MINIMUM REDUNDANCY MULTICARRIER AND SINGLE-CARRIER SYSTEMS BASED ON HARTLEY TRANSFORMS

Wallace A. Martins and Paulo S. R. Diniz

LPS – Signal Processing Laboratory
COPPE/DEL-Poli/Federal University of Rio de Janeiro
P.O. Box 68504, Rio de Janeiro, RJ, 21941-972, Brazil
Email: {wallace,diniz}@lps.ufrj.br

ABSTRACT

Four efficient block-based transceivers exploiting the displacement structure approach are proposed. In terms of computational burden, the resulting systems are asymptotically as simple as orthogonal frequency-division multiplex (OFDM) and single-carrier with frequency-domain (SC-FD) equalization transceivers. Even though the effective channel impulse response must be symmetric, the novel schemes are appealing since they only use discrete Hartley transforms (DHTs) and diagonal matrices in their structures, which results in numerically efficient algorithms for the equalization process. The key feature of the proposed transceivers is their higher throughput, since they require only half the number of symbols of redundancy in comparison to the standard OFDM and SC-FD systems.

1. INTRODUCTION

OFDM and SC-FD are the simplest and most widely used implementations of fixed and memoryless multicarrier and single-carrier transceivers. The standard design of these systems requires, at least, L elements for redundancy, where L stands for the channel order. The redundancy eliminates the inherent interblock interference (IBI), which is part of all block-based transceivers, and turns the channel matrix circulant. This property allows the use of *superfast*[1] algorithms for designing intersymbol-interference-free (ISI-free) or zero-forcing (ZF), and minimum mean squared error (MMSE) equalizers, by means of the spectral decomposition of the circulant channel matrix using the discrete Fourier transform (DFT) [1]. Other alternative real-valued transforms may also be successfully employed by inducing a Toeplitz–plus–Hankel structure in the effective channel matrix [2].

The role of redundancy in quite general transceivers was extensively studied in [1], [3], [4], and [5]. When dealing with block-based or memoryless[2] systems it was shown in [4] that the minimum required redundancy for IBI-free designs is $\lceil L/2 \rceil$. However, the solution relies on inversion of matrices, which in general requires $\mathcal{O}(n^3)$ operations.

So far, the only effective and practical solutions employing minimum redundancy were proposed in [6]. The referred solutions require $\mathcal{O}(n \log n)$ computations for equalization since they are based on standard DFT and diagonal matrices. This paper complements those recent results by solving the problem of designing fixed and memoryless transceivers with minimum redundancy for frequency-selective channels, utilizing DHTs and diagonal matrices.

When compared to OFDM and SC-FD, the proposed multicarrier and single-carrier transceivers have comparable computational complexity for the equalization process, i.e., $\mathcal{O}(n \log n)$, and substantially higher throughput for channels with long impulse responses due to their minimum required redundancy. *However, as a drawback of the proposed transceivers, the finite impulse response (FIR) filter that models the effective channel is constrained to be symmetric.*

In order to achieve our goals, this paper discusses the properties of structured matrices [7], [8] required to derive superfast transceivers with minimum redundancy. The Sylvester and Stein displacements [7] are utilized to exploit the structural properties of channel matrix representations. By using adequate displacement properties it is possible to conceive DHT-based representations of Bezoutians [8], which are the key to reach solutions for block-based transceivers requiring minimum redundancy. Some structured matrix properties are presented in the paper, where those directly available from the literature include no proof, whereas those adapted or modified are presented along with their proofs.

2. SYSTEM MODEL

Assume that we transmit a vector $\mathbf{s} \in \mathbb{C}^{M\times 1}$, with $M \in \mathbb{N}$, through an $(L+1)$-tap channel whose discrete-time block model is given by the pseudo-circulant matrix $\mathbf{H}(z) = \mathbf{H}_{\mathrm{ISI}}(z) + z^{-1}\mathbf{H}_{\mathrm{IBI}}(z)$ [4]. It is possible to eliminate the IBI caused by the matrix $\mathbf{H}_{\mathrm{IBI}}(z) \in \mathbb{C}^{N\times N}$, where $N = M + K \in \mathbb{N}$, by using at least $K \geq L/2$ elements for redundancy [4]. This can be achieved by the following transmitter and receiver matrices, respectively [4]: $\mathbf{F} = [\,\mathbf{F}_0^T \quad \mathbf{0}_{M\times K}]^T$, with $\mathbf{F}_0 \in \mathbb{C}^{M\times M}$, and $\mathbf{G} = [\,\mathbf{0}_{M\times(L-K)} \quad \mathbf{G}_0\,]$, with $\mathbf{G}_0 \in \mathbb{C}^{M\times(M+2K-L)}$. Thus, the transfer matrix of this transceiver model is given by $\mathbf{T}(z) = \mathbf{G}\mathbf{H}(z)\mathbf{F} = \mathbf{G}_0\mathbf{H}_0\mathbf{F}_0 = \mathbf{T}$, in which $\mathbf{H}_0$ is a Toeplitz matrix, whose ijth coefficient is given by $h(K+i-j), \forall i,j \in \mathcal{M} = \{0, 1, \cdots, M-1\}$ such that $0 \leq K+i-j \leq L$, and otherwise $h(K+i-j) = 0$. We will assume that $\mathbf{H}_0$ is already symmetric, that is $K = L/2$ (even order) and $h(K+i-j) = h(K+j-i)$.[3] Notice that we considered the special case where the noise $\mathbf{v}$ is null, motivated by the design of ZF systems [4].

The aim of this work is to design the matrices $\mathbf{F}_0$ and $\mathbf{G}_0$ by using only DHTs and diagonal matrices. The following definitions for the orthogonal DHTs and the unitary DFTs [8] are employed in this work.

Definition 1. Given $\theta_{\mathrm{I}}(i,j) = 2ij\pi/M$, $\theta_{\mathrm{II}}(i,j) = i(2j+1)\pi/M$, $\theta_{\mathrm{III}}(i,j) = (2i+1)j\pi/M$, and $\theta_{\mathrm{IV}}(i,j) = (2i+1)(2j+1)\pi/2M$, for $(i,j) \in \mathcal{M}^2$, we define the orthogonal DHT-X matrix as $[\boldsymbol{\mathcal{H}}_X]_{ij} = (\sin[\theta_X(i,j)] + \cos[\theta_X(i,j)])/\sqrt{M}$, and the unitary DFT-$X$ matrix as $[\mathbf{W}_X]_{ij} = (\sin[\theta_X(i,j)] - \jmath\cos[\theta_X(i,j)])/\sqrt{M}$, where $X \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$ and $\jmath^2 = -1$.

3. DISPLACEMENT STRUCTURE

There are ways to measure the degree of structure of a matrix, such as through the *displacement operator*. Defi-

[1] That is, it requires $\mathcal{O}(n \log^d n)$ operations, for $d \leq 3$ [7].

[2] We do not distinguish between these terms as done in [4].

[3] When L is not even, we can consider the channel model zero padded with one zero in order to achieve an even order. Besides, the symmetric channel can be approximated by using a front-end prefilter [9], [2].

nition 2 contains a formal statement about displacement operators [7].

Definition 2. For $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{C}^{M\times M}$, the operators $\nabla_{\mathbf{A},\mathbf{B}}, \Delta_{\mathbf{A},\mathbf{B}} : \mathbb{C}^{M\times M} \to \mathbb{C}^{M\times M}$, defined by $\nabla_{\mathbf{A},\mathbf{B}}(\mathbf{C}) = \mathbf{AC} - \mathbf{CB}$ and $\Delta_{\mathbf{A},\mathbf{B}}(\mathbf{C}) = \mathbf{C} - \mathbf{ACB}$, are the *displacement linear operator of Sylvester and Stein types*, respectively.

The rank of the resulting matrices $\nabla_{\mathbf{A},\mathbf{B}}(\mathbf{C})$ and $\Delta_{\mathbf{A},\mathbf{B}}(\mathbf{C})$ are the so-called *displacement ranks*. It is very important to choose correctly the *operator matrices* $\mathbf{A}$ and $\mathbf{B}$ in order to obtain a relatively small displacement rank. The most common operator matrices are the λ-circulant matrix $\mathbf{Z}_\lambda = [\,\mathbf{e}_2 \cdots \mathbf{e}_M\ \lambda\mathbf{e}_1\,]$ and the diagonal matrix $\mathbf{D}_{\boldsymbol{\nu}} = \mathrm{diag}\{\boldsymbol{\nu}\}$, where $\lambda \in \mathbb{C}$, $\mathbf{e}_m$ is a vector having its mth element equal to 1 and all others equal to 0, and $\boldsymbol{\nu} = [\,\nu_0\ \nu_1 \cdots \nu_{M-1}\,]^T$, with $\nu_m^M = \nu \in \mathbb{C}, \forall m \in \mathcal{M}$. Note that $\mathbf{Z}_\lambda^{-1} = \mathbf{Z}_{1/\lambda}^T, \forall\lambda \in \mathbb{C}\setminus\{0\}$ [7].

The proposed design for block-based transceivers relies on the displacement rank approach, which is characterized by the following key features [7]: (*i*) *Compression*: the displacement rank of a structured matrix $\mathbf{C}$ must be small compared to the dimension of $\mathbf{C}$. In this case, the displacement can be *compressed* by using the so-called *displacement generator* for the matrix $\mathbf{C}$. The displacement generator, given by the pair $(\mathbf{P},\mathbf{Q})$, has the following characteristic: by considering that we are dealing with a Sylvester operator, being R its rank, we have that $\nabla_{\mathbf{A},\mathbf{B}}(\mathbf{C}) = \sum_{r=1}^{R}\mathbf{p}_r\mathbf{q}_r^T = \mathbf{PQ}^T$, where $\mathbf{P} = [\,\mathbf{p}_1 \cdots \mathbf{p}_R\,] \in \mathbb{C}^{M\times R}$ and $\mathbf{Q} = [\,\mathbf{q}_1 \cdots \mathbf{q}_R\,] \in \mathbb{C}^{M\times R}$, with $\mathbf{p}_r = [\,p_{0r}\ p_{1r} \cdots p_{(M-1)r}\,]^T$ and $\mathbf{q}_r = [\,q_{0r}\ q_{1r} \cdots q_{(M-1)r}\,]^T$, $\forall r \in \{1,\cdots,R\}$; (*ii*) *Operation*: once compressed, operations with structured matrices can be performed much faster by using their displacements; and (*iii*) *Decompression*: after the operation stage, the original matrices can be recovered through *decompression* from their displacement.

The first important result employing displacement operators is the equivalence of the Sylvester and Stein displacements when at least one of the two operator matrices, $\mathbf{A}$ or $\mathbf{B}$, is non-singular [7].

Proposition 1. *If the operator matrix* $\mathbf{B}$ *is invertible, then* $\nabla_{\mathbf{A},\mathbf{B}}(\mathbf{C}) = -\Delta_{\mathbf{A},\mathbf{B}^{-1}}(\mathbf{C})\mathbf{B}$.

The second result relates a Sylvester displacement of a matrix with a Sylvester displacement of its inverse [7]. This result shows that the compression of the inverse of a matrix can be achieved through operation on the compressed representation of the original matrix.

Proposition 2. *For an invertible matrix* $\mathbf{C} \in \mathbb{C}^{M\times M}$*, we have that* $\nabla_{\mathbf{B},\mathbf{A}}(\mathbf{C}^{-1}) = -\mathbf{C}^{-1}\nabla_{\mathbf{A},\mathbf{B}}(\mathbf{C})\mathbf{C}^{-1}$*, where* $\mathbf{A},\mathbf{B} \in \mathbb{C}^{M\times M}$.

Propositions 3 and 4 describe how traditional operations, such as linear combinations and products of matrices, transform the displacement generators of the original matrices [7].

Proposition 3. *For* $\alpha \in \mathbb{C}$, $\nabla_{\mathbf{A},\mathbf{B}}(\mathbf{C}) = \mathbf{PQ}^T$*, and* $\nabla_{\mathbf{A},\mathbf{B}}(\mathbf{D}) = \mathbf{P}'\mathbf{Q}'^T$*, we have that* $\nabla_{\mathbf{A},\mathbf{B}}(\mathbf{C}+\alpha\mathbf{D}) = \bar{\mathbf{P}}\bar{\mathbf{Q}}^T$*, where* $\bar{\mathbf{P}} = [\,\mathbf{P}\quad \alpha\mathbf{P}'\,]$ *and* $\bar{\mathbf{Q}} = [\,\mathbf{Q}\quad \mathbf{Q}'\,]$.

Proposition 4. *For* $\nabla_{\mathbf{A},\mathbf{B}}(\mathbf{C}) = \mathbf{PQ}^T$ *and* $\nabla_{\mathbf{B},\mathbf{D}}(\mathbf{E}) = \mathbf{P}'\mathbf{Q}'^T$*, we have that* $\nabla_{\mathbf{A},\mathbf{D}}(\mathbf{CE}) = \bar{\mathbf{P}}\bar{\mathbf{Q}}^T$*, where* $\bar{\mathbf{P}} = [\,\mathbf{P}\quad \mathbf{CP}'\,]$ *and* $\bar{\mathbf{Q}} = [\,\mathbf{E}^T\mathbf{Q}\quad \mathbf{Q}'\,]$.

Now, it is possible to apply the displacement operators on Toeplitz matrices in order to verify if they can be compressed. Consider the Sylvester operator $\nabla_{\mathbf{Z}_\eta,\mathbf{Z}_\xi}$ applied to a symmetric Toeplitz matrix $\boldsymbol{T} = (t_{|i-j|})_{i,j=0}^{M-1}$, with $\xi,\eta \in \mathbb{R}$:

$$\begin{aligned}
\nabla_{\mathbf{Z}_\eta,\mathbf{Z}_\xi}(\boldsymbol{T}) &= \mathbf{Z}_\eta\boldsymbol{T} - \boldsymbol{T}\mathbf{Z}_\xi \\
&= \underbrace{\mathbf{e}_1}_{\hat{\mathbf{p}}_1}\underbrace{[\eta t_{M-1} - t_1 \quad \cdots \quad \eta t_1 - t_{M-1} \quad \eta t_0]}_{\hat{\mathbf{q}}_1^T} \\
&\quad + \underbrace{[-\xi t_0 \quad t_{M-1} - \xi t_1 \quad \cdots \quad t_1 - \xi t_{M-1}]^T}_{\hat{\mathbf{p}}_2}\underbrace{\mathbf{e}_M^T}_{\hat{\mathbf{q}}_2^T} \\
&= \hat{\mathbf{p}}_1\hat{\mathbf{q}}_1^T + \hat{\mathbf{p}}_2\hat{\mathbf{q}}_2^T = [\,\hat{\mathbf{p}}_1\ \hat{\mathbf{p}}_2\,]\begin{bmatrix}\hat{\mathbf{q}}_1^T\\ \hat{\mathbf{q}}_2^T\end{bmatrix} = \hat{\mathbf{P}}\hat{\mathbf{Q}}^T. \qquad (1)
\end{aligned}$$

Hence, it is obvious that *a symmetric Toeplitz matrix can be compressed*, when $M \gg R = 2$.

Proposition 5 contains an important result: the relationship between the displacement generators of a Toeplitz matrix and its inverse [6]. The inverse of a Toeplitz matrix is called a *T-Bezoutian matrix* [8].

Proposition 5. *For a invertible Toeplitz matrix* $\boldsymbol{T} \in \mathbb{C}^{M\times M}$ *such that its displacement generator pair related to the Sylvester displacement operator* $\nabla_{\mathbf{Z}_\eta,\mathbf{Z}_\xi}$*, with* $\xi,\eta \in \mathbb{C}$*, is given by* $(\hat{\mathbf{P}},\hat{\mathbf{Q}})$*, we have that the displacement generator pair* $(\mathbf{P},\mathbf{Q})$ *related to the Sylvester displacement operator* $\nabla_{\mathbf{Z}_\xi,\mathbf{Z}_\eta}$ *applied to the T-Bezoutian* $\boldsymbol{B} = \boldsymbol{T}^{-1}$ *is given by* $(-\boldsymbol{B}\hat{\mathbf{P}}, \boldsymbol{B}^T\hat{\mathbf{Q}})$.

4. DHT REPRESENTATION OF BEZOUTIANS

In this section, we develop the mathematical background required for deriving the main contribution of this work related to the design of practical block-based transceivers with minimum redundancy. Inspired by traditional ZF-OFDM and ZF-SC-FD systems that decompose inverses of circulant matrices by using DFT, IDFT, and diagonal matrices, we now describe some results related to the decomposition of a *Bezoutian matrix* by using DHTs and diagonal matrices. A matrix $\mathbf{C} \in \mathbb{C}^{M\times M}$ is a Bezoutian matrix if $R = \mathrm{rank}\{\nabla_{\mathbf{Z}_\xi,\mathbf{Z}_\eta}(\mathbf{C})\} \ll M$ [6]. Notice that a T-Bezoutian matrix has rank two.

Proposition 6 is our first contribution. It is based on a similar result of [8]. Unlike the polynomial approach adopted in [8], we use a matrix approach based on the Sylvester and Stein displacement operators. Our approach allows us to derive transformations without requiring extension with zeros of the involved matrices as in [8]. This eventually allows us to design multicarrier transceivers (see Section 5), which is not possible by using the same formulation presented in [8].

Proposition 6. *Given a centro-symmetric matrix* $\mathbf{C} \in \mathbb{C}^{M\times M}$*, i.e.,* $\mathbf{C} = \mathbf{JCJ}$*, with* $\mathbf{J} = [\,\mathbf{e}_M\ \mathbf{e}_{M-1}\ \cdots\ \mathbf{e}_2\ \mathbf{e}_1\,]$*, and given that* $\nabla_{\mathbf{Z}_1,\mathbf{Z}_{-1}}(\mathbf{C}) = \mathbf{PQ}^T$*, where* $(\mathbf{P},\mathbf{Q}) \in \mathbb{C}^{M\times R}\times\mathbb{C}^{M\times R}$ *and* $R \in \mathbb{N}$*, then* $\boldsymbol{\mathcal{H}}_{\mathrm{II}}\mathbf{C}\boldsymbol{\mathcal{H}}_{\mathrm{IV}} = \tilde{\mathbf{C}}$ *is such that* $\Delta_{\mathbf{D}_1,\mathbf{D}_{-1}}(\tilde{\mathbf{C}}) = \tilde{\mathbf{P}}\tilde{\mathbf{Q}}^T$*, with* $(\tilde{\mathbf{P}},\tilde{\mathbf{Q}}) = (-\jmath\mathbf{W}_{\mathrm{II}}\mathbf{P}, \mathbf{W}_{\mathrm{IV}}\mathbf{Z}_{-1}\mathbf{Q})$*. Furthermore,* $\tilde{\mathbf{C}}$ *can be expressed as* $[\tilde{\mathbf{C}}]_{ij} = [(-\mathbf{W}_{\mathrm{I}}\mathbf{P})(\mathbf{W}_{\mathrm{III}}\mathbf{Z}_{-1}\mathbf{Q})^T]_{ij}/2\sin\left(\frac{(2i+2j+1)\pi}{2M}\right)$.

Proof. See the appendix. ☐

Proposition 7 is our second contribution. It is also based on a similar result of [8]. However, our approach allows us to work with complex matrices, which is not possible in [8].

Proposition 7. *Given* $(\mathbf{P},\mathbf{Q}) \in \mathbb{C}^{M\times R}\times\mathbb{C}^{M\times R}$*, with* $R \in \mathbb{N}$*, we have that* $-\mathbf{W}_{\mathrm{I}}\mathbf{P} = \boldsymbol{\mathcal{H}}_{\mathrm{I}}(-\mathbf{P}_+ + \jmath\mathbf{P}_-) = \bar{\mathbf{P}}$ *and* $\mathbf{W}_{\mathrm{III}}\mathbf{Z}_{-1}\mathbf{Q} = \boldsymbol{\mathcal{H}}_{\mathrm{III}}(-\jmath\mathbf{Q}_+ + \mathbf{Q}_-) = \bar{\mathbf{Q}}$*, where* $\mathbf{P}_\pm = (\mathbf{P} \pm \mathbf{J}'\mathbf{P})/2$*,* $\mathbf{Q}_\pm = (\mathbf{Z}_{-1}\mathbf{Q} \pm \mathbf{J}''\mathbf{Z}_{-1}\mathbf{Q})/2$*,* $\mathbf{J}' = [\,\mathbf{e}_1\ \mathbf{e}_M\ \cdots\ \mathbf{e}_3\ \mathbf{e}_2\,]$*, and* $\mathbf{J}'' = [\,-\mathbf{e}_1\ \mathbf{e}_M\ \cdots\ \mathbf{e}_3\ \mathbf{e}_2\,]$.

Proof. See the appendix. ☐

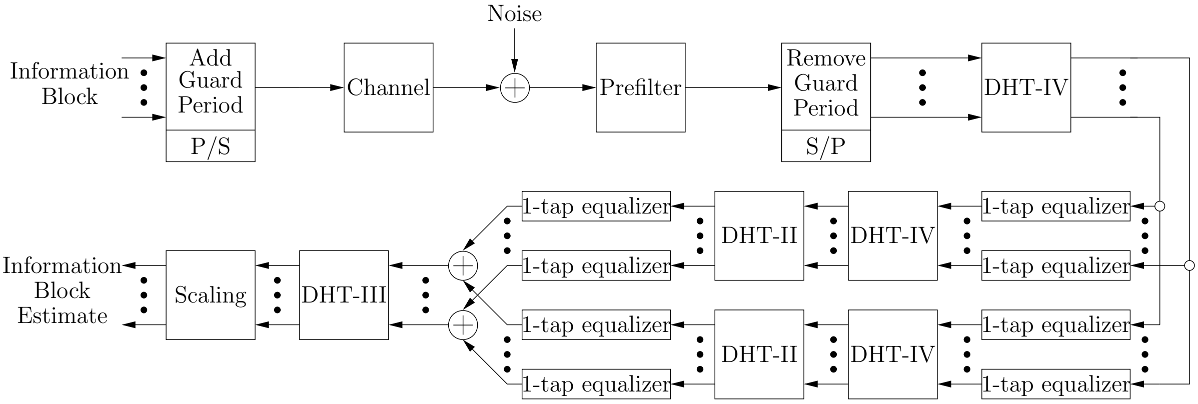

Figure 1: Single-carrier minimum redundancy block transceiver: ZF-SC-MRBT.

Proposition 8 is a result taken from [8] that is required to prove the main mathematical contribution of this work, Theorem 1. Once again, Theorem 1 is based on a similar result of [8], but only with our solution it is possible to design multicarrier transceivers for complex channel models.

Proposition 8. *The Hartley transforms* $\boldsymbol{\mathcal{H}}_{\mathrm{II}}$ *and* $\boldsymbol{\mathcal{H}}_{\mathrm{IV}}$ *obey the following relationship:* $[\boldsymbol{\mathcal{H}}_{\mathrm{II}}\boldsymbol{\mathcal{H}}_{\mathrm{IV}}]_{ij} = 1/M \sin\left(\frac{(2i+2j+1)\pi}{2M}\right)$.

Theorem 1. *Given* $\mathbf{C}$ *as in Proposition 6 and* $(\bar{\mathbf{P}}, \bar{\mathbf{Q}})$ *as in Proposition 7, it follows that:*

$$\mathbf{C} = \frac{M}{2}\boldsymbol{\mathcal{H}}_{\mathrm{III}}\left(\sum_{r=1}^{R}\mathbf{D}_{\bar{\mathbf{p}}_r}\boldsymbol{\mathcal{H}}_{\mathrm{II}}\boldsymbol{\mathcal{H}}_{\mathrm{IV}}\mathbf{D}_{\bar{\mathbf{q}}_r}\right)\boldsymbol{\mathcal{H}}_{\mathrm{IV}}, \qquad (2)$$

where $\bar{\mathbf{p}}_r$ *is the* rth *column vector of* $\bar{\mathbf{P}}$ *and* $\bar{\mathbf{q}}_r$ *is the* rth *column vector of* $\bar{\mathbf{Q}}$.

Proof. See the appendix. □

5. DESIGN OF SUPERFAST TRANSCEIVERS

5.1 ZF Solution

5.1.1 Single-Carrier System

As in SC-FD, let us define $\mathbf{F}_0 = \mathbf{I}_M$, in such a way that we must have $\mathbf{G}_0 = \mathbf{H}_0^{-1}$ in order to achieve the ZF solution. Of course, this ISI-free solution can only be implemented if $\mathbf{H}_0$ is square and invertible.

Since $\mathbf{H}_0^{-1} = \mathbf{J}\mathbf{H}_0^{-1}\mathbf{J}$ is a centro-symmetric T-Bezoutian, we can apply Theorem 1, in such a way that

$$\mathbf{G}_0 = \mathbf{H}_0^{-1} = \frac{M}{2}\boldsymbol{\mathcal{H}}_{\mathrm{III}}\left(\sum_{r=1}^{2}\mathbf{D}_{\bar{\mathbf{p}}_r}\boldsymbol{\mathcal{H}}_{\mathrm{II}}\boldsymbol{\mathcal{H}}_{\mathrm{IV}}\mathbf{D}_{\bar{\mathbf{q}}_r}\right)\boldsymbol{\mathcal{H}}_{\mathrm{IV}}, \qquad (3)$$

where $\bar{\mathbf{p}}_r, \bar{\mathbf{q}}_r$ can be easily found from $\hat{\mathbf{p}}_r, \hat{\mathbf{q}}_r$ by using Propositions 7 and 5. The generator vectors $\hat{\mathbf{p}}_r, \hat{\mathbf{q}}_r$ can be determined by using eq. (1), with $\xi = 1$, $\eta = -1$, and, $\forall m \in \mathcal{M}$, $t_m = h(L/2 \pm m)$, for $0 \leq m \leq L/2$, otherwise $t_m = 0$.

Figure 1 depicts the resulting single-carrier transceiver structure. In this transceiver, the guard period consists of $L/2$ zeros. The prefilter turns the channel symmetric [9]. After removing the guard period, the DHT-IV is applied to the received vector. The first equalization step on the data vector is performed, that is, the resulting data vector is simultaneously processed by two different branches of the transceiver. The 1-tap equalizers in this stage are the elements of the vectors $\bar{\mathbf{q}}_1$ and $\bar{\mathbf{q}}_2$. A final equalization step is performed in each branch, after the application of the DHT-IV and DHT-II. The 1-tap equalizers in this stage are the elements of the vectors $\bar{\mathbf{p}}_1$ and $\bar{\mathbf{p}}_2$.

5.1.2 Multicarrier System

Similarly, it is also possible to design the ZF solution for a multicarrier minimum redundancy block transceiver (ZF-MC-MRBT). This system is characterized by the following matrices: $\mathbf{F}_0 = \boldsymbol{\mathcal{H}}_{\mathrm{III}}$ and $\mathbf{G}_0 = \boldsymbol{\mathcal{H}}_{\mathrm{III}}^T\mathbf{H}_0^{-1}$, that is:

$$\mathbf{G}_0 = \frac{M}{2}\left(\sum_{r=1}^{2}\mathbf{D}_{\bar{\mathbf{p}}_r}\boldsymbol{\mathcal{H}}_{\mathrm{II}}\boldsymbol{\mathcal{H}}_{\mathrm{IV}}\mathbf{D}_{\bar{\mathbf{q}}_r}\right)\boldsymbol{\mathcal{H}}_{\mathrm{IV}}. \qquad (4)$$

5.2 MMSE Solution

5.2.1 Single-Carrier System

Given the equivalent model $\mathbf{y} = \mathbf{H}_0\mathbf{u} + \mathbf{v}$, with $\mathbf{u} = \mathbf{F}_0\mathbf{s}$, the linear MMSE solution $\mathbf{K}_{\mathrm{MMSE}} \in \mathbb{C}^{M\times M}$ is given by $\mathbf{K}_{\mathrm{MMSE}} = \mathbf{H}_0^*\left[\mathbf{H}_0\mathbf{H}_0^* + (\sigma_v^2/\sigma_s^2)\mathbf{I}\right]^{-1} = \mathbf{J}\mathbf{K}_{\mathrm{MMSE}}\mathbf{J}$ (centro-symmetric), where we considered that the transmitted symbols and the noise are i.i.d., drawn from white zero-mean stochastic processes, and mutually independent. Besides, it was considered that $\mathsf{E}[ss^*] = \sigma_s^2$ and $\mathsf{E}[vv^*] = \sigma_v^2$.

Consider that $\nabla_{\mathbf{Z}_{1/\eta},\mathbf{Z}_{\xi}}(\mathbf{H}_0) = \hat{\mathbf{P}}\hat{\mathbf{Q}}^T$ and $\nabla_{\mathbf{Z}_{\xi},\mathbf{Z}_{\rho}}(\mathbf{H}_0^*) = \hat{\mathbf{P}}'\hat{\mathbf{Q}}'$, for $(\rho,\xi,\eta) \in \mathbb{C}^3$ and $\eta \neq 0$. Now, by applying Proposition 4, we have that $\nabla_{\mathbf{Z}_{1/\eta},\mathbf{Z}_{\rho}}(\mathbf{H}_0\mathbf{H}_0^*) = \mathring{\mathbf{P}}\mathring{\mathbf{Q}}^T$, with $\mathring{\mathbf{P}} = [\,\hat{\mathbf{P}} \quad \mathbf{H}_0\hat{\mathbf{P}}'\,]$ and $\mathring{\mathbf{Q}} = [\,\mathbf{H}_0^*\hat{\mathbf{Q}} \quad \hat{\mathbf{Q}}'\,]$.

Define $\mathbf{A} = \mathbf{H}_0\mathbf{H}_0^* + (\sigma_v^2/\sigma_s^2)\mathbf{I}$. Thus, by supposing that $\nabla_{\mathbf{Z}_{1/\eta},\mathbf{Z}_{\rho}}(\mathbf{I}) = \mathring{\mathbf{p}}\mathring{\mathbf{q}}^T$ and employing Proposition 3, we obtain: $\nabla_{\mathbf{Z}_{1/\eta},\mathbf{Z}_{\rho}}(\mathbf{A}) = \check{\mathbf{P}}\check{\mathbf{Q}}^T$, with $\check{\mathbf{P}} = [\,\mathring{\mathbf{P}} \quad (\sigma_v^2/\sigma_s^2)\mathring{\mathbf{p}}\,]$ and $\check{\mathbf{Q}} = [\,\mathring{\mathbf{Q}} \quad \mathring{\mathbf{q}}\,]$. In addition, from Proposition 2, we have $\nabla_{\mathbf{Z}_{\rho},\mathbf{Z}_{1/\eta}}\left(\mathbf{A}^{-1}\right) = \check{\mathbf{P}}'\check{\mathbf{Q}}'^T$, with $\check{\mathbf{P}}' = -\mathbf{A}^{-1}\check{\mathbf{P}}$ and $\check{\mathbf{Q}}' = \mathbf{A}^{-T}\check{\mathbf{Q}}$.

Thus, by again applying Proposition 4, we obtain $\nabla_{\mathbf{Z}_{\xi},\mathbf{Z}_{1/\eta}}(\mathbf{K}_{\mathrm{MMSE}}) = \mathbf{P}\mathbf{Q}^T$, with $\mathbf{P} = [\,\hat{\mathbf{P}}' \quad \mathbf{H}_0^*\check{\mathbf{P}}'\,]$ and $\mathbf{Q} = [\,\mathbf{A}^{-T}\hat{\mathbf{Q}}' \quad \check{\mathbf{Q}}'\,]$.

Hence, the displacement generator of the MMSE solution is given by the pair

$$\begin{aligned}\mathbf{P} &= \left[\hat{\mathbf{P}}' \quad -\mathbf{K}_{\mathrm{MMSE}}\hat{\mathbf{P}} \quad -\mathbf{K}_{\mathrm{MMSE}}\mathbf{H}_0\hat{\mathbf{P}}' \quad -\frac{\sigma_v^2}{\sigma_s^2}\mathbf{K}_{\mathrm{MMSE}}\mathring{\mathbf{p}}\right]_{M\times 7}, \\ \mathbf{Q} &= \left[\mathbf{A}^{-T}\hat{\mathbf{Q}}' \quad \mathbf{K}_{\mathrm{MMSE}}^T\hat{\mathbf{Q}} \quad \mathbf{A}^{-T}\hat{\mathbf{Q}}' \quad \mathbf{A}^{-T}\mathring{\mathbf{q}}\right]_{M\times 7}.\end{aligned}$$

By applying the matrix inversion lemma, it is possible to show that $\mathbf{P}\mathbf{Q}^T$ can be expressed as

$$\frac{\sigma_v^2}{\sigma_s^2}\left(\mathbf{A}^*\right)^{-1}\hat{\mathbf{P}}'\hat{\mathbf{Q}}'^T\mathbf{A}^{-1} - \mathbf{K}_{\mathrm{MMSE}}\hat{\mathbf{P}}\hat{\mathbf{Q}}^T\mathbf{K}_{\mathrm{MMSE}} - \frac{\sigma_v^2}{\sigma_s^2}\mathbf{K}_{\mathrm{MMSE}}\mathring{\mathbf{p}}\mathring{\mathbf{q}}^T\mathbf{A}^{-1}.$$

A more compact definition for $\mathbf{P}$ and $\mathbf{Q}$ is:

$$\mathbf{P} = \begin{bmatrix} \frac{\sigma_v^2}{\sigma_s^2}(\mathbf{A}^*)^{-1}\hat{\mathbf{P}}' & -\mathbf{K}_{\text{MMSE}}\hat{\mathbf{P}} & -\frac{\sigma_v^2}{\sigma_s^2}\mathbf{K}_{\text{MMSE}}\mathring{\mathbf{p}} \end{bmatrix}_{M\times 5} \quad (5)$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{A}^{-T}\hat{\mathbf{Q}}' & \mathbf{K}_{\text{MMSE}}^T\hat{\mathbf{Q}} & \mathbf{A}^{-T}\mathring{\mathbf{q}} \end{bmatrix}_{M\times 5}. \quad (6)$$

Hence, by using the result in Theorem 1 and by considering that $(\rho, \xi, \eta) = (0, 1, -1)$, we have that

$$\mathbf{K}_{\text{MMSE}} = \frac{M}{2}\boldsymbol{\mathcal{H}}_{\text{III}}\left(\sum_{r=1}^{5}\mathbf{D}_{\bar{\mathbf{p}}_r}\boldsymbol{\mathcal{H}}_{\text{II}}\boldsymbol{\mathcal{H}}_{\text{IV}}\mathbf{D}_{\bar{\mathbf{q}}_r}\right)\boldsymbol{\mathcal{H}}_{\text{IV}}. \quad (7)$$

The displacement generator pairs $(\hat{\mathbf{P}}, \hat{\mathbf{Q}}), (\hat{\mathbf{P}}', \hat{\mathbf{Q}}') \in \mathbb{C}^{M\times 2} \times \mathbb{C}^{M\times 2}$ are easily found by using eq. (1). Moreover, $\mathring{\mathbf{p}} = [1\ 0\ \cdots\ 0]^T$ and $\mathring{\mathbf{q}} = [0\ 0\ \cdots\ -2]^T$.

Thus, in the single-carrier transmission, we can define $\mathbf{F}_0 = \mathbf{I}_M$ and

$$\mathbf{G}_0 = \frac{M}{2}\boldsymbol{\mathcal{H}}_{\text{III}}\left(\sum_{r=1}^{5}\mathbf{D}_{\bar{\mathbf{p}}_r}\boldsymbol{\mathcal{H}}_{\text{II}}\boldsymbol{\mathcal{H}}_{\text{IV}}\mathbf{D}_{\bar{\mathbf{q}}_r}\right)\boldsymbol{\mathcal{H}}_{\text{IV}} \quad (8)$$

in order to achieve the linear MMSE solution.

Note that the equalization process of the MMSE-SC-MRBT requires almost the same processing time of the ZF solution, since the structures of the receivers are very similar. It is also possible to take advantage of the inherent parallel structures (the MMSE entails five parallel branches instead of only two. See Figure 1).

5.2.2 Multicarrier System

In the multicarrier transmission (MMSE-MC-MRBT), we can define $\mathbf{F}_0 = \boldsymbol{\mathcal{H}}_{\text{III}}$ and

$$\mathbf{G}_0 = \frac{M}{2}\left(\sum_{r=1}^{5}\mathbf{D}_{\bar{\mathbf{p}}_r}\boldsymbol{\mathcal{H}}_{\text{II}}\boldsymbol{\mathcal{H}}_{\text{IV}}\mathbf{D}_{\bar{\mathbf{q}}_r}\right)\boldsymbol{\mathcal{H}}_{\text{IV}}. \quad (9)$$

6. SIMULATION EXPERIMENTS

In this section, we present two simulation examples in order to compare the performance of our proposed designs against the standard OFDM and SC-FD systems.

Example 1 (Symmetric Random Rayleigh Channel). In this example, it is transmitted 100 blocks, each one containing $M = 32$ BPSK data symbols (without taking redundancy into account), and it is computed the throughput by using a Monte-Carlo averaging process with 1000 simulations. These symbols are sampled at a frequency $f_s = 1.0$ MHz and they are transmitted through a channel with a model operating at the same frequency as the symbols and with impulse response of order $L = 8$. Both the imaginary and real parts of the channel are independently drawn from a white and Gaussian process. Besides, the channels are already considered symmetric. The throughput performance of the proposed transceivers is much better than the traditional ones, as illustrates Figure 2. Such favorable result originates from the choices for M and L (delay constrained applications in quite dispersive environments). These types of applications are suitable for the proposed transceivers. In the cases where $M \gg L$, the traditional OFDM and SC-FD solutions are more adequate.

Example 2 (ADSL Shortened Channel). In this example, it was transmitted 1000 blocks, each one containing $M = 256$ QPSK data symbols (without taking redundancy into account), and we compute the resulting BER for such transmission. The symbols are also sampled at a frequency $f_s = 1.0$ MHz and they are transmitted through an ADSL channel[4] whose model operates at the same frequency. This channel is represented by the FIR approximation with 93 coefficients of

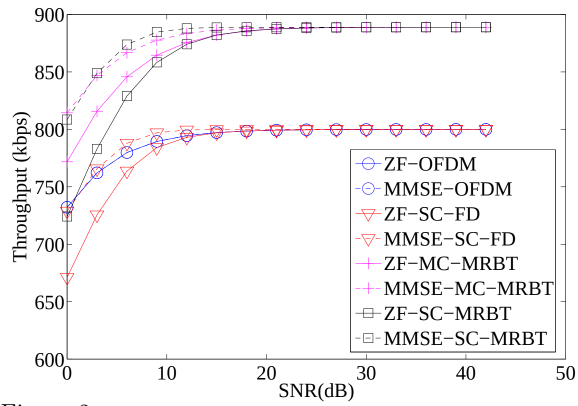

Figure 2: Throughput × SNR for symmetric Rayleigh channels.

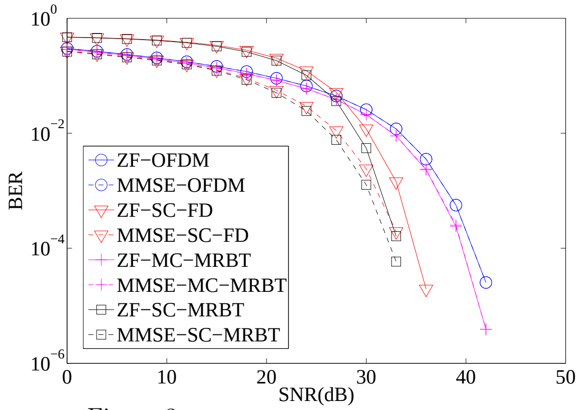

Figure 3: BER × SNR for an ADSL channel.

the transfer function $H(z) = (0.1z^{-2} - 0.1)/(z^{-2} - 1.5z^{-1} + 0.54)$ [10].

Considering delay constrained applications with very dispersive environment, it is mandatory for OFDM and SC-FD systems to employ a front-end prefilter in order to shorten the channel [9], [2]. Thus, in order to have a fair comparison in terms of throughput, we designed a shortening FIR filter of order 64, considering the SNR fixed at 30 dB for the design of this filter. The length of the TIR (target impulse response) was set to $L/2 + 1 = 47$, in such a way that the amount of redundancy for both the traditional and the proposed systems was $L/2$. On the other hand, the length of the TIR for the proposed systems was set the same of the channel length $L + 1 = 93$, but with symmetry constraint[5]. Figure 3 depicts a BER curve for all the systems, considering the existence of the prefilter. It is possible to observe that the proposed transceivers outperform their related pairs for both ZF and MMSE designs.

7. CONCLUDING REMARKS

In this paper we proposed transceivers with minimum redundancy for block data transmission. The ZF and MMSE solutions employ only DHTs and diagonal matrices. This feature makes the new transceivers computationally efficient. Our approach relied on the properties of structured matrices using the concepts of Sylvester and Stein displacements.

[4]In practice, an ADSL system applies bit and power loading to the subchannels, rather than transmitting equal power signals on every subchannel as done here. But, the problem of power loading when using the proposed transceivers has not been addressed and appears to be more complex than in the traditional DMT schemes, since the effective channel matrix is not diagonalized.

[5]Additional details about the degrees of freedom required by the front-end prefilter in order to shorten the channel and to make it symmetric can be found in [9], [2].

These concepts aimed at exploiting the structural properties of typical channel matrix representations. It was derived new DHT-based representations of centro-symmetric Bezoutians, which were the key tools to reach the proposed solutions for the multicarrier and single-carrier systems. A possible future work is to verify if the channel capacity can be achieved as the number of subcarries increases (with ideal Gaussian codes). This is a desirable feature inherent to OFDM-based schemes.

Acknowledgment

The authors would like to thank the anonymous reviewers that contributed to the improvement of this paper. This work was made possible by the grants from CNPq, CAPES and FAPERJ, Brazilian research councils.

APPENDIX

Before demonstrating Proposition 6, it will be helpful to state some results, as follows:

Lemma 1. *The four DFT matrices obey the following identities:* $\mathbf{Z_1} = \mathbf{W}_{\mathrm{I}}^H \mathbf{D_1} \mathbf{W}_{\mathrm{I}} = \mathbf{W}_{\mathrm{II}}^H \mathbf{D_1} \mathbf{W}_{\mathrm{II}}$ *and* $\mathbf{Z}_{-1} = \mathbf{W}_{\mathrm{III}}^H \mathbf{D_{-1}} \mathbf{W}_{\mathrm{III}} = \mathbf{W}_{\mathrm{IV}}^H \mathbf{D_{-1}} \mathbf{W}_{\mathrm{IV}}$, *where* $\mathbf{D_1} = \mathrm{diag}\left\{W_M^m\right\}_{m=0}^{M-1}$ *contains all the* Mth *unit roots, and* $\mathbf{D_{-1}} = \mathrm{diag}\left\{W_M^m \exp\left(-\jmath\frac{\pi}{M}\right)\right\}_{m=0}^{M-1}$ *contains all the* Mth *roots of* -1, *with* $W_M = \exp(-\jmath\frac{2\pi}{M})$.

Proof. First, consider that $j \in \mathcal{M} \setminus \{M-1\}$. Thus, $[\mathbf{D_1}\mathbf{W}_{\mathrm{I}}]_{ij} = W_M^i W_M^{ij} = W_M^{i(j+1)} = [\mathbf{W}_{\mathrm{I}}]_{i(j+1)} = [\mathbf{W}_{\mathrm{I}}\mathbf{Z_1}]_{ij}$. Second, consider that $j = M-1$. In this case, we have $[\mathbf{D_1}\mathbf{W}_{\mathrm{I}}]_{i(M-1)} = W_M^i W_M^{i(M-1)} = W_M^{iM} = 1 = [\mathbf{W}_{\mathrm{I}}]_{i0} = [\mathbf{W}_{\mathrm{I}}\mathbf{Z_1}]_{i(M-1)}$. The other three identities can be analogously proved. $\square$

As discussed in [8], it is possible to verify that $\mathbf{W}_{\mathrm{II}} = \mathrm{diag}\{\exp(-\jmath m\pi/M)\}_{m=0}^{M-1}\mathbf{W}_{\mathrm{I}}$ and $\mathbf{W}_{\mathrm{IV}} = \mathrm{diag}\{\exp(-\jmath(2m+1)\pi/2M)\}_{m=0}^{M-1}\mathbf{W}_{\mathrm{III}}$. In addition, Lemma 2 holds [8].

Lemma 2. *Given that* $\mathbf{C} \in \mathbb{C}^{M\times M}$ *is a centro-symmetric matrix, we have that* $\boldsymbol{\mathcal{H}}_{\mathrm{II}}\mathbf{C}\boldsymbol{\mathcal{H}}_{\mathrm{IV}} = \jmath\mathbf{W}_{\mathrm{II}}\mathbf{C}\mathbf{W}_{\mathrm{IV}}$.

Proof of Proposition 6. By applying the results of Lemma 1, Lemma 2, Proposition 1, and the fact that $\mathbf{Z}_\lambda^{-1} = \mathbf{Z}_{1/\lambda}^T, \forall\lambda \in \mathbb{C}\setminus\{0\}$, we have that the Stein displacement $\Delta_{\mathbf{D_1},\mathbf{D_{-1}}}$ applied to $\tilde{\mathbf{C}}$ is given by $\Delta_{\mathbf{D_1},\mathbf{D_{-1}}}(\boldsymbol{\mathcal{H}}_{\mathrm{II}}\mathbf{C}\boldsymbol{\mathcal{H}}_{\mathrm{IV}}) = \Delta_{\mathbf{D_1},\mathbf{D_{-1}}}(\jmath\mathbf{W}_{\mathrm{II}}\mathbf{C}\mathbf{W}_{\mathrm{IV}}) = \jmath\mathbf{W}_{\mathrm{II}}\mathbf{C}\mathbf{W}_{\mathrm{IV}} - (\mathbf{W}_{\mathrm{II}}\mathbf{Z}_1\mathbf{W}_{\mathrm{II}}^H)(\jmath\mathbf{W}_{\mathrm{II}}\mathbf{C}\mathbf{W}_{\mathrm{IV}})(\mathbf{W}_{\mathrm{IV}}^*\mathbf{Z}_{-1}^T\mathbf{W}_{\mathrm{IV}}^T) = \jmath\mathbf{W}_{\mathrm{II}}(\mathbf{C} - \mathbf{Z}_1\mathbf{C}\mathbf{Z}_{-1}^T)\mathbf{W}_{\mathrm{IV}} = \jmath\mathbf{W}_{\mathrm{II}}\Delta_{\mathbf{Z}_1,\mathbf{Z}_{-1}^T}(\mathbf{C})\mathbf{W}_{\mathrm{IV}} = -\jmath\mathbf{W}_{\mathrm{II}}\nabla_{\mathbf{Z}_1,\mathbf{Z}_{-1}}(\mathbf{C})\mathbf{Z}_{-1}^T\mathbf{W}_{\mathrm{IV}} = (-\jmath\mathbf{W}_{\mathrm{II}}\mathbf{P})(\mathbf{W}_{\mathrm{IV}}\mathbf{Z}_{-1}\mathbf{Q})^T$. Thus, by using this fact, it is straightforward to verify that

$$[\tilde{\mathbf{C}}]_{ij} = \frac{[(-\jmath\mathbf{W}_{\mathrm{II}}\mathbf{P})(\mathbf{W}_{\mathrm{IV}}\mathbf{Z}_{-1}\mathbf{Q})^T]_{ij}}{\left(1 - e^{-\jmath\frac{(2i+2j+1)\pi}{M}}\right)} \tag{10}$$

$$= \frac{e^{\jmath\frac{i\pi}{M}}[(-\jmath\mathbf{W}_{\mathrm{II}}\mathbf{P})(\mathbf{W}_{\mathrm{IV}}\mathbf{Z}_{-1}\mathbf{Q})^T]_{ij}e^{\jmath\frac{(2j+1)\pi}{2M}}}{e^{\jmath\frac{(2i+2j+1)\pi}{2M}} - e^{-\jmath\frac{(2i+2j+1)\pi}{2M}}} \tag{11}$$

$$= \frac{[(-\mathbf{W}_{\mathrm{I}}\mathbf{P})(\mathbf{W}_{\mathrm{III}}\mathbf{Z}_{-1}\mathbf{Q})^T]_{ij}}{2\sin\left(\frac{(2i+2j+1)\pi}{2M}\right)}. \tag{12}$$

$\square$

Now, we state some useful equalities related to Proposition 7. A vector $\boldsymbol{\nu}$ is *even* if $\mathbf{J}'\boldsymbol{\nu} = \boldsymbol{\nu}$, it is *odd* if $\mathbf{J}'\boldsymbol{\nu} = -\boldsymbol{\nu}$, it is *quasi-even* if $\mathbf{J}''\boldsymbol{\nu} = \boldsymbol{\nu}$, and it is *quasi-odd* if $\mathbf{J}''\boldsymbol{\nu} = -\boldsymbol{\nu}$. The definitions of quasi-even and quasi-odd were necessary in order to correct a slip in the following lemma stated in [8]. The authors of the referred paper did not distinguish between quasi-even/odd and even/odd vectors.

Lemma 3. *Given an even vector* $\boldsymbol{\nu}_e \in \mathbb{C}^{M\times 1}$, *an odd vector* $\boldsymbol{\nu}_o \in \mathbb{C}^{M\times 1}$, *a quasi-even vector* $\boldsymbol{\nu}_{qe} \in \mathbb{C}^{M\times 1}$, *and a quasi-odd vector* $\boldsymbol{\nu}_{qo} \in \mathbb{C}^{M\times 1}$, *we have that* $\mathbf{W}_{\mathrm{I}}\boldsymbol{\nu}_e = \boldsymbol{\mathcal{H}}_{\mathrm{I}}\boldsymbol{\nu}_e$, $\mathbf{W}_{\mathrm{I}}\boldsymbol{\nu}_o = -\jmath\boldsymbol{\mathcal{H}}_{\mathrm{I}}\boldsymbol{\nu}_o$, $\mathbf{W}_{\mathrm{III}}\boldsymbol{\nu}_{qe} = -\jmath\boldsymbol{\mathcal{H}}_{\mathrm{III}}\boldsymbol{\nu}_{qe}$, *and* $\mathbf{W}_{\mathrm{III}}\boldsymbol{\nu}_{qo} = \boldsymbol{\mathcal{H}}_{\mathrm{III}}\boldsymbol{\nu}_{qo}$.

Proof of Proposition 7. Since $\mathbf{P}_\pm = (\mathbf{P} \pm \mathbf{J}'\mathbf{P})/2$ and $\mathbf{Q}_\pm = (\mathbf{Z}_{-1}\mathbf{Q} \pm \mathbf{J}''\mathbf{Z}_{-1}\mathbf{Q})/2$, then each column vector of $\mathbf{P}_+$ is an even vector, whereas each column vector of $\mathbf{Q}_+$ is a quasi-even vector. In addition, those columns of $\mathbf{P}_-$ and $\mathbf{Q}_-$ are odd and quasi-odd vectors, respectively. By applying Lemma 3, we have that $-\mathbf{W}_{\mathrm{I}}\mathbf{P} = -\boldsymbol{\mathcal{H}}_{\mathrm{I}}\mathbf{P}_+ + \jmath\boldsymbol{\mathcal{H}}_{\mathrm{I}}\mathbf{P}_- = \boldsymbol{\mathcal{H}}_{\mathrm{I}}(-\mathbf{P}_+ + \jmath\mathbf{P}_-) = \bar{\mathbf{P}}$ and $\mathbf{W}_{\mathrm{III}}\mathbf{Z}_{-1}\mathbf{Q} = -\jmath\boldsymbol{\mathcal{H}}_{\mathrm{III}}\mathbf{Q}_+ + \boldsymbol{\mathcal{H}}_{\mathrm{III}}\mathbf{Q}_- = \boldsymbol{\mathcal{H}}_{\mathrm{III}}(-\jmath\mathbf{Q}_+ + \mathbf{Q}_-) = \bar{\mathbf{Q}}$. $\square$

Proof of Theorem 1. Considering that $\bar{\mathbf{P}} = [\,\bar{\mathbf{p}}_1 \ \cdots \ \bar{\mathbf{p}}_R\,]$ and $\bar{\mathbf{Q}} = [\,\bar{\mathbf{q}}_1 \ \cdots \ \bar{\mathbf{q}}_R\,]$, then, based on Propositions 6, 7, and 8, we have that:

$$\tilde{\mathbf{C}} = \left[\frac{[\bar{\mathbf{P}}\bar{\mathbf{Q}}^T]_{ij}}{2\sin\left(\frac{(2i+2j+1)\pi}{2M}\right)}\right]_{i,j=0}^{M-1} \tag{13}$$

$$\tilde{\mathbf{C}} = \frac{M}{2}\sum_{r=1}^{R}\mathbf{D}_{\bar{\mathbf{p}}_r}\boldsymbol{\mathcal{H}}_{\mathrm{II}}\boldsymbol{\mathcal{H}}_{\mathrm{IV}}\mathbf{D}_{\bar{\mathbf{q}}_r}, \tag{14}$$

leading to the required result using the fact that $\boldsymbol{\mathcal{H}}_{\mathrm{II}}^T = \boldsymbol{\mathcal{H}}_{\mathrm{III}}$. $\square$

REFERENCES

[1] A. Scaglione, G. B. Giannakis, and S. Barbarossa, "Redundant filterbank precoders and equalizers part I: unification and optimal designs", *IEEE Trans. Signal Processing*, vol. 47, no. 7, pp. 1988–2006, July 1999.

[2] N. Al-Dhahir, H. Minn, and S. Satish, "Optimum DCT-based multicarrier transceivers for frequency-selective channels", *IEEE Trans. Commun.*, vol. 54, no. 5, pp. 911–921, May 2006.

[3] C. B. Ribeiro, M. L. R. Campos, and P. S. R. Diniz, "FIR equalizers with minimum redundancy", *Proc. 2002 IEEE Intern. Conf. on Acoust. Speech and Signal Processing*, Orlando, Florida, Vol. III, pp. 2673-2676, May 2002.

[4] Y.-P. Lin and S.-M. Phoong, "Minimum redundancy for ISI free FIR filterbank transceivers", *IEEE Trans. Signal Processing*, vol. 50, no. 4, pp. 842–853, April 2002.

[5] C. B. Ribeiro, M. L. R. Campos, and P. S. R. Diniz, "Time-varying FIR transmultiplexers with minimum redundancy", *IEEE Trans. Signal Processing*, vol. 57, no. 3, pp. 1113–1127, March 2009.

[6] W. A. Martins and P. S. R. Diniz, "Block-based transceivers with minimum redundancy", submitted for publication, March 2009.

[7] V. Y. Pan, *Structured Matrices and Polynomials: Unified Superfast Algorithms.* New York: Springer, 2001.

[8] G. Heinig and K. Rost, "Hartley transform representations of symmetric Toeplitz matrix inverses with application to fast matrix-vector multiplication", *SIAM J. Matrix Anal. Appl.*, vol. 22, no. 1, pp. 86–105, May 2000.

[9] C. Ribeiro, V. Silva, and P. S. R. Diniz, "Linear phase impulse response shortening for xDSL DMT modems", *IEEE Int. Telecom. Symp.*, Natal, RN, Brazil, pp. 368-371, September 2002.

[10] J. M. Cioffi, *A Multicarrier Primer.* Amati Commun. Corp., Stanford Univ., Tutorial.