A SUBBAND ALGORITHM FOR ESTIMATING THE PARAMETERS OF TWO-DIMENSIONAL EXPONENTIAL SIGNALS

El-Hadi Djermoune, David Brie and Marc Tomczak

CRAN, Nancy-Université, CNRS
Boulevard des Aiguillettes, BP 239, 54506 Vandoeuvre Cedex, France
phone: + (33) 383 68 44 72, fax: + (33) 383 68 44 62, email: {firstname.lastname}@cran.uhp-nancy.fr
web: www.cran.uhp-nancy.fr

ABSTRACT

We propose an adaptive subband decomposition scheme designed to estimate the parameters of two-dimensional (2D) exponential signals from large data sets. The principle of the method consists to perform recursive 2D decimation and estimation steps. At each resulting subband, a stopping rule is evaluated to decide whether the decomposition should be continued or stopped. The proposed rule is based on the Fisher g-test applied on the estimation error. The advantages and limitations of the subband approach are discussed through a simulation example. The method is then demonstrated on an experimental 2D nuclear magnetic resonance spectroscopy signal.

1. INTRODUCTION

In many applications, such as sonar, radar, mobile communications, and nuclear magnetic resonance (NMR) spectroscopy, signals can be modelled as the sum of two-dimensional (2D) damped or undamped complex exponentials (called modes) in additive noise. From a 2D data set, it is desired to estimate the parameters of the model. For instance, in multidimensional NMR spectroscopy, the frequencies and damping factors of damped sinusoids are crucial in determining protein structures [8]. For this issue, Fourier-based algorithms such as the correlogram or periodogram methods are quite simple and fast, but suffer from the Rayleigh resolution limit. Hence, to improve resolution, several methods have been proposed in the signal processing literature. They are often extensions of 1D approaches to the 2D case, such as 2D IQML [1], 2D MUSIC [8], TLS-Prony [13], Matrix Pencil [5], etc. All these methods provide high-resolution estimation and achieve the Cramér-Rao bound under some mild conditions. In particular, a rule of thumb for subspace-based methods to achieve a minimum frequency variance is to fix the so-called "prediction order" to about third to half the number of samples in each dimension [6, 7, 3]. So, when the data set is relatively small, all the methods apply well. On the other hand, if the data set or the number of modes is very large, it is often difficult to obtain satisfactory results due to prohibitive computational costs (inversion of large matrices, rooting of high-order polynomials, etc.).

In this paper, we propose a subband estimation scheme in which the data set is decomposed into a certain number of data sets, each being much smaller than the original one and more favorable from a numerical point of view. The use of subband decomposition prior to the estimation process has been known for several years [10, 12, 15]. The originality of the method proposed here lies in the fact that the decomposition is carried out adaptively. The purpose of this scheme is to select the set of subbands in which the spectral information (frequencies) is localised. Hence, the decomposition is performed recursively. As soon as a resulting subband is numerically tractable, an estimation procedure is performed. Then, to detect possible missed modes, the Fisher g-test [9] is evaluated using the estimation residuals: the decomposition of the subband is stopped only if no periodicity is found in the residual signal.

In the next Section, we present the 2D damped exponential model and discuss an estimation method. In Section 3, the proposed approach is described. In particular, we state the principles of the subband decomposition and the stopping rule. A simulation example and an application to an experimental data set is given in Section 4. Finally, conclusions are given in Section 5.

2. PARAMETER ESTIMATION OF 2D EXPONENTIAL SIGNALS

We consider the following 2D exponential model:

$$d(n,m) = \sum_{i=1}^{I} \lambda_i z_i^n w_i^m + e(n,m) \tag{1}$$

for $n = 0,\dots,N-1$ and $m = 0,\dots,M-1$. Here, $z_i = \exp(-\alpha_{1,i} + j\omega_{1,i})$ and $w_i = \exp(-\alpha_{2,i} + j\omega_{2,i})$ are the components of the mode (z_i, w_i) with complex amplitude λ_i. A pure sinusoidal model involves $\alpha_{1,i} = \alpha_{2,i} = 0, \forall i$, and a completely damped one corresponds to $\alpha_{1,i} > 0$ and $\alpha_{2,i} > 0$. The term $e(n,m)$ is assumed to be a two-dimensional Gaussian complex white noise. The problem is to estimate the number of modes I and the set of parameters $\{z_i, w_i, \lambda_i\}_{i=1}^{I}$, given the noisy measurements $d(n,m)$.

Several algorithms have been developed to solve this problem. Without loss of generality, let us focus on the 2D TLS-Prony algorithm proposed in [13] (see [17] and references therein for more details on 2D subspace-based methods and their performances). The starting point of the TLS-Prony method is the following form of model (1):

$$d(n,m) = \sum_{k=1}^{K} c_{k,m} p_{x_k}^n + e(n,m) \tag{2}$$

where

$$c_{k,m} = \sum_{l=1}^{L_k} a_{k,l} p_{y_{k,l}}^m. \tag{3}$$

p_{x_k} is the kth x-mode (x-component of 2D exponential), $p_{y_{k,l}}$ is the k,lth y-mode, $a_{k,l}$ is the k,lth amplitude coefficient

and L_k is the number of y-modes corresponding to the kth x-mode. In order to estimate the 2D signal parameters, the idea is to perform a set of 1D estimations using (2) and (3). Indeed, it is clear from (2) that the sequence obtained for a fixed value of m is a 1D exponential signal whose parameters may be estimated with a 1D high-resolution technique.
The TLS-Prony algorithm for 2D frequency estimation in [13] consists of the following four steps:

- Using (2), form the backward linear prediction equation:

$$\mathbf{S}\begin{bmatrix} 1 \\ \mathbf{b} \end{bmatrix} = 0 \tag{4}$$

where $\mathbf{b} = [b_1, b_2, \cdots, b_P]^T$ is the prediction vector and $P \geqslant K$ is the prediction order. The matrix $\mathbf{S}$ is formed from the data set as $\mathbf{S} = [\mathbf{D}_0^T | \mathbf{D}_1^T | \cdots | \mathbf{D}_{M-1}^T]^T$, where $\mathbf{D}_i$ is a Hankel matrix:

$$\mathbf{D}_i = \begin{bmatrix} d(0,i) & d(1,i) & \cdots & d(P,i) \\ d(1,i) & d(2,i) & \cdots & d(P+1,i) \\ \vdots & \vdots & & \vdots \\ d(N-P-1,i) & d(N-P,i) & \cdots & d(N-1,i) \end{bmatrix}$$

for $i = 0, \cdots, M-1$. In order to achieve a minimum frequency variance, P have to be chosen in the range $[N/3, N/2]$, with $P \approx N/3$ for undamped sinusoids [6] and $P \approx N/2$ for damped ones [3]. By performing the singular value decomposition (SVD) of the matrix $\mathbf{S}$, one can estimate the number of x-modes $\hat{K}$ using a theoretical information criterion, such as MDL or AIC (see [16]). Then, (4) should be solved in the total least squares (TLS) sense [11] with an SVD truncation to obtain $\hat{b}$. Finally, the estimated x-modes are found by

$$\hat{p}_{x_k} = \frac{1}{\text{zero}_k(\hat{B}(z))}, \quad k = 1, 2, ..., \hat{K}$$

where $B(z) = 1 + b_1 z + ... + b_P z^P$ (the $P - \hat{K}$ zeros of $\hat{B}(z)$ lying inside the unit circle must be discarded).
- For each time index $m = 0, ..., M-1$, compute the x-amplitude coefficients $\hat{c}_{k,m}$ in the least squares sense using (2) and the estimated modes $\hat{p}_{x_k}$.
- For each x-mode $\hat{p}_{x_k}$, $k = 1, ..., \hat{K}$, obtain the corresponding $\hat{L}_k$ y-modes $\hat{p}_{y_{k,l}}$ from (3) using once again the 1D TLS-Prony approach (here the prediction equations are made over the m index, for a fixed k).
- Compute the amplitude coefficients $\hat{a}_{k,l}$ for $k = 1, ..., \hat{K}$ by solving the set of Vandermonde equations obtained from (3) in the least squares sense.

Finally, the 2D signal parameters $(\hat{z}_i, \hat{w}_i)$ with amplitudes $\hat{\lambda}_i$ correspond to the set of couples $(\hat{p}_{x_k}, \{\hat{p}_{y_{k,l}}\}_{l=1}^{\hat{L}_k})$ with amplitudes $\{\hat{a}_{k,l}\}_{l=1}^{\hat{L}_k}$. The total number of estimated modes is then:

$$\hat{I} = \sum_{k=1}^{\hat{K}} \hat{L}_k.$$

Generally speaking, the use of a high-resolution technique to estimate the 2D parameters leads to good performances in terms of precision and resolution. Unfortunately, when the number of samples is large, it is often difficult to benefit from these performances. For instance, the dimension of the matrix $\mathbf{S}$ in (4) is about $NM \times P$. So it is clear that, for large signals, it is necessary to reduce the problem complexity by using, for example, subband decomposition.

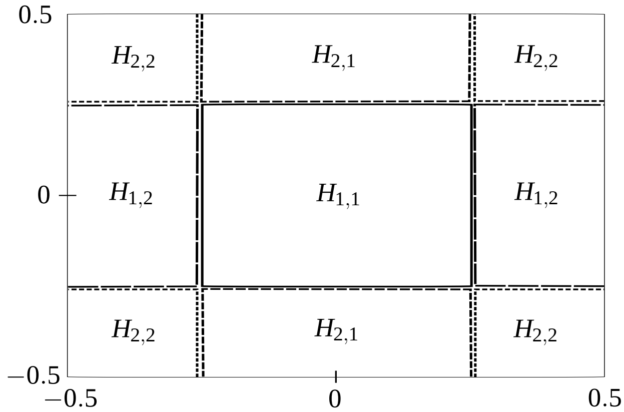


Figure 1: Schematic representation of the ideal frequency responses $\{H_{i,j}(f_1, f_2)\}_{i,j=1}^{2}$ of the subband 2D filters.

3. ADAPTIVE SUBBAND ESTIMATION

3.1 2D Signal Decimation

Subband decomposition is achieved classically through successive filtering and decimation stages, say, by a factor 2. Four 2D decimation filters are necessary as illustrated in figure 1. The same result can be obtained using a single lowpass 2D filter (denoted by $H_{1,1}$ in figure 1), but conveniently centered on each of the four subbands. So, without loss of generality, we consider here a lowpass filter with impulse response $h(n,m)$. The corresponding subband signal is obtained as follows:

$$d'(n,m) = \sum_{n'} \sum_{m'} h(n', m') d(2n - n', 2m - m')$$

for $n = 0, 1, \cdots, N'-1$ and $m = 0, 1, \cdots, N'-1$ ($N' = N/2$, $M' = M/2$). Note that the decimation reduces the data set size and possibly the number of modes due to the filtering operation. The previous filtering and decimation process may then be repeated as long as the desired subband signal size is not reached.

Note that, in practice, the subband filters overlap each other and thus a single frequency may appear in two contiguous subbands. To avoid this problem, we use the overdetermined filterbank structure presented in [14]. Moreover, the transient introduced by the filter is suppressed so that the model of the subband signals is still a sum of a (possibly reduced) number of modes which can be estimated by the TLS-Prony method presented in the previous section. Assume that the number of mode in a given subband is $I' \leqslant I$:

$$d'(n,m) = \sum_{i=1}^{I'} a'_i z_i'^n w_i'^m + e'(n,m).$$

If $\hat{I}'$ modes are detected by the TLS-Prony approach, then the residual signal is defined by:

$$r(n,m) = d'(n,m) - \sum_{i=1}^{\hat{I}'} \hat{a}'_i \hat{z}_i'^n \hat{w}_i'^m \tag{5}$$

for $n = 0,\ldots,N'-1$ and $m = 0,\ldots,M'-1$. Ideally, if all subband modes are correctly retrieved, the residuals are close to a white noise. If one or more modes are missed, then the signal $r(n,m)$ is no more white. The stopping rule described below is based on this observation.

3.2 Stopping Rule

The adaptive subband decomposition method we propose here aims at satisfying the following constraint. If in any subband, the estimation procedure misses some modes, the decomposition should continue. On the contrary, if all spectral information has been retrieved, there is no need to carry on the decomposition, which thus should be stopped. In order to check for the presence of some "hidden periodicities" in the residual signal, several measures may considered such as the MDL criterion, whiteness test [4, 2], etc. Here, we propose to use the well-known Fisher g-statistic [9]. It was originally developed for 1D sequences, but it is easily extendable to 2D signals. Denote by $Q(k_1,k_2)$ the periodogram of the residual signal in (5), evaluated at standard frequencies:

$$Q(k_1,k_2) = \frac{1}{N'M'}\left|\sum_{n=0}^{N'-1}\sum_{m=0}^{M'-1} r(n,m)e^{-j2\pi nk_1/N'}e^{-j2\pi mk_2/M'}\right|^2.$$

Let $\{q_k\}_{k=0}^{N'M'}$ the sequence obtained by unfolding $Q(k_1,k_2)$ into a 1D signal. The Fisher test is based on the statistic [9]:

$$g = \frac{\max(q_k)}{\sum_{k=0}^{N'M'-1} q_k}. \quad (6)$$

The distribution of g under the null hypothesis (i.e. when $r(n,m)$ is a Gaussian white noise) is:

$$p\left[g > \frac{z}{N'M'}\right] \approx N'M'\exp(-z/2).$$

Hence, if one chooses a significance level (false alarm rate) α, the detection threshold is then given by:

$$z_\alpha = \frac{2}{N'M'}\ln(N'M'/\alpha).$$

We conclude that $r(n,m)$ contains a periodic component if $g > z_\alpha$.

3.3 Algorithm

The proposed algorithm can be summarised in the following steps:

1. Initialisation: choose a decimation filter and a significance level α.
2. Split the data set into 4 subbands.
3. For each of the resulting subbands, do the following:
   (a) If the subband data set is still large, go to step 4.
   (b) Estimate the subband parameters using the TLS-Prony algorithm presented in Section 2.
   (c) Test for missed modes in the residual signal using the statistic (6). The current subband should not be decomposed again if $g \leqslant z_\alpha$, otherwise it is marked "decomposable".
4. Search for a decomposable band from the whole tree, obtain its children by further decimation and go to step 3.
5. Convert the subband parameters to their fullband values.

Table 1: Parameters of the simulation signal.

Mode	$\ln z_i$	$\ln w_i$	λ_i
1	$-0.05+j2\pi 0.10$	$-0.05+j2\pi 0.10$	1.0
2	$-0.05+j2\pi 0.14$	$-0.05+j2\pi 0.14$	1.0

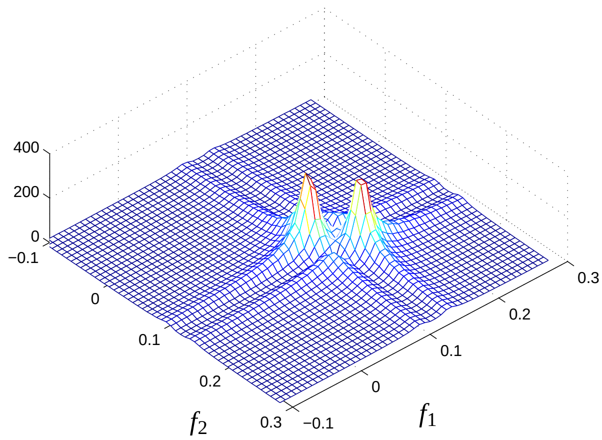


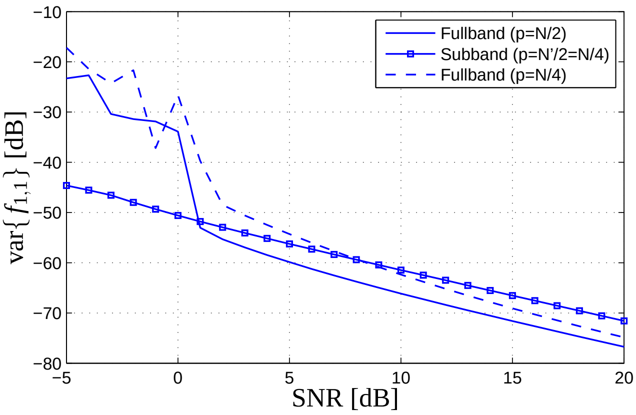
Figure 2: Fourier transform of the noiseless simulation signal.

4. EXPERIMENTS

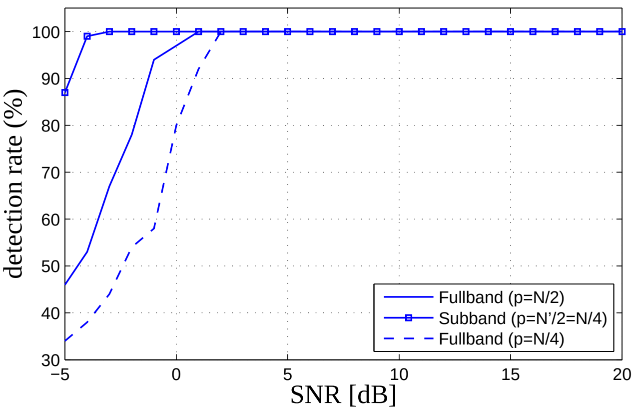
4.1 Simulation Signal

The purpose of this section is to compare the fullband and subband estimations. The simulation signal contains two 2D modes (z_1,w_1) and (z_2,w_2) with parameters given in table 1. The generated signal forms a data set of size 60×60 ($N = M = 60$), whose Fourier transform is shown in figure 2. The signal-to-noise ratio (SNR) varies between -5 dB and 20 dB. For each SNR, a Monte Carlo simulation is made using 200 realisations of the additive white noise. For subband decomposition, the decimation filter is a separable FIR filter (i.e. $h(n,m) = h_1(n)h_2(m)$) of order 6×6 and the decimation level is 1, in both dimensions. The filter is designed so as to separate the two modes into two different subbands. The comparison of fullband and subband estimations will be achieved through two points of view: the variance of the frequency, and the detection rate (using the MDL criterion).

The variance of the y-frequency of the first mode (i.e. $f_{1,1} = \omega_{1,1}/2\pi$) and the detection rate of z_1 are shown in figures 3(a) and 3(b), respectively. The estimation is made with $\mathbf{p} = [N/2, N/2]^T$ in fullband and $\mathbf{p}' = [N'/2, N'/2]^T$ in subband ($\mathbf{p} = [P_1, P_2]^T$, with P_1 and P_2 are the prediction orders in dimension 1 and 2, respectively). First, at high SNR it is obvious that the fullband estimation variance is better. Now, if one decreases the prediction order in fullband ($P_1 = P_2 = N/4$) in order to reduce the computational time, the fullband and subband variances reach almost the same value. The advantage of subband estimation appears clearly at low SNR: in addition to the computational cost, the variance and detection rate are better, and the threshold SNR is smaller in subband. So, at a small expense of the variance at high SNR, the subband estimation is generally more efficient.



(a)



(b)

Figure 3: Results achieved on the simulation signal. (a) First mode frequency variance; (b) First mode detection rate.

4.2 Experimental NMR Signal

In this second experiment, we consider an experimental 2D NMR signal of size 64×2048. Here again, the decimation filter is separable into two identical 1D filters in each dimension. The 1D filter is designed with an equiripple routine: the ripple amplitude in passband is 0.1 and the stopband attenuation is -60 dB. Since the NMR data set has a large amount of data in the second dimension, we fixed the minimum decimation level (without estimation) to $[1,2]^T$ and the maximum one to $[2,4]^T$. The prediction orders are fixed to $\mathbf{p} = [N'/2, M'/2]^T$, where $N' \times M'$ is the size of the subband signal. The final subbands obtained with our algorithm in the spectral region $[-.25, 0] \times [-.25, -.25]$ are shown in figure 4. One can observe that the decomposition is generally deeper in the spectral regions where several modes are located. On the other hand, for remote modes, the decomposition is stopped at an early level. This is the case for instance with the mode located in the band $[-.0625, 0] \times [-.25, -.125]$. So the method is able to adapt itself to the local complexity of a signal, allowing one to reduce the calculation time, as compared to a uniform decomposition in which several small subbands need to be analysed. The results obtained in some subbands are represented in figure 5, where the estimated modes are indicated with thick circles. For the subband in figure 5(a), 13 modes have been detected among which 10 correspond to

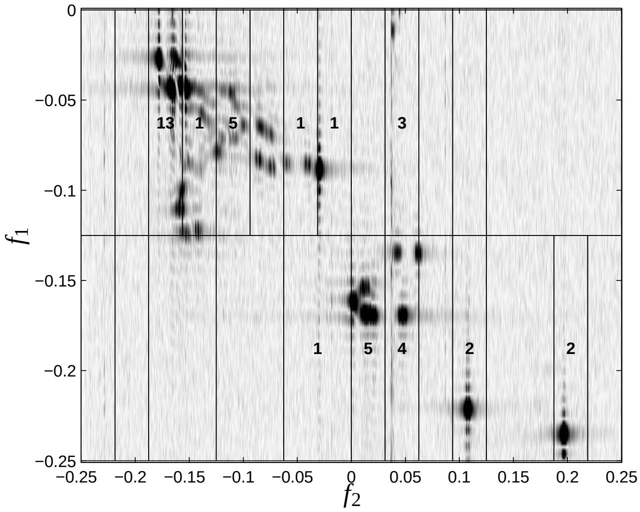


Figure 4: Spectral region $[-.25, 0] \times [-.25, -.25]$ of the NMR signal with the final subbands and the number of estimated modes.

true peaks. We observe also in figure 5(b) that the approach tend to fit a very large peak by several small ones. Finally, thanks to the subband decomposition, some modes that are very close have been resolved, this is shown for example in figure 5(c).

5. CONCLUSION

In this paper, we have proposed an adaptive subband decomposition approach for the analysis of two-dimensional damped or undamped exponentials. This method uses a stopping rule based on the Fisher *g*-test applied on the subband residuals: the decomposition is stopped on a given subband only if the test does not detect any component. Using a simulation signal, it was shown that the subband approach is better than the fullband one at low SNR in terms of detection rate, but the variance is slightly worse. The great advantage of the approach is its low computational complexity as compared to a global one especially for large data sets. This was confirmed on an experimental NMR signal.

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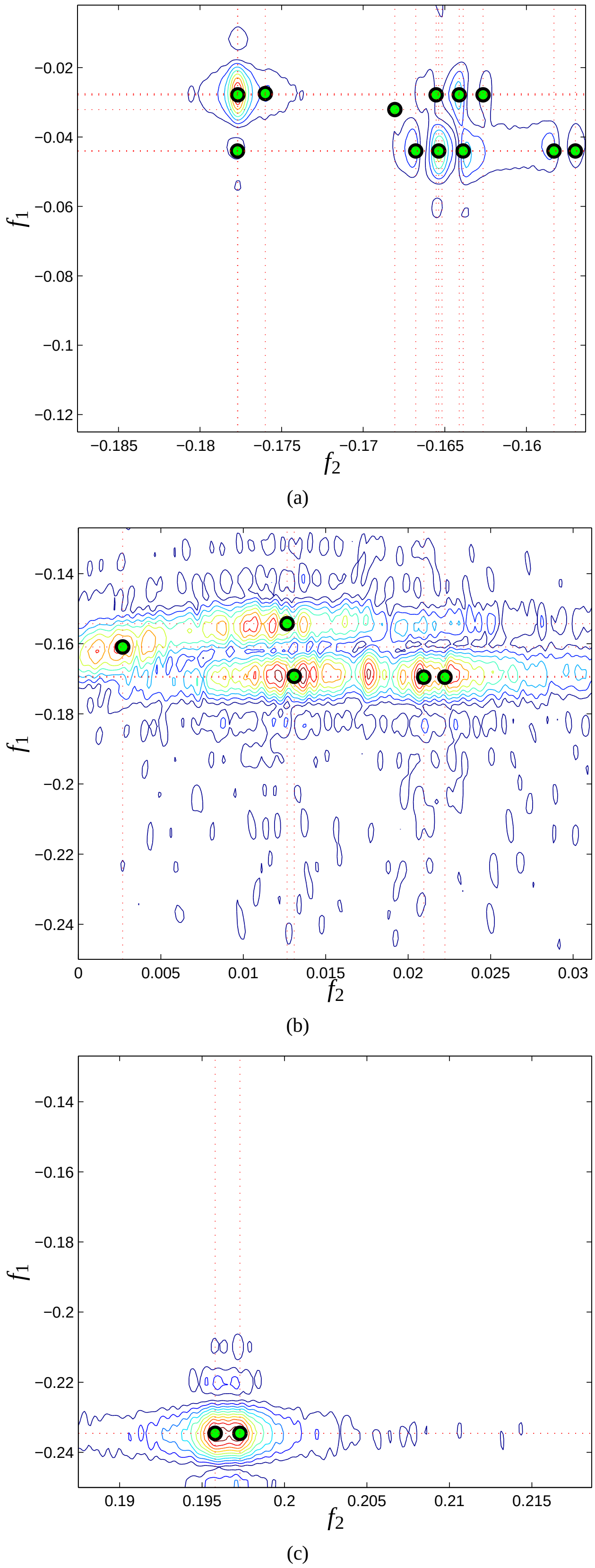

Figure 5: Results on the NMR signal. Reconstructed contour plot in (a) band $[-.125, 0] \times [.19, .155]$, (b) band $[-.25, -.125] \times [0, .03]$, and (c) band $[-.25, -.125] \times [.185, .22]$.