REDUCED-RANK TRANSFORM-DOMAIN LMS ALGORITHM FOR STABILIZING FRACTIONALLY-SPACED CHANNEL EQUALIZERS

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ABSTRACT

Fractionally-spaced channel equalizers suffer from stability problems due to ill-conditioning of the input signal. Pulse shaping is the root cause of signal ill-conditioning, which manifests itself as lack of persistent excitation, poor convergence and coefficient drifts. The traditional solutions to ill-conditioning involve regularization of the input signal autocorrelation matrix using a tap-leakage adaptive filter, which improves the eigenvalue spread of the input signal at the expense of increased steady-state mean-squared error (MSE). In this paper we propose a new solution based on the transform-domain least-mean-square (TD-LMS) algorithm. The proposed algorithm exploits the unitary transform of TD-LMS to identify and update only the equalizer coefficients that fall within the passband of the pulse shape. The new algorithm improves the eigenvalue spread of the input signal without compromising the MSE performance, which in turn eliminates stability problems and produces a much improved convergence performance.

1. INTRODUCTION

High-speed transmission of data over bandlimited channels causes undesirable signal distortion which is referred to as *intersymbol interference* (ISI) [1]. Channel equalization at the receiver aims to undo ISI introduced by the bandlimited channel. Timing and carrier recovery from the received signal is significantly simplified if it is sampled faster the symbol (baud) rate. The use of sampling at rates higher than the baud rate results in so-called fractionally-spaced channel equalization [2]. An important feature of fractionally-spaced channel equalization is its ability to incorporate the matched filter into the the adaptive equalizer. A major problem with fractionally-spaced equalization when combined with pulse shaping is the lack of persistent excitation at the equalizer input, which has the potential to lead to instability, coefficient drift and poor convergence performance. Pulse shaping is employed to maintain a narrow bandwidth for the modulated signal and helps avoid interference and cross-talk. Sampling of a pulse shaped signal beyond the baud rate creates an undesirable spectral band with close-to-zero spectral content. This in turn significantly increases the eigenvalue spread of the signal applied to the adaptive equalizer.

Traditional solutions to the lack of persistent excitation arising from pulse shaping involve the use of leaky-LMS (least-mean-square) type adaptive filters (see e.g. [3, 2, 4, 5, 6, 7]) and time-domain interpolation [8]. The former algorithms, which have proved to be more popular, essentially introduce some regularization in order to decrease the eigenvalue spread of the input signal autocorrelation matrix. A shortcoming of this approach is to reduce the input signal-to-noise ratio (SNR), leading to increased steady-state mean-squared error (MSE). In this paper we propose a new approach to alleviate lack-of-persistent excitation problems predicated on rank reduction via transform-domain adaptive filtering. Using a rough knowledge of excess bandwidth for the pulse shape, we implement a reduced-rank transform-domain LMS algorithm, which uses those outputs of the unitary transform that correspond to the passband of the pulse shape while discarding the other transform outputs. By so doing the adaptive filter only deals with the input signal in the passband of the pulse shape, thereby entirely ignoring the higher frequencies that carry no or little information. Consequently the eigenvalue spread of the input signal is reduced without necessarily causing channel noise amplification.

The paper is organized as follows. Section 2 describes pulse shaping, fractionally-spaced channel equalization and persistent excitation problems associated with fractionally-spaced equalizers. The bandwidth limited signal created by pulse shaping is shown to cause numerical ill-conditioning and convergence problems. In Section 3 we review the transform-domain LMS algorithm and develop a modified version of the transform-domain LMS algorithm to avoid ill-conditioning arising from pulse shaping. Section 4 presents simulation studies to demonstrate the effectiveness of the proposed reduced-rank transform-domain LMS algorithm.

2. FRACTIONALLY-SPACED CHANNEL EQUALIZATION

2.1 Pulse shaping

In digital communications each symbol is replaced by a continuous pulse during the transmission process. The objective of pulse shaping is to comply with the bandwidth requirements of the transmission channel, which is also related to tolerable interference levels for neighbouring channels. For a symbol (baud) rate of $1/T$ symbols/s, the ideally bandlimited pulse is a sinc pulse

$$p(t) = \frac{\sin Wt}{Wt}, \quad W = \frac{\pi}{T}. \tag{1}$$

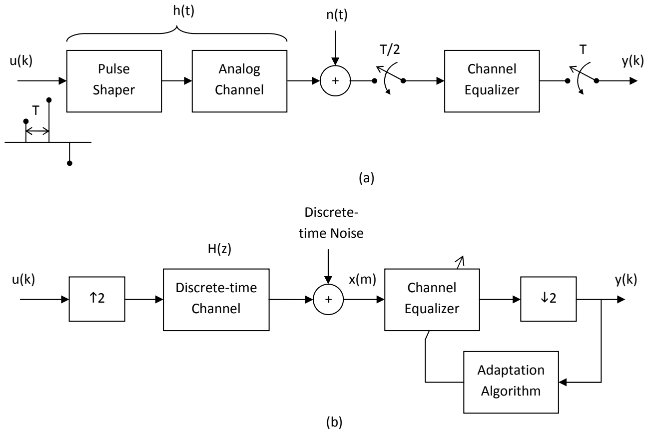

Figure 1: Baseband fractionally spaced equalization: (a) Continuous-time model, and (b) equivalent discrete-time model.

In addition to achieving a perfectly bandlimited signal with bandwidth W, the sinc pulse does not introduce any ISI to the digital bit stream, which is a property obeyed by all *Nyquist pulses* [2]. However, being of infinite length, the sinc pulse is not practical. A commonly used Nyquist pulse is the *raised-cosine pulse* which also has infinite length, but can be approximated by FIR filters. The bandwith of the raised-cosine pulse is controlled by the *roll-off* factor α between W and $2W$. For $\alpha = 0$ it has bandwidth W (0% excess bandwidth). As α increases from 0 to 1 the excess bandwidth increases from 0% to 100%. For α close to 1 the raised-cosine pulse has vanishingly small tails, allowing a good approximation by a sufficiently high-order FIR filter.

2.2 Fractionally-Spaced Equalization

The block diagram of a baseband digital communication system employing fractionally spaced equalization is shown in Fig. 1 [9]. A fractionally-spaced equalizer (FSE) samples the received signal at a rate higher than the symbol rate T (usually $T/2$) in order to avoid aliasing of the bandlimited received signal for non-zero excess bandwidth. If the pulse shape had zero-excess bandwidth, then baud-rate sampling would suffice. However this is not achievable in practice. Since the received signal is sampled alias-free, it can be reconstructed perfectly from its fractionally-spaced samples. This in turn allows for better timing and carrier recovery at the receiver. FSE combines matched filtering and adaptive equalization, thereby doing away with the need to have a separate matched (receive) filter. Another advantage of FSE is that it is capable of providing perfect zero-forcing equalization for an FIR channel with no common subchannel zeros by using an FIR equalizer [10].

The normalized least-mean-square (NLMS) algorithm for FSE with $N \times 1$ coefficient vector $\boldsymbol{\theta}(k) = [\theta_1(k), \ldots, \theta_N(k)]^T$ is

$$\boldsymbol{\theta}(k+1) = \boldsymbol{\theta}(k) + \frac{\mu}{\|\boldsymbol{x}(2k)\|^2 + \epsilon} e(k)\boldsymbol{x}^*(2k), \quad k = 0, 1, \ldots \tag{2}$$

where μ is the step-size parameter, ϵ is a small positive number that avoids numerical problems with division when $\|\boldsymbol{x}(2k)\|^2 = \boldsymbol{x}^H(2k)\boldsymbol{x}(2k)$ is close to zero. The regressor vector $\boldsymbol{x}(2k)$ contains fractionally-spaced ($T/2$-sampled) channel outputs

$$\boldsymbol{x}(2k) = [x(2k), x(2k-1), \ldots, x(2k-N+1)]^T \tag{3}$$

and $e(k)$ is the equalizer output error

$$e(k) = u(k-D) - y(k). \tag{4}$$

Here D is a delay for the desired equalizer response that takes into account pulse shape and channel delays. The FSE output is related to the fractionally-spaced regressor vector through

$$y(k) = \boldsymbol{x}^T(2k)\boldsymbol{\theta}(k). \tag{5}$$

It is clear from (2) that FSE operates as a linear adaptive filter, but adapts its coefficients at the symbol rate as implied by downsampling of its output in Fig. 1(b).

2.3 Lack of Persistent Excitation

As a result of pulse shaping the bandwidth of the $T/2$-sampled signal applied to FSE is between $\pi/2$ and π rad. Depending on the roll-off factor α, part of the sampled signal spectrum between $(1+\alpha)\pi/2$ and π will be zero or very small. This results in an ill-conditioned autocorrelation matrix for the input signal since some of its eigenvalues will be very small. To see this consider the $N \times N$ input correlation matrix $\boldsymbol{R} = E\{\boldsymbol{x}(2k)\boldsymbol{x}^T(2k)\}$. The eigenvalues of $\boldsymbol{R}$, λ_i, are bounded by [2]

$$\min_\omega S(\omega) < \lambda_i < \max_\omega S(\omega) \tag{6}$$

where $S(\omega)$ is the power spectral density of the input signal. As $N \to \infty$ the maximum and minimum eigenvalues of $\boldsymbol{R}$ are given by

$$\lambda_{\max} \to \max_\omega S(\omega) \quad \text{and} \quad \lambda_{\min} \to \min_\omega S(\omega). \tag{7}$$

Thus if the input signal spectrum has zero regions, $\boldsymbol{R}$ will have large eigenvalue spread $\lambda_{\max}/\lambda_{\min}$, adversely affecting the convergence of the adaptive filter and in some cases resulting in catastrophic coefficient drift.

Several solutions have been offered to address FSE stability problems caused by lack of persistent excitation. What is common to these solutions is introduction of some regularization to $\boldsymbol{R}$ to improve its conditioning at the expense of compromising the steady-state MSE. We next propose a much better solution that does not draw on regularization or tap-leakage adaptive filter implementation.

3. REDUCED-RANK TRANSFORM-DOMAIN LMS EQUALIZER

The transform-domain LMS (TD-LMS) algorithm has been proposed to speed up the convergence of the LMS algorithm for strongly correlated input signals (see e.g. [11, 12]). It is based on the intuitive notion of pre-whitening the input signal. The whitening process involves approximate decorrelation by a unitary transform followed by power normalization. Fig. 2 shows the

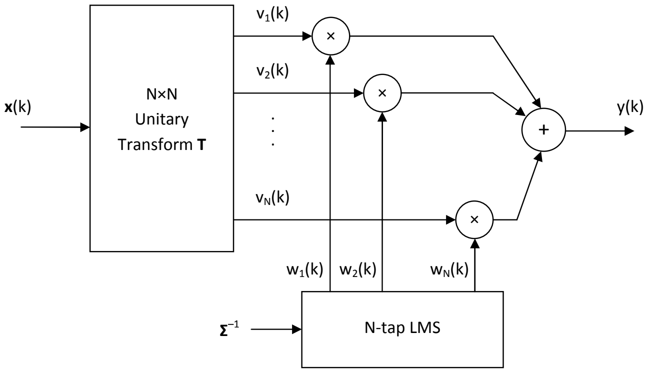

Figure 2: Transform-domain LMS.

building blocks of the TD-LMS algorithm. The input regressor vector $\boldsymbol{x}(k)$ is applied to an $N \times N$ unitary transform $\boldsymbol{T}$, which is usually a fixed data-independent matrix obtained from a discrete transform such as the discrete Fourier transform (DFT), discrete cosine transform (DCT), Hartley transform, etc. DCT is particularly attractive since it is real-valued. It also provides spectral analysis of the input signal. For example one can determine the spectral content of a certain frequency band by checking the corresponding transform (DCT bin) output. The output of the transform

$$\boldsymbol{v}(k) = \boldsymbol{T}\boldsymbol{x}(k) \tag{8}$$

has the approximately diagonalized autocorrelation matrix $\boldsymbol{\Sigma} = E\{\boldsymbol{v}^*(k)\boldsymbol{v}^T(k)\}$. Power normalization of the entries of $\boldsymbol{v}(k)$ produces an approximately pre-whitened regressor vector with almost unit eigenvalue spread. Applying LMS to this pre-whitened regressor with adaptive coefficients $w_1(k), \ldots, w_N(k)$ improves the convergence rate dramatically for coloured $x(k)$.

Using TD-LMS to adapt FSE coefficients in Fig. 1(b) results in the following update equation

$$\boldsymbol{\theta}(k+1) = \boldsymbol{\theta}(k) + \mu\boldsymbol{\Sigma}^{-1}(k)\boldsymbol{v}^*(k)e(k), \quad k = 0, 1, \ldots \tag{9}$$

where

$$\begin{aligned} \boldsymbol{v}(k) &= \boldsymbol{T}\boldsymbol{x}(2k) \\ y(k) &= \boldsymbol{v}^T(k)\boldsymbol{\theta}(k) \\ e(k) &= u(k-D) - y(k). \end{aligned}$$

The diagonal matrix $\boldsymbol{\Sigma}(k) = \text{diag}\{\sigma_1^2(k), \sigma_2^2(k), \ldots, \sigma_N^2(k)\}$ contains the power estimate for each DCT bin output $v_i(k)$. Using exponentially weighted averaging, the diagonal entries of $\boldsymbol{\Sigma}(k)$ can be computed as

$$\sigma_i^2(k) = (1-\lambda)|v_i(k)|^2 + \lambda\sigma_i^2(k-1), \quad k = 0, 1, \ldots \tag{10}$$

where $0 < \lambda < 1$ is the exponential forgetting factor.

As we saw before, zero or small spectral content of the input signal is responsible for large eigenvalue spread, convergence problems and possible coefficient drift. All these problems can be rectified by designing the equalizer in such a way that it only deals with the

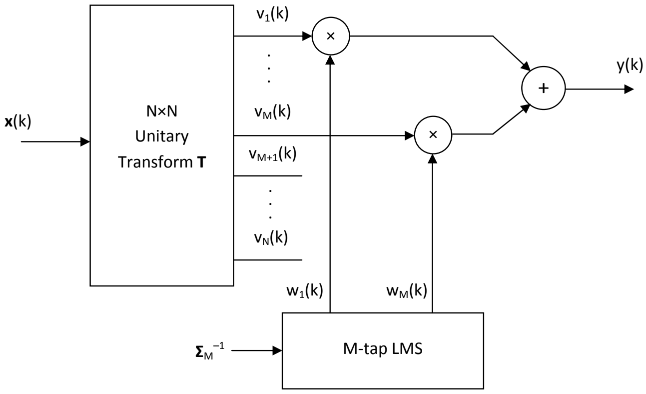

Figure 3: Reduced-rank TD-LMS ($M < N$).

non-zero spectral part of the input signal and ignores the zero spectrum. This is also equivalent to approximating $\boldsymbol{R}$ with a smaller size autocorrelation matrix that is full rank. This type of reduced-rank matrix approximation is often performed when solving ill-conditioned least-squares problems [13]. Rather than use computationally expensive singular value decomposition (SVD) to obtain a reduced-rank matrix approximation, we propose the following modification to TD-LMS that achieves roughly the same objective albeit in a much simpler way: Use only the first M DCT bins ($M < N$) that cover the passband of the pulse shape. This leads to the following *reduced-rank TD-LMS (RR-TD-LMS) algorithm*:

$$\boldsymbol{\theta}_M(k+1) = \boldsymbol{\theta}_M(k) + \mu\boldsymbol{\Sigma}_M^{-1}(k)\boldsymbol{v}_M^*(k)e_M(k), \quad k = 0, 1, \ldots \tag{11}$$

where

$$\begin{aligned} \boldsymbol{\theta}_M(k) &= [\theta_1(k), \ldots, \theta_M(k)]^T \\ \boldsymbol{\Sigma}_M(k) &= \text{diag}\{\sigma_1^2(k), \sigma_2^2(k), \ldots, \sigma_M^2(k)\} \\ \boldsymbol{v}_M(k) &= [v_1(k), \ldots, v_M(k)]^T \\ y_M(k) &= \boldsymbol{v}_M^T(k)\boldsymbol{\theta}_M(k) \\ e_M(k) &= u(k-D) - y_M(k). \end{aligned}$$

As shown in Fig. 3, RR-TD-LMS still uses the same $N \times N$ unitary transform $\boldsymbol{T}$ as in TD-LMS, but only requires an M-tap LMS adaptive filter rather than the full N-tap adaptive filter where $M < N$. The selection of the rank parameter M depends on the value of α. As a rule of thumb, we have

$$M \approx \frac{(1+\alpha)N}{2}. \tag{12}$$

The rationale for this expression is explained in Fig. 4. By using a shorter adaptive filter than TD-LMS, RR-TD-LMS also enjoys some reduction in computational complexity.

For bandlimited input signals TD-LMS attempts to normalize power in all DCT bins even though some might have zero or very small signal as a result of pulse shaping (see Fig. 4). Consequently the channel noise in those bins that contain almost zero signal get amplified much more than other bins. The noise amplification in DCT bins with zero output tends to increase the

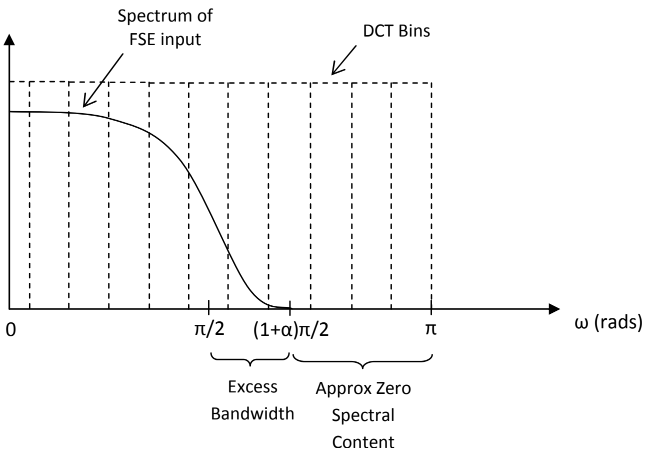

Figure 4: Fractionally spaced equalizer input spectrum and DCT bins from 0 to π rad. The bandwidth of the input signal is approximately $(1+\alpha)\pi/2$ rad which corresponds to DCT bin $(1+\alpha)N/2$.

steady-state MSE level for the adaptive filter. On the other hand, the RR-TD-LMS algorithm reduces the adverse effect of noise amplification by not attempting to power-normalize the DCT bins with almost zero output.

4. SIMULATION RESULTS

In this section we present computer simulations for fractionally-spaced equalization using NLMS, TD-LMS and the proposed RR-TD-LMS algorithm. The simulated channel is a $T/2$-sampled terrestrial microwave channel (channel #3) obtained from the Rice University Signal Processing Information Base (SPIB) at `http://spib.rice.edu/spib/microwave.html`. The impulse response and frequency response of this channel are shown in Fig. 5. The transmitted data symbols are white and QPSK modulated. The pulse shape is an FIR raised-cosine pulse of order 20. The roll-off factor is set to $\alpha = 0.2$. The frequency response of the $T/2$-sampled raised-cosine pulse shape is shown in Fig. 6. Note that the pulse is approximately bandlimited to $1.2W$. The additive channel noise is complex Gaussian with 25-dB signal-to-noise ratio. The FSE length is set to $N = 200$ and the delay for training is $D = 30$. The TD-LMS and RR-TD-LMS use a forgetting factor of $\lambda = 0.99$ for power estimation. The rank parameter for RR-TD-LMS is set to $M = 120$ which is consistent with $\alpha = 0.2$ (see (12)). The regularization parameter for NLMS is $\epsilon = 10^{-4}$. All adaptive filters use centre-tap initialization given by $\boldsymbol{\theta}(0) = [0, \ldots, 0, 1, 0, \ldots, 0]^T$.

Fig. 7 shows ensemble-averaged learning curves for NLMS, TD-LMS and RR-TD-LMS where MSE $= E\{|e(k)|^2\}$. For ensemble averaging 200 simulation runs were used. The step-size parameters were set to $\mu = 1$ for NLMS and $\mu = 0.006$ for both TD-LMS and RR-TD-LMS. These step-sizes were chosen to ensure identical initial convergence rates. We observe that the proposed algorithm RR-TD-LMS outperforms both TD-LMS and NLMS by achieving a much lower steady-state MSE. This outcome is a result of rank reduction by discarding zero-spectral region of the input signal, which improves the conditioning of the input signal autocorrelation matrix and alleviates noise amplification. Thanks to the absence of regularization, RR-TD-LMS also avoids undesirable elevation of steady-state MSE. The extremely slow convergence of the NLMS algorithm resulting from a flat cost function associated with large eigenvalue spread is clearly visible from Fig. 7.

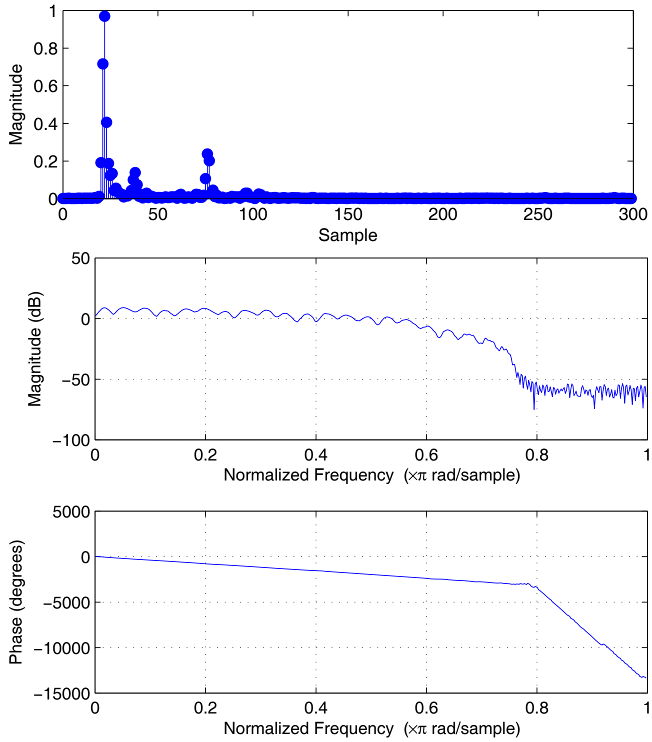

Figure 5: Magnitude impulse response and frequency response for SPIB microwave channel #3.

The rank parameter M controls the steady-state MSE achievable by the RR-TD-LMS algorithm. This is shown for the previous microwave channel simulation in Fig. 8. If M is chosen too small, then part of the useful spectrum of the input signal is discarded, which has the effect of increasing the steady-state MSE. The optimal value for M is seen to be approximately $M = 110$. If a larger M is used, the MSE begins to climb gradually. Note that $M = 200$ is equivalent to the TD-LMS algorithm.

5. CONCLUSIONS

In this paper we have proposed a new fractionally-spaced channel equalization algorithm that avoids problems arising from pulse shaping of transmitted signals. The proposed algorithm modifies the TD-LMS by exploiting the bank-of-bandpass-filter interpretation of the DCT unitary transform. The transform outputs mapped to the frequencies larger than the bandwidth are ignored, thereby improving the eigenvalue spread of the signal and alleviating noise amplification. The effectiveness of the new algorithm was illustrated with simulation examples.

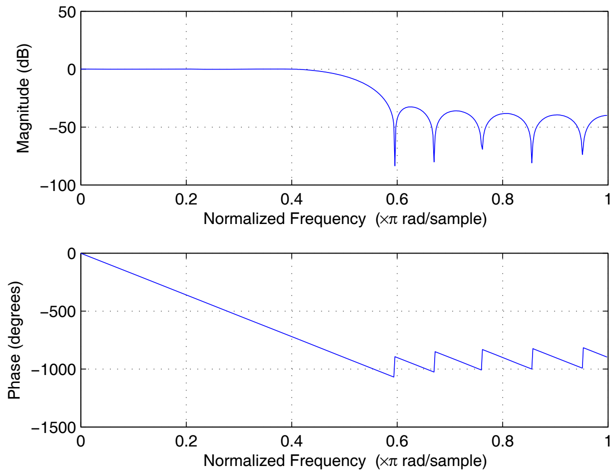

Figure 6: Frequency response of the simulated fractionally-spaced raised-cosine pulse shape with roll-off factor $\alpha = 0.2$.

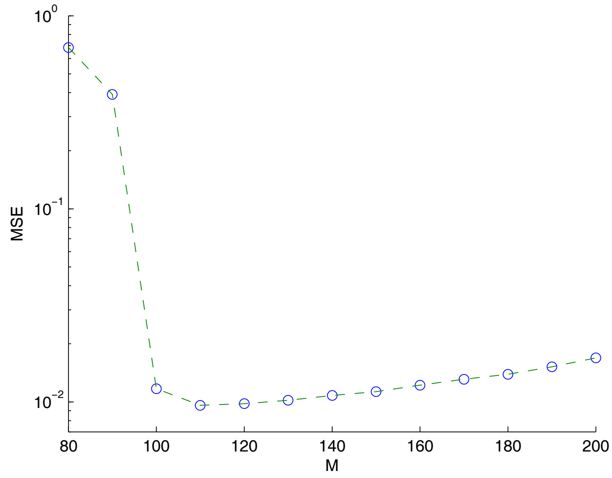

Figure 8: Steady-state MSE versus M for the RR-TD-LMS algorithm. MSE is minimized at approximately $M = 110$.

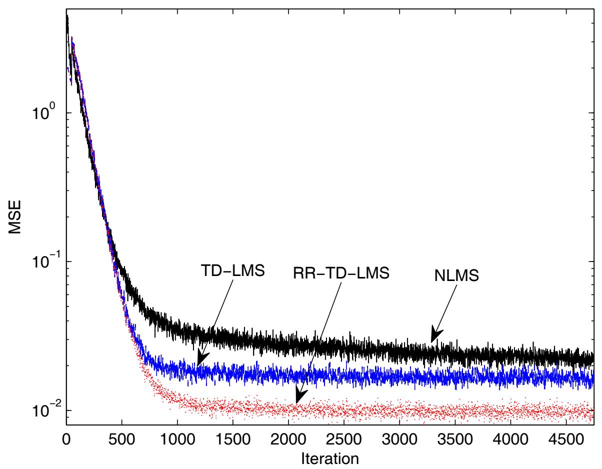

Figure 7: Learning curves for FSE using NLMS, TD-LMS and RR-TD-LMS.

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