# OPTIMAL SUPRATHRESHOLD STOCHASTIC RESONANCE BASED NONLINEAR DETECTOR

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## ABSTRACT

*Stochastic resonance and suprathreshold stochastic resonance are nonlinear phenomena that yield a non-monotonic variation of system performance measures such as SNR gain, with respect to input noise variance [1, 2]. There exists an optimal strength of the added noise at which the performance measure is maximised. In this work, an analytical method is proposed to find the optimal input noise variance to be added to the array of quantizers in an SSR detector to optimize the performance. The theory is verified experimentally for different noise environments and added SSR noise.*

## 1. INTRODUCTION

The phenomena of stochastic resonance (SR) and suprathreshold stochastic resonance (SSR) have generated considerable interest in the field of signal processing. These phenomena imply that a system has non-monotonic variation of its performance measures such as output SNR, SNR gain, Fisher information or mutual information with respect to the input noise variance. In static nonlinear systems such as a quantizer, it has been shown that adding a small amount of noise at the input of the quantizer along with the signal that is in general smaller than the quantizer threshold, tends to aid the performance of the system. This phenomenon is referred to as SR. When an array of quantizers is used and independent and identically distributed (i.i.d) noise is added to each quantizer along with the signal that may be larger than the threshold, the performance is found to be better than that obtained using a single quantizer. This phenomenon is referred to as SSR [3, 4]. These phenomena can be employed to aid in signal detection, especially in an environment contaminated with non-Gaussian noise with a heavy-tailed distribution.

An SR/SSR detector may be optimized in two ways : by tuning system parameters (such as the value of threshold) for the best performance [5] or by optimizing the noise added at the input of the quantizers [6]. In this paper, an analytical method of optimizing the noise to be added to the array of quantizers constituting the SSR detector is proposed and the performance of the optimal SSR detector is investigated. The paper is organized as follows. In Section 2, the input data and the SSR detector models are described. In Section 3, a novel analytical method is presented for choosing the optimal amount of SSR noise to be added. In Section 4, experimental (simulation) results to verify the theory are presented and the performance of the optimal SSR detector is discussed. Conclusions are presented in Section 5.

## 2. SSR DETECTOR

The SSR system shown in Fig.1 consists of a parallel array of M one-bit quantizers with a common input x(t), t = 0, 1, ... ,N -1. The input x(t) consists of a signal As(t) and an additive environmental noise e(t), where A is the signal amplitude. Independent and identically distributed white noises $a_1(t)$, $a_2(t)$,.. . $a_M(t)$ that are independent of e(t) are added separately to the quantizer inputs. The quantizer outputs {$q_M$ (t), m =1, 2, ..., M} are averaged to obtain the output Y(t) of the quantizer array. To implement the detection scheme, Y(t) is now applied to a matched filter or correlator detector. Presence or absence of the signal is decided by comparing the output of the matched filter with the detector threshold. We refer to this nonlinear detector as the SSR detector. The test statistic of the SSR detector is thus given by

$$T(Y) = \sum_{t=0}^{N-1} Y(t)s(t) \,. \qquad (1)$$

Figures 2(a) and 2(b) show the conventional linear matched filter (LMF) and the SSR detector respectively.

The receiver operating characteristic (ROC) of the SSR detector depends on the signal s(t), the probability density function (pdf) of the environmental noise e(t), and the pdf and variance of the i.i.d added noises $a_1(t)$, $a_2(t)$,.. . $a_M(t)$. We shall designate the added noise as SSR noise. The variance $\sigma^2_a$ of the SSR noises is to be chosen so as to maximize the probability of detection $P_D$. In general, the optimal value of $\sigma_a$ and the performance of the optimal SSR detector depend on the SSR noise pdf and the environmental noise pdf.

We will compare the performance of the SSR detectors with that of the LMF and the locally optimal detector (LOD).The LOD is the optimal detector under the weak signal approximation [7]. We will assume that the environmental noise is heavy-tailed (leptokurtic) in nature. This characteristic is typical of the noise encountered in underwater acoustic channels.

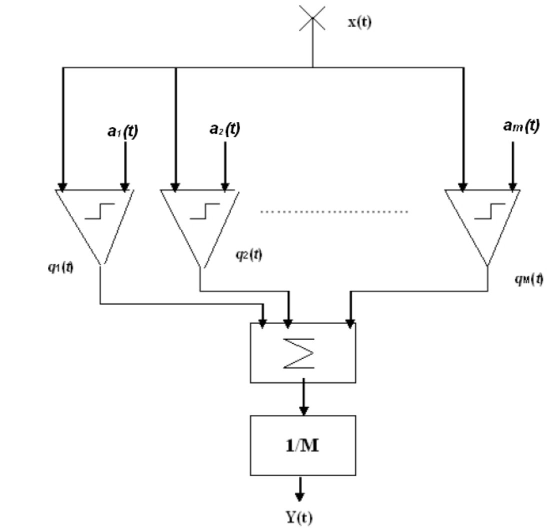


Figure - 1: Schematic of an SSR system

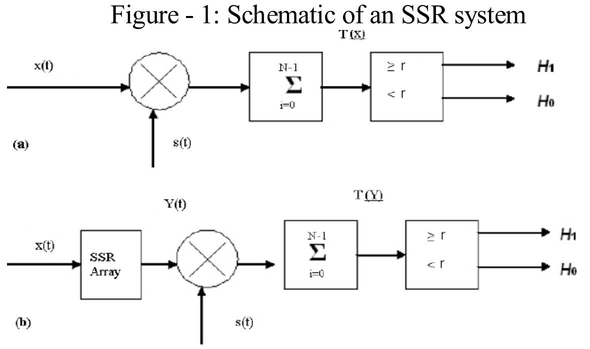


Figure – 2(a) Conventional linear matched filter detector, 2(b) SSR detector.

There exist several ways of modelling leptokurtic noise. Two such models will be considered here, viz. the generalised Gaussian noise (GGN) and the mixture of Gaussians (MOG). The pdf of a GGN random variable X with variance $\sigma^2$ is given by

$$f_X(x) = \frac{p}{2\,A(p)\,\Gamma(1/p)} \exp\left\{-\left[\frac{|x|}{A(p)}\right]^p\right\},\ p>0,$$

where $A(p) = \left[\sigma^2 \dfrac{\Gamma(1/p)}{\Gamma(3/p)}\right]^{1/2}$, $\Gamma(.)$ is the gamma function.

This pdf is Gaussian for p=2 and leptokurtic for p<2. The pdf of a MOG random variable with variance $\sigma^2$ consisting of a mixture of two zero-mean Gaussians, parameterized by a single parameter u is:

$$f_X(x) = \left(\frac{u^3}{\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{ux^2}{\sigma^2}\right) + \left(\frac{(1-u)^3}{\pi\sigma^2}\right)^{1/2} \exp\left[-\frac{(1-u)x^2}{\sigma^2}\right],$$

$0<u<1.$

The above pdf is Gaussian for u = 0.5, and it becomes progressively more leptokurtic as |u-0.5| increases.

## 3. DETERMINATION OF OPTIMAL SSR NOISE VARIANCE

Let the input to the system be the noisy signal x(t) given by

$$x(t) = A\,s(t) + e(t) = A\,s(t) + \sigma_e\, w(t),\ t = 0,1,\dots N-1 \quad (2)$$

where w(t) is a zero-mean unit-variance noise. Let $f_w(x)$ and $F_w(x)$ denote respectively the pdf and cumulative distribution function (cdf) of w(t). We assume that

$$\frac{1}{N}\sum_{t=0}^{N-1} s^2(t) = 1\ , \quad (3)$$

so that the parameter $A^2$ denotes the signal power. Let the added SSR noise at the $m^{th}$ quantizer be

$$a_m(t) = \sigma_a\, v_m(t),\ \ m = 1,2,\dots M$$

where $v_m(t)$ are i.i.d zero-mean unit-variance noises with pdf $f_v(x)$ and cdf $F_v(x)$. The output of the $m^{th}$ quantizer is

$$q_m(t) = sgn\ [x(t) + \sigma_a\, v_m(t)], \quad (4)$$

where *sgn(.)* denotes the signum function. The quantizer outputs are averaged to get the outputs of the quantizer array

$$Y(t) = \frac{1}{M}\sum_{m=1}^{M} q_m(t)\,. \quad (5)$$

Assuming samples of w(t) are i.i.d, it can be shown that [6]

$$E[Y(t)] = 1 - 2\int_{-\infty}^{\infty}\{F_v(-\frac{x(t)}{\sigma_a})\}\frac{1}{\sigma_e} f_w(\frac{x(t)}{\sigma_e} - \frac{A}{\sigma_e}s(t))dx(t)\,, \quad (6)$$

$$E[Y^2(t)] = 1 + 4M_1\int_{-\infty}^{\infty}\frac{1}{\sigma_e}\{F_v^2(-\frac{x(t)}{\sigma_a}) - F_v(-\frac{x(t)}{\sigma_a})\}\, f_w(\frac{x(t)}{\sigma_e} - \frac{A}{\sigma_e}s(t))dx(t), \quad (7)$$

where $M_1 = \dfrac{M-1}{M}$. Derivation of (6) and (7) follows the procedure described in [6]. We shall now proceed to determine the optimal value of $\sigma_a$ under the weak signal assumption $A/\sigma_e \ll 1$.On defining

$$A_1 = \frac{A}{\sigma_e}, u = \frac{x(t)}{\sigma_e}, \sigma = \frac{\sigma_a}{\sigma_e}, \quad (8)$$

and assuming that $f_v(x)$ is an even function so that

$$F_v(-\frac{u}{\sigma}) = 1 - F_v(\frac{u}{\sigma}),$$

equation (6) reduces to

$$E[Y(t)] = 2\int_{-\infty}^{\infty} F_v(\frac{u}{\sigma})\, f_w(u - A_1 s(t))du - 1\,. \quad (9)$$

Under the weak-signal assumption, we now have $A_1 \ll 1$. Expanding $f_w(u-A_1s(t))$ in a Taylor series around u and truncating the series after two terms,

$$f_w\left(u - A_1 s(t)\right) = f_w(u) - A_1 s(t)\frac{df_w(u)}{du}, \quad (10)$$

and assuming that $f_w(x)$ is an even function, (9) reduces to

$$E[Y(t)] = 2A_1 s(t) K(\sigma)\,, \quad (11)$$

$$\text{where } K(\sigma) = 2\int_0^{\infty} f_v(u)\, f_w(\sigma u)du. \quad (12)$$

Since both the factors in (12) are non-negative functions, we conclude that $K(\sigma) \geq 0$. We also note that $K(0) = f_w(0)$ is the maximum value of $K(\sigma)$ if $f_w(x)$ is unimodal.

Similarly, assuming that $f_w(x)$ and $f_v(x)$ are even functions, and employing the small signal approximation, equation (7) reduces to

$$E[Y^2(t)] = 1 - 8M_1\sigma \int_0^\infty F_v(u)\{1 - F_v(u)\} f_w(\sigma u) du. \quad (13)$$

Consider the expression for the variance,

$$\mathrm{var}(Y(t)) = E[Y^2(t)] - \{E[Y(t)]\}^2. \quad (14)$$

Since $A_1 << 1$, the second term in (14) may be ignored to obtain

$$\mathrm{var}(Y(t)) \approx 1 - 8M_1\sigma \int_0^\infty F_v(u)\{1 - F_v(u)\} f_w(\sigma u) du. \quad (15)$$

Thus var(Y(t)) is independent of t and $A_1$. We define

$$L(\sigma) = \mathrm{var}(Y(t)). \quad (16)$$

Since $F_v(u)\{1 - F_v(u)\} \leq \frac{1}{4}$ for all u, it follows that $L(\sigma) \geq 0$, and the maximum value of $L(\sigma)$ is $L(0) = 1$.

Now consider the detection problem, given by the binary hypotheses $H_0$ and $H_1$ where $H_1$ corresponds to the presence of signal, represented as

$H_0 : x(t) = \sigma_e w(t)$,
$H_1 : x(t) = A s(t) + \sigma_e w(t)$.

The test statistic T(Y) for the detection problem is given in (1). The means and variances of T(Y) under the 2 hypotheses are

$$m_0 = E[T; H_0] = \sum_{t=0}^{N-1} E[Y(t) ; H_0]\, s(t) = 0, \quad (17)$$

$$m_1 = E[T; H_1] = \sum_{t=0}^{N-1} E[Y(t) ; H_1]\, s(t)$$
$$= 2 A_1 K(\sigma) \sum_{t=0}^{N-1} s^2(t)$$
$$= 2 A_1 N K(\sigma), \quad (18)$$

$$\sigma_0^2 = \sigma_1^2 = var(T;H_0) = var(T;H_1)$$
$$= \sum_{t=0}^{N-1} s^2(t)\, var(Y(t)) = N L(\sigma). \quad (19)$$

The test statistic T is asymptotically ($N \to \infty$) Gaussian and hence the ROC for large data set is given by the equation

$$P_D = Q(Q^{-1}(P_F) - d), \quad (20)$$

where d is the deflection coefficient defined as

$$d = \frac{m_1 - m_0}{\sigma_1}. \quad (21)$$

From (17),(18) and (19), equation (21) reduces to

$$d = d(\sigma) = 2A_1\sqrt{N} D(\sigma), \quad (22)$$

where $D(\sigma)$ is the detection index defined as

$$D(\sigma) = K(\sigma) L^{-1/2}(\sigma). \quad (23)$$

It follows from (20) and (22) that $P_D$ is a monotonically increasing function of $D(\sigma)$. Hence, to maximize $P_D$ for a given $P_F$, we only need to maximize the detection index $D(\sigma)$. Hence, the optimal value of σ, denoted by $\sigma_{opt}$, is the value of σ that maximizes D, i.e.,

$$\sigma_{opt} = arg\,[Max_\sigma D(\sigma)]. \quad (23)$$

Thus we may obtain the optimal variance ($\sigma_{opt}^2$) of SSR noise by maximizing $D(\sigma)$ for any given pdf pair $\{f_w(x), f_v(x)\}$. We shall denote the maximum value of D by $D_{max}$. It may be noted that, for the SSR detector, the detection index D is a measure of performance that is independent of $A_1$ and N.

## 4. EXPERIMENTAL RESULTS

Theoretical and experimental (simulation) results are presented in this section for different models of environmental noise pdf $f_w(x)$ and added (SSR) noise pdf $f_v(x)$. All simulation results have been obtained from 20000 Monte Carlo simulations.

Figure 3 shows the experimentally plotted variation of probability of detection $P_D$ with standard deviation $\sigma$ of the SSR noise, assuming without loss of generality that the environmental noise has unit variance. Environmental noise was considered to be MOG with u=0.025 (Kurtosis 27.8) and SSR noise is Gaussian. The signal is DC of amplitude $A_1$=0.1, the number of quantizers is M = 200, data length is N = 80, and false alarm probability is $P_F$ = 0.1. The value of $\sigma$ that maximizes $P_D$ is 0.825

Figure 4(a) shows the variation of detection index $D(\sigma)$ with σ at the same experimental conditions as in Fig 3. It can be seen that the theoretical value of σ that maximizes D in Fig. 4(a) is the same as the experimental value of $\sigma_{opt}$ that maximizes $P_D$ in Fig.3. In Fig.4(b), the environmental noise is GGN with p = 0.5 (kurtosis = 22.2), other conditions being the same as in Fig.4(a). In this case, D is a monotonically decreasing function of σ, and hence $\sigma_{opt} = 0$. This difference in behavior may be explained as follows. We have $D(0) = f_w(0)$. For MOG pdf, $f_w(0)$ shows only a small increase from $\frac{1}{\sqrt{2\pi}}$ to $\frac{1}{\sqrt{\pi}}$ as u decreases from 0.5 to 0. But for GGN pdf, $f_w(0)$ increases without bound as $p \to 0$.

Variation of $D_{max}$ with the number of quantizers M is shown in Figs.5(a) and 5(b) for MOG(u=0.01) and GGN(p=0.5) environmental noise. The saturation with increasing M is clearly seen.

Experimentally plotted Receiver operating characteristics (ROC) of SSR detectors with different values of σ are plotted in Fig. 6 to compare their performance with that of the LOD and the LMF. The ROC curves are plotted using 25000 Monte Carlo trials. Environmental noise is MOG (u=0.025) in Fig.6(a) and GGN(p=0.5) in Fig.6(b). We have also verified that the experimental ROCs are practically indistinguishable from the theoretical ROCs defined by (20).

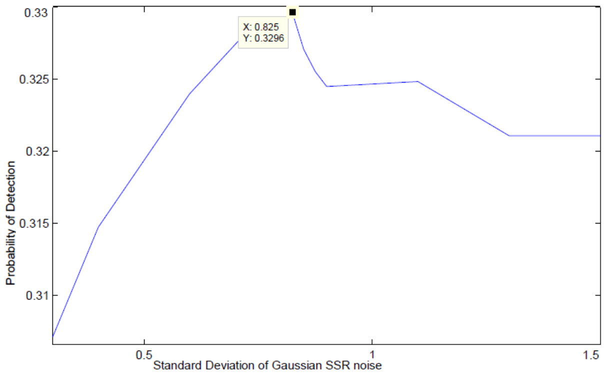
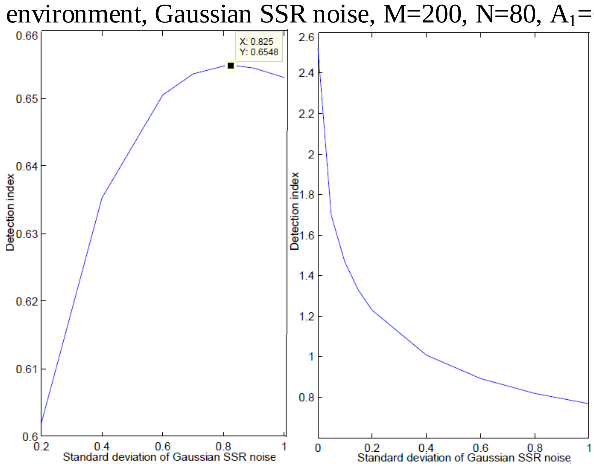



Figure-3:Plot of $P_D$ vs σ for SSR detector. MOG (u=0.025) environment, Gaussian SSR noise, M=200, N=80, $A_1$=0.1




Figure–4: Plot of D vs σ for Gaussian SSR noise, M=200. Environmental noise is (a)MOG with u = 0.025, (b)GGN with p=0.5

In each of the Figs. 6(a) and 6(b), the thick outer line shows the ROC of LOD which is the optimal detector for weak signals. The ROC of the optimal SSR detector (SSR detector with optimal σ) shown by dashed lines is very close to that of LOD. The SSR detectors with other values of σ have a slightly lower performance compared to that of the optimal SSR detector. Hence the experimental ROCs demonstrate that the theoretically predicted values of noise variance are also the optimal ones. All the SSR detectors are observed to outperform the LMF (dotted lines). It is also seen that improvement in performance of optimal SSR detector with respect to LMF is higher for GGN (p = 0.5, kurtosis = 22.2) environmental noise than for MOG (u = 0.025, kurtosis = 27.8) environmental noise even though the kurtosis of the former pdf is slightly lower than that of the latter.

Figure 7 shows the variation of $P_D$ with parameter u of the MOG and parameter p of the GGN environmental noise. It is seen that with increasing kurtosis, the performance of the optimal SSR detector shows an increasing trend similar to that of the LOD. In other words, in strongly leptokurtic environments the optimal SSR detector provides a significant performance enhancement compared to the LMF.

In all the results presented in this paper, the SSR noise is Gaussian. We have also examined the effect of using SSR

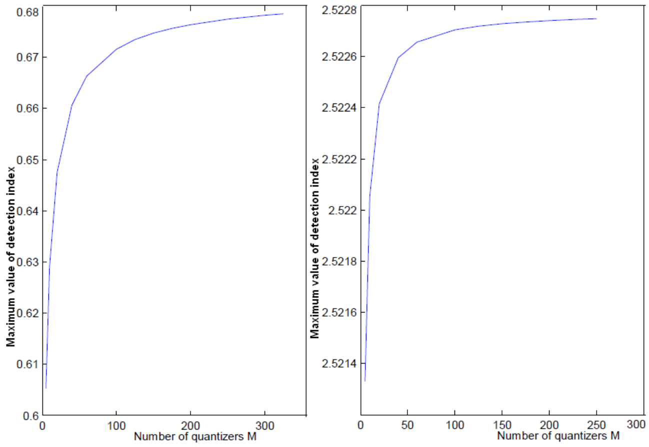



Figure-5: Plots of $D_{max}$vs M for Gaussian SSR noise. Environmental noise (a) MOG with u=0.01, (b) GGN with p=0.5

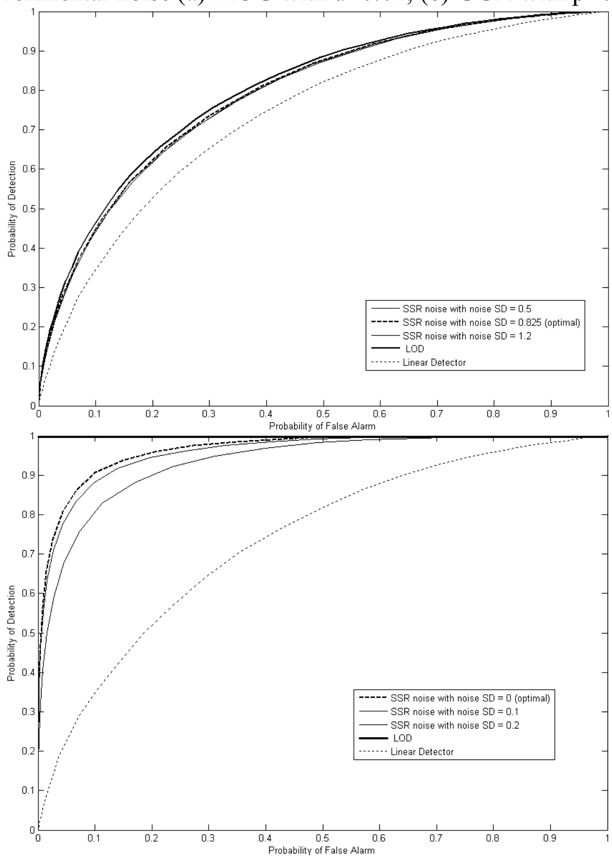
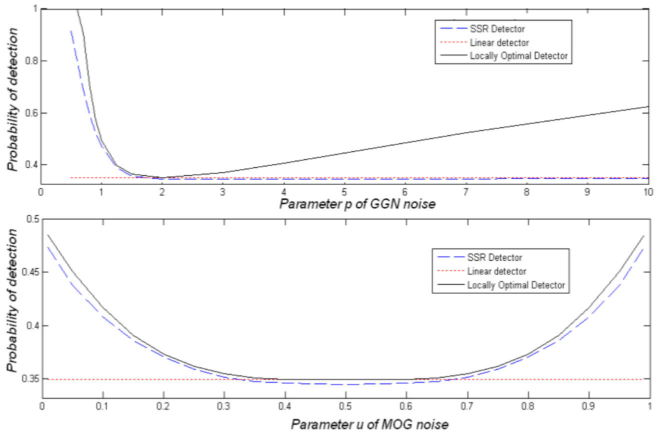


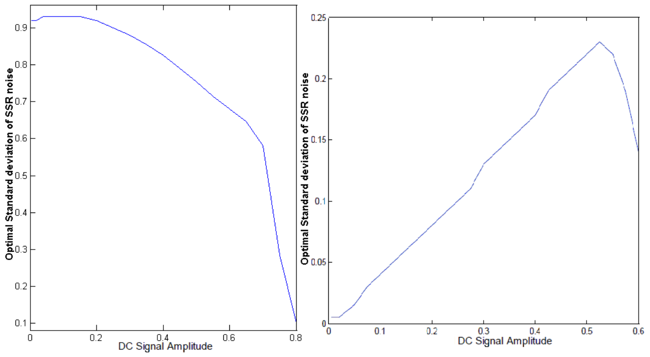

Figure-6: Experimentally plotted ROCs of SSR detectors (Gaussian SSR noise, M=200), LOD, and LMF for N =80, $A_1$=0.1, 25000 MC simulations. Environmental noise is (a)MOG with u=0.025,(b)GGN with p=0.5

noise having GGN pdf with different values of p in the interval (0.5, 10). This limited study indicates that changes in the value of $D_{max}$ (and the consequent changes in the value of $P_D$) due to changes in the pdf of SSR noise are very small. But a more detailed analysis is required to fully understand the relationship between the pdf of the SSR noise (added noise) and the performance of the SSR detector.

The foregoing analysis is based on the assumption that the




Figure–7: $P_D$ VS a) parameter p of GGN noise, b) parameter u of MOG environment noise.




Figure–8: Plots of $\sigma_{opt}$ vs $A_1$ for $P_F$=0.1, M=200,N=80, Gaussian SSR noise. Environmental noise is (a) MOG with u = 0.01, (b) GGN with p = 0.5

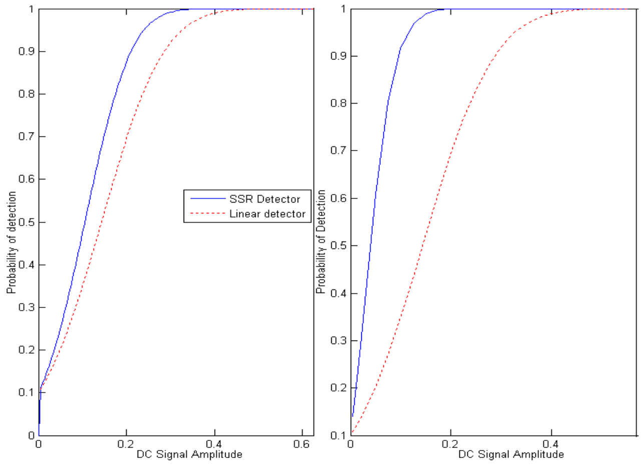



Figure-9: Plots of $P_D$ vs $A_1$ for optimal SSR detector and LMF. $P_F$=0.1, M=200, N=80, Gaussian SSR noise. Environmental noise is (a) MOG with u=0.01, (b) GGN with p=0.5

signal is weak, i.e. $A_1$<<1. It is of interest to consider the performance of the optimal SSR detector when the signal is not weak. The value of $\sigma_{opt}$ in the case of non-weak signals can be determined either theoretically by computing E[Y(t)] and E[$Y^2$(t)] from (6) and (7) respectively (without invoking the weak signal approximation) and finding the value of σ that maximizes the deflection coefficient d(σ) defined in (21), or experimentally by determining the value of σ that maximizes $P_D$. Either approach is quite straightforward but tedious. The variation of $\sigma_{opt}$ with $A_1$ is shown in Fig.8(a) for MOG (u = 0.01) environmental noise and in Fig 8(b) for GGN (p = 0.5) environmental noise. The SSR noise is Gaussian. The corresponding plots of $P_D$ vs $A_1$ for $P_F$ = 0.1 are shown in Figs.9(a) and 9(b). It is seen that the optimal SSR detector offers a significantly better performance than the LMF even when the signal is not weak.

## 5. CONCLUSIONS

In this paper, the performance of the SSR detector for detection of weak signals in leptokurtic (heavy-tailed) noise has been investigated. A method has been proposed to analytically obtain the optimal SSR noise to be added to the system to maximize the SSR detector performance. The method has been verified with experimental results that show excellent agreement between predicted and experimental performance for a wide selection of environmental and SSR noise pdfs. It has been shown that the performance of the optimal SSR detector is very close to that of the locally optimal detector. The performance of the optimal SSR detector is significantly better than that of the linear matched filter in leptokurtic noise environments, and the degree of improvement increases as the kurtosis of the environmental noise becomes higher. The performance of the optimal SSR detector depends on the choice of the SSR noise pdf, but this dependence appears to be quite weak. Finally, it has been shown that in leptokurtic environments the optimal SSR detector performs significantly better than the linear matched filter even when the signal is not weak.

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