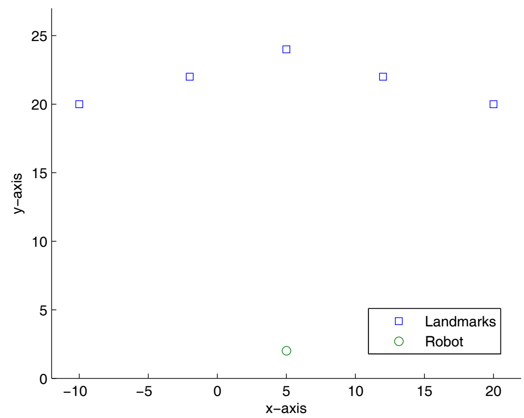
$$\underbrace{\begin{bmatrix} \hat{\boldsymbol{\nu}}_1^T & \hat{\boldsymbol{\nu}}_1^T \boldsymbol{r}_{\psi,1} & \hat{\boldsymbol{\gamma}}_1^T \boldsymbol{r}_{\psi,1} \\ \hat{\boldsymbol{\nu}}_2^T & \hat{\boldsymbol{\nu}}_2^T \boldsymbol{r}_{\psi,2} & \hat{\boldsymbol{\gamma}}_2^T \boldsymbol{r}_{\psi,2} \\ & \vdots & \\ \hat{\boldsymbol{\nu}}_N^T & \hat{\boldsymbol{\nu}}_N^T \boldsymbol{r}_{\psi,N} & \hat{\boldsymbol{\gamma}}_N^T \boldsymbol{r}_{\psi,N} \end{bmatrix}}_{\boldsymbol{A}_\psi} \boldsymbol{w}_\psi \approx \boldsymbol{0}. \tag{20}$$

It is obvious that $\boldsymbol{\psi}$ only changes the noise in the last two columns of $\boldsymbol{A}$. Define the TLS error as

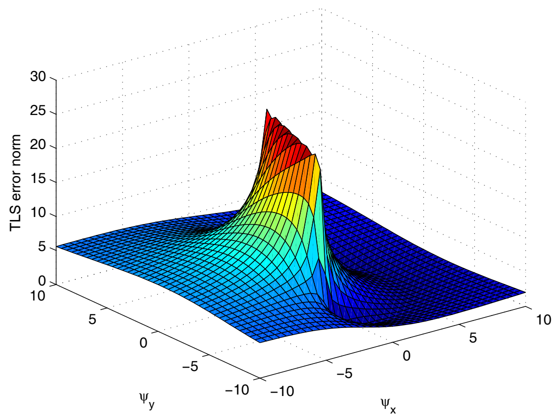
$$\boldsymbol{e}_\psi = \hat{\boldsymbol{p}}_\psi - \boldsymbol{p}_\psi \tag{21}$$

where $\hat{\boldsymbol{p}}_\psi$ is the TLS estimate of robot location after a geometric shift by $\boldsymbol{\psi}$. Since for different $\boldsymbol{\psi}$ we have different noise in $\boldsymbol{A}_\psi$, we argue that the TLS estimation error will change with $\boldsymbol{\psi}$ for a given set of bearing measurements.

To illustrate the dependence of the TLS error on coordinate shifts, consider the robot localization scenario shown in Fig. 2(a) with $N = 5$, $\phi = 30°$ and $\sigma = 2°$. For one realization of noisy bearing measurements, Fig. 2(b) shows the norm of the TLS error $\|\boldsymbol{e}_\psi\|$ as a function of coordinate shifts $\boldsymbol{\psi}$. It is clearly seen that $\boldsymbol{e}_\psi$ changes with $\boldsymbol{\psi}$. The MLE error norm is 3.92 regardless of coordinate shifts.



(a)



(b)

Figure 2: (a) Original robot localization geometry with $\boldsymbol{\psi} = [0, 0]^T$, and (b) TLS estimation error norm as a function of geometric shifts. The TLS error is clearly dependent on coordinate shifts.

5. IMPROVED TLS ESTIMATE UTILIZING COORDINATE SHIFTS

The dependence of TLS error on coordinate shifts can be viewed as an opportunity for improving TLS localization performance. The objective is to find a coordinate shift or the location of the coordinate system origin that minimizes the bias and MSE of the resulting TLS estimate.

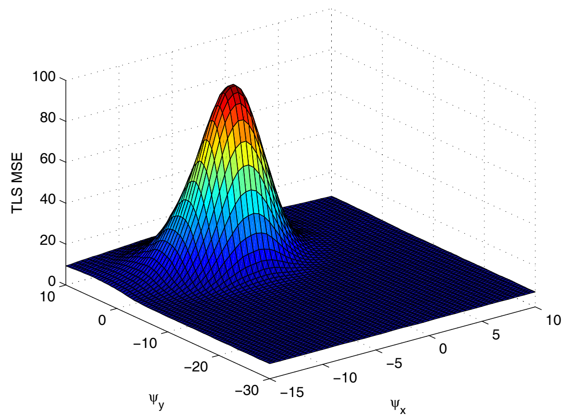
Unfortunately the lack of a closed-form expression for the TLS bias and MSE makes it rather challenging to find such optimal coordinate shift. Formally we have the following optimization problem to solve:

$$\min_{\boldsymbol{\psi}} E\{\|\hat{\boldsymbol{p}}_\psi - \boldsymbol{p}_\psi\|^2\} \tag{22}$$

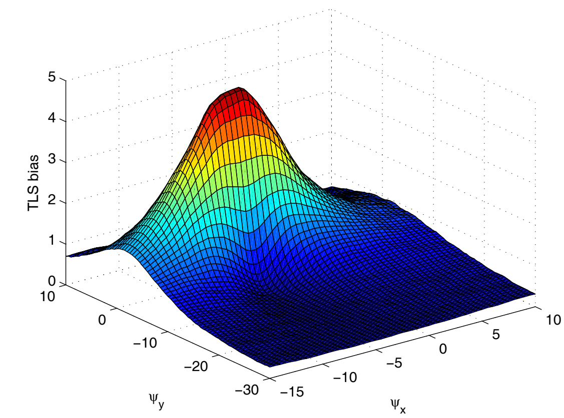
where $E\{\|\hat{\boldsymbol{p}}_\psi - \boldsymbol{p}_\psi\|^2\}$ is the MSE of the TLS estimate. Likewise one can also employ bias minimization as an optimization criterion:

$$\min_{\boldsymbol{\psi}} \|E\{\hat{\boldsymbol{p}}_\psi - \boldsymbol{p}_\psi\}\| \tag{23}$$

where $\|E\{\hat{\boldsymbol{p}}_\psi - \boldsymbol{p}_\psi\}\|$ is the TLS bias norm.



(a)



(b)

Figure 3: TLS MSE and bias norm as a function of $\boldsymbol{\psi}$. The bias norm is minimized at $\boldsymbol{\psi} = [-5.5, -12.5]^T$.

In Fig. 3 the TLS MSE and bias norm are plotted as a function of $\boldsymbol{\psi}$ for the the robot localization scenario shown in Fig. 2(a). The MSE and bias were estimated using 2000 Monte Carlo runs. The bias norm is minimized at $\boldsymbol{\psi} = [-5.5, -12.5]^T$. The MSE appears to be relatively flat outside the region around $\boldsymbol{\psi} = -[5, 2]^T$.

The centre of gravity of landmark locations is a good choice for the coordinate origin in general as this tends to reduce the noise effects on $\boldsymbol{A}$ as a result of decreasing the norm of landmark location vectors. In this case the geometric shift vector is simply given by

$$\boldsymbol{\psi} = -\bar{\boldsymbol{r}} = -\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{r}_i. \tag{24}$$

Table 1 summarizes the bias and MSE of performance of the TLS and MLE for the robot localization problem in Fig. 2(a) before and after shifting the geometry by $\boldsymbol{\psi} = -\bar{\boldsymbol{r}}$. The bias and MSE were estimated using 5000 Monte Carlo runs. The TLS estimate achieves a significant performance improvement after the geometric shift by $\boldsymbol{\psi} = -\bar{\boldsymbol{r}}$, attaining bias and MSE levels comparable to the MLE. The adoption of (24) as a desirable geometric shift was based on the belief that making $\|\boldsymbol{r}_{\psi,i}\|$ small would be advantageous in terms of noise minimization in the matrix $\boldsymbol{A}_\psi$. Following the same line of reasoning the use of excessively large $\|\boldsymbol{\psi}\|$ would be discouraged. However it turns out that for $\boldsymbol{\psi} = [50000, 50000]^T$ the bias norm and MSE of the TLS estimate are 0.2833 and 7.5245, respectively. These results compare well both with TLS at $\boldsymbol{\psi} = -\bar{\boldsymbol{r}}$ and MLE.

Table 1: Bias and MSE comparison

	Bias Norm		MSE	
	Org. Geom.	Shift by $\boldsymbol{\psi} = -\bar{\boldsymbol{r}}$	Org. Geom.	Shift by $\boldsymbol{\psi} = -\bar{\boldsymbol{r}}$
TLS	2.7687	0.2101	29.1679	7.4466
MLE	0.2282	0.2282	7.5711	7.5711

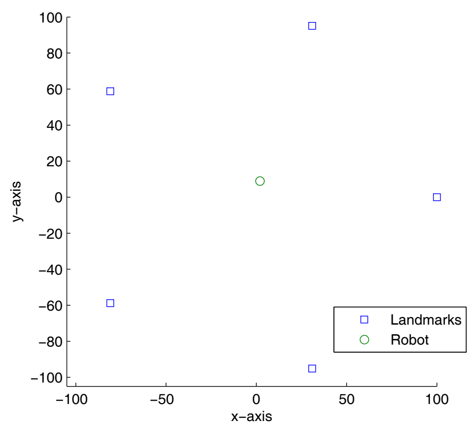
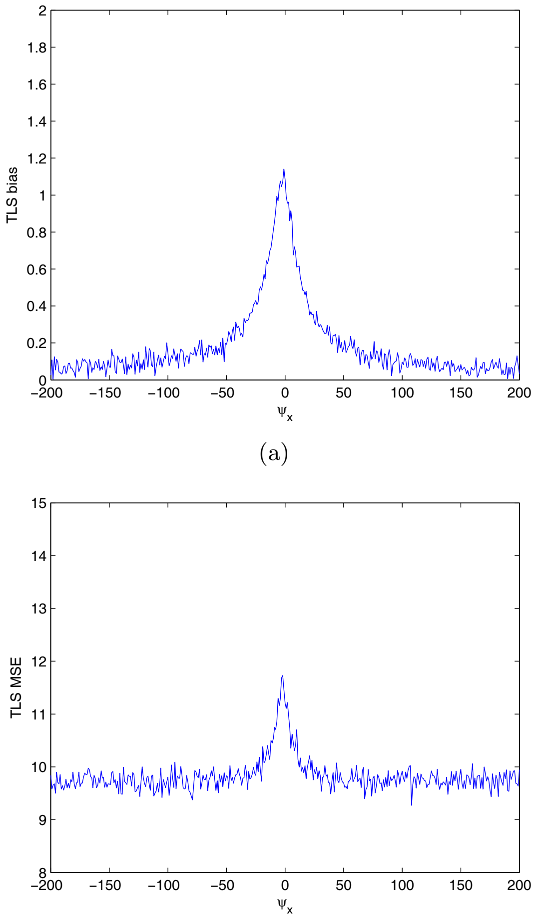


Figure 4: Robot localization geometry with circular landmark configuration.

Despite its intuitive appeal the geometric shift by $\boldsymbol{\psi} = -\bar{\boldsymbol{r}}$ does not always yield good estimation performance. This is particularly evident in robot localization scenarios where the robot is surrounded by landmarks. In such configurations the coordinate origin should be placed far away from the center of gravity of landmarks in order to improve the TLS estimation performance. We illustrate this observation with an example. Consider the robot localization geometry shown in Fig. 4. There are $N = 5$ landmarks uniformly distributed on a circle of radius 100 centred at the origin. The robot is at $\boldsymbol{p} = [2, 9]^T$. The robot orientation angle is $\phi = 30°$ and the bearing noise standard deviation is $\sigma = 2°$. Fig. 5 shows the TLS bias norm and MSE estimates as a function of ψ_x with ψ_y fixed at 0. Note that in this simulation $\bar{\boldsymbol{r}} = \boldsymbol{0}$ and therefore $\boldsymbol{\psi} = \boldsymbol{0}$ corresponds to $\boldsymbol{\psi} = -\bar{\boldsymbol{r}}$. As is evident from Fig. 5, neither bias nor MSE are minimized at $\boldsymbol{\psi} = -\bar{\boldsymbol{r}}$. In fact both bias and MSE appear to settle at their minimum value as $|\psi_x| \to \infty$.



(a)

(b)

Figure 5: TLS bias norm and MSE as a function of ψ_x with $\psi_y =$ fixed at 0.

Table 2 lists the bias and MSE of the TLS and MLE for the robot localization problem in Fig. 4 for two geometric shifts, viz., $\boldsymbol{\psi} = -\bar{\boldsymbol{r}} = \boldsymbol{0}$ and $\boldsymbol{\psi} = [50000, 50000]^T$. The bias and MSE were estimated using 5000 Monte Carlo runs. The TLS estimate at $\boldsymbol{\psi} = -\bar{\boldsymbol{r}} = \boldsymbol{0}$ performs worse than at $\boldsymbol{\psi} = [50000, 50000]^T$. What is more the geometric shift $\boldsymbol{\psi} = [50000, 50000]^T$ enables TLS estimate to achieve bias and MSE results comparable to those of the MLE. We conclude that contrary to the line of reasoning that has led to (24), one may place the coordinate origin far away from the landmarks to achieve significant performance improvement for the TLS estimate in most cases on par with the MLE.

Table 2: Bias and MSE comparison

	Bias Norm		MSE	
	Shift by $\boldsymbol{\psi} = -\bar{\boldsymbol{r}}$	Large Shift	Shift by $\boldsymbol{\psi} = -\bar{\boldsymbol{r}}$	Large Shift
TLS	1.0972	0.0602	11.5736	9.8526
MLE	0.0327	0.0327	9.7439	9.7439

6. CONCLUSION

This paper has investigated the dependence of the TLS estimation performance on local coordinate translations in robot localization from landmark bearings. Several numerical examples were presented to compare the bias and MSE performance of the TLS and MLE. It was observed that the TLS estimate performs almost as well as MLE as the origin of the local coordinates is placed sufficiently away from the landmarks. However the optimal translation for TLS is in general unknown. For non-circular landmark configurations an effective way of achieving good TLS estimation performance is to place the origin of the local coordinates at the centre of gravity of the landmarks. The same is not necessarily true for circular landmark configurations surrounding the robot.

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