CLOSED-FORM AUTOMATICALLY PAIRED 2-D DIRECTION-OF-ARRIVAL ESTIMATION WITH ARBITRARY ARRAYS

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ABSTRACT

A new approach is proposed for two-dimensional (2-D) direction-of-arrival (DOA) estimation with arbitrary array geometries. The proposed method is based on array interpolation and it provides automatically paired source azimuth and elevation angle estimates. It is a fast algorithm and there is no need for search. In addition, the number of sensors required for such an operation is decreased due to array interpolation. Wiener array interpolation is used to improve the interpolation error. Proposed method is applied to the uniform circular array (UCA). Simulation results are presented and it is shown that the method is very effective with limited number of sensors.

1. INTRODUCTION

High resolution azimuth and elevation DOA estimation is an important problem in array signal processing. It has considerable amount of applications in wireless communications, radar, sonar and radio astronomy.

2-D DOA estimation algorithms are generally divided into two categories; search-based and search free methods. Examples of search-based methods are maximum likelihood and MUSIC algorithms. These techniques requires multidimensional search, therefore they are not suitable for practical applications due to their high computational load. The development of search free methods in 2-D is not straightforward. In general, this problem is decomposed into two independent fast 1-D DOA estimation which is based on root-MUSIC [1], ESPRIT [2], propagator method [3], etc. The main drawback of the extended versions of these 1-D algorithms for 2-D angle estimation is the fact that resulting azimuth and elevation angle estimates should be paired appropriately. This pairing operation is a challenging task. Several pairing methods are proposed in the literature [4] (and it's references). Some of these methods are based on searching all the possible pairs. However the performance of these pairing methods is not satisfactory especially at low SNR. On the other hand, fast and automatically paired 2-D DOA estimation methods have no pairing requirement and gives azimuth and elevation angles in closed form expressions. Unitary ESPRIT based automatically paired 2-D angle estimation method, is proposed in [5] which uses uniform rectangular array (URA). In [6] the special structure of two parallel uniform linear arrays (ULA) is for 2-D paired angle estimation. Automatically paired 2-D angle estimation methods are more attractive than the pair matching based methods but these methods are based on some special array structures such as URA, or two parallel ULA.

The proposed method in this paper uses array interpolation in order to decrease the number of sensors required for a fast algorithm and presents closed form expressions for automatically paired 2-D angle estimation. The main advantages of the proposed technique are that it is applicable to the arbitrary planar array geometries and it is possible to estimate correlated or coherent sources. 2-D array interpolation errors are minimized by using the recently proposed Wiener formulation [7]. The proposed method's DOA performance is improved by iteratively improving the array interpolation mapping matrix.

The array interpolation is first applied to ESPRIT based DOA estimation in [8]. But for the 2-D DOA estimation, two identical sub-arrays are necessary and it is required to pair the angles correctly which is not always possible. In this study, an automatically pairing method is presented. This method simply shifts the base array into two appropriate sub-arrays. These virtual sub-array measurements are found with improved 2-D array interpolation. Then these measurements are combined and used with the base array measurements in order to find the DOA estimates. The eigenvalues of the matrix that relates the base array measurements and combined measurements has a magnitude and phase component which is related with azimuth and elevation angles.

The proposed method is evaluated by using UCA. It is known that isotropic and uncoupled planar array geometries are usually desired for 2-D DOA estimation. It is shown that the proposed method can effectively pair and resolve azimuth and elevation angles. The simulations show that the iterative improvements on the mapping matrix improve the performance.

2. PROBLEM FORMULATION

We assume that there are D narrowband plane waves impinging on an arbitrary planar array composed of M sensors located at the positions $[x_l, y_l]$, l=1,…,M. The DOA's of the sources are $\Theta_d=[\phi_d,\theta_d]$ d=1,…,D, where ϕ and θ are the azimuth and elevation angles respectively as shown in Figure 1. If the sensors are identical omnidirectional and far-field assumption is made, the sensor output, $\mathbf{y}$(t), can be written as,

$$\mathbf{y}(t) = \mathbf{A}(\Theta)\mathbf{s}(t) + \mathbf{n}(t), \quad t = 1,\dots,N \tag{1}$$

where N is the number of snapshots and $\mathbf{s}$(t) is a D×1 signal vector which represents a stationary, zero-mean random process uncorrelated with noise. It is assumed that the noise, $\mathbf{n}$(t), is both spatially and temporally white with variance σ^2 and it is also assumed that the source number D is known. $\mathbf{A}(\Theta) = [\mathbf{a}(\phi_1,\theta_1)\dots\mathbf{a}(\phi_D,\theta_D)]$ is the $M \times D$ steering matrix for the planar array and the vectors $\mathbf{a}(\phi_d,\theta_d)$ are given as,

$$\mathbf{a}(\phi_d,\theta_d) = \left[\exp\left\{j\frac{2\pi}{\lambda}(x_1\cos\alpha_d + y_1\cos\beta_d)\right\}\dots\right.$$

$$\exp\left\{j\frac{2\pi}{\lambda}(x_M\cos\alpha_d + y_M\cos\beta_d)\right\}\Bigg]^T, \quad (2)$$

where

$$\cos\alpha_d = \cos\phi_d \sin\theta_d \quad (3)$$
$$\cos\beta_d = \sin\phi_d \sin\theta_d. \quad (4)$$

Note that the array is positioned on a plane with $z = 0$ for simplicity. The output covariance matrix, $\mathbf{R}$, is

$$E\{\mathbf{y}(t)\mathbf{y}(t)^H\} = \mathbf{R} = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \sigma^2\mathbf{I}, \quad (5)$$

where $(.)^H$ denotes the conjugate transpose of a matrix, $\mathbf{R}_s$ is the source correlation matrix and $\mathbf{I}$ is the identity matrix.

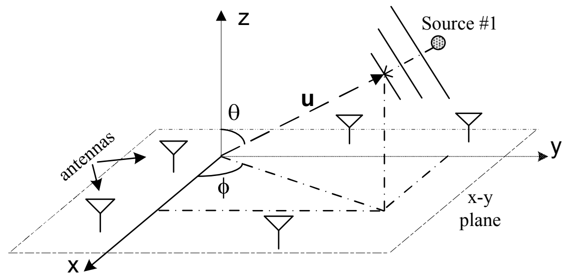

Figure 1: Coordinate system for 2-D angle estimation.

3. 2-D DOA ESTIMATION WITH ARRAY INTERPOLATION

3.1 2-D Array Interpolation

Array interpolation is a well known technique in DOA estimation [9], [10]. It is used to map the covariance matrix of a real array to a virtual array.

2-D array interpolation is implemented by considering a two dimensional interpolation sector for azimuth $[\phi_s, \phi_f]$ and elevation $[\theta_s, \theta_f]$. Let $\mathbf{A}_1(\tilde{\phi}, \tilde{\theta})$ and $\mathbf{A}_2(\tilde{\phi}, \tilde{\theta})$ be the steering matrices for real and virtual array, respectively. The mapping matrix for the conventional array interpolation, $\mathbf{B}_{12}$, is given as,

$$\mathbf{B}_{12} = \mathbf{A}_2(\tilde{\phi}, \tilde{\theta})\mathbf{A}_1(\tilde{\phi}, \tilde{\theta})^H \left(\mathbf{A}_1(\tilde{\phi}, \tilde{\theta})\mathbf{A}_1(\tilde{\phi}, \tilde{\theta})^H\right)^{-1} \quad (6)$$

Given the real array measurements $\mathbf{y}_1 = \mathbf{A}_1\mathbf{s} + \mathbf{n}$ for planar array, we need to find $\mathbf{y}_2 = \mathbf{A}_2\mathbf{s}$ of the virtual planar array. If we define the error as $\mathbf{e} = \mathbf{y}_2 - \mathbf{B}_{12}\mathbf{y}_1$ and find the MSE optimum solution for $\mathbf{B}_{12}$, we obtain,

$$\mathbf{B}_{12} = \mathbf{A}_2(\tilde{\phi}, \tilde{\theta})\mathbf{R}_s\mathbf{A}_1(\tilde{\phi}, \tilde{\theta})^H \big(\mathbf{A}_1(\tilde{\phi}, \tilde{\theta})\mathbf{R}_s \\ \mathbf{A}_1(\tilde{\phi}, \tilde{\theta})^H + \mathbf{R}_n\big)^{-1} \quad (7)$$

If we assume $\mathbf{R}_n = \sigma_n^2\mathbf{I}$ and $\mathbf{R}_s = \sigma_s^2\mathbf{I}$ for uncorrelated source signals, we have,

$$\mathbf{B}_{12} = \sigma_s^2\mathbf{A}_2(\tilde{\phi}, \tilde{\theta})\mathbf{A}_1(\tilde{\phi}, \tilde{\theta})^H \big(\sigma_s^2\mathbf{A}_1(\tilde{\phi}, \tilde{\theta}) \\ \mathbf{A}_1(\tilde{\phi}, \tilde{\theta})^H + \sigma_n^2\mathbf{I}\big)^{-1} \quad (8)$$

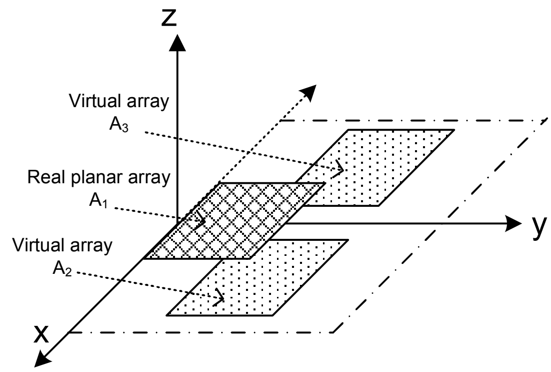

Figure 2: A_1: Real array, A_2: (dx,dy) shifted array and A_3: (-dx,dy) shifted array.

The formulation in (8) is recently proposed in [7]. It provides some important advantages according to the conventional array interpolation. Usually the size of the interpolation sector should be decreased in order to obtain acceptable DOA performance [8]. Wiener formulation in (8) allows one to choose larger interpolation sector. It also improves the interpolation performance especially at low SNR.

3.1.1 Iterative Improvement

If an initial DOA estimate is available, array interpolation error can be decreased by selecting a narrow sector in the neighborhood of the initial estimate. Assume that an initial 2-D DOA estimate is available $\hat{\Theta}$=[$(\hat{\phi}_1, \hat{\theta}_1)$, ..., $(\hat{\phi}_D, \hat{\theta}_D)$]. $\mathbf{B}_{12}$ can be constructed by considering narrow sectors in the neighborhood of each azimuth and elevation angles. In this case, the calibration angles in these sectors can be defined as $\tilde{\phi}_{i,j} \in [\hat{\phi}_i - \phi_\varepsilon,\ \hat{\phi}_i + \phi_\varepsilon]$, $\tilde{\phi}_{i,j}$=$\hat{\phi}_i - \phi_\varepsilon + j\Delta\phi$ and $\tilde{\theta}_{i,m} \in [\hat{\theta}_i - \theta_\varepsilon,\ \hat{\theta}_i + \theta_\varepsilon]$, $\tilde{\theta}_{i,m}$=$\hat{\theta}_i - \theta_\varepsilon + m\Delta\theta$. ϕ_ε and θ_ε are the sector sizes for azimuth and elevation angles respectively. $\tilde{\Theta}$ is a collection of the azimuth and elevation angle pairs in the interpolation sectors corresponding to each initial DOA estimate, i.e.,

$$\tilde{\Theta} = [\tilde{\Theta}_1, \tilde{\Theta}_2, \ldots, \tilde{\Theta}_D] \quad (9)$$

where $\tilde{\Theta}_i$=[$(\tilde{\phi}_{i1}, \tilde{\theta}_{i1})$, ..., $(\tilde{\phi}_{ij}, \tilde{\theta}_{im})$, ..., $(\tilde{\phi}_{iN_\phi}, \tilde{\theta}_{iN_\theta})$] for $j = 0, 1, \cdots, N_\phi = \lfloor \frac{2\phi_\varepsilon}{\Delta\phi} \rfloor$ and $m = 0, 1, \cdots, N_\theta = \lfloor \frac{2\theta_\varepsilon}{\Delta\theta} \rfloor$. By using the definition of $\tilde{\Theta}$, $\mathbf{A}_1(\tilde{\phi}, \tilde{\theta})$ and $\mathbf{A}_2(\tilde{\phi}, \tilde{\theta})$ are found as explained in the previous section. Then the interpolation matrix is found from (8). As the DOA estimation is iterated, it is also possible to decrease the sector size.

3.2 Fast 2-D DOA Estimation

In this study, array interpolation is used to generate virtual shifted sub-arrays. These sub-arrays are required to estimate the azimuth and elevation angles without pairing problem. The virtual measurements from the two virtual sub-arrays are combined to generate angle information in magnitude and phase. This is then merged with the real array measurements matrix in order to use ESPRIT algorithm for a fast computation. This formulation allows us to obtain azimuth and elevation angle estimated from the magnitude and phase of the eigenvalues of a matrix.

3.2.1 Automatically Pairing Azimuth and Elevation Angles

It is asssumed that the only available data comes from the real array as shown in Figure 2. The sensor output for the real array is $\mathbf{y}_1 = \mathbf{A}_1\mathbf{s} + \mathbf{n}_1$ where the positions of the M sensors are $[x_l, y_l]$, l=1,...,M. The data for the two virtual array are obtained by applying array interpolation as shown in Figure 2. The displacement from the real array is $[x_l \pm d_x, y_l + d_y]$ where d_x and d_y are the constant terms and $\mathbf{B}_{12}$ and $\mathbf{B}_{13}$ are the mapping matrices for this purpose. The sensor output for the second virtual array is obtained as,

$$\begin{aligned} \mathbf{y}_2 &= \mathbf{B}_{12}\mathbf{y}_1 = \mathbf{B}_{12}\mathbf{A}_1\mathbf{s} + \mathbf{B}_{12}\mathbf{n}_1 \\ \mathbf{y}_2 &= \mathbf{A}_2\mathbf{s} + \mathbf{B}_{12}\mathbf{n}_1 \end{aligned} \quad (10)$$

The array output for the third virtual array is obtained as,

$$\begin{aligned} \mathbf{y}_3 &= \mathbf{B}_{13}\mathbf{y}_1 = \mathbf{B}_{13}\mathbf{A}_1\mathbf{s} + \mathbf{B}_{13}\mathbf{n}_1 \\ \mathbf{y}_3 &= \mathbf{A}_3\mathbf{s} + \mathbf{B}_{13}\mathbf{n}_1 \end{aligned} \quad (11)$$

As it is easily seen, virtual array steering matrices ($\mathbf{A}_2$ and $\mathbf{A}_3$) are related to the real array as,

$$\mathbf{A}_2 = \mathbf{A}_1\mathbf{\Phi}_1 \quad (12)$$

$$\mathbf{A}_3 = \mathbf{A}_1\mathbf{\Phi}_2 \quad (13)$$

and if we take

$$d_x = d_y = d \leq \lambda/2 \quad (14)$$

and $\frac{2\pi}{\lambda}d = \tau_{xy}$, $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ can be written as,

$$\begin{aligned} \mathbf{\Phi}_1 = diag\{&\exp(j\tau_{xy}(\cos\alpha_1 + \cos\beta_1))\ldots \\ &\exp(j\tau_{xy}(\cos\alpha_D + \cos\beta_D))\} \end{aligned} \quad (15)$$

$$\begin{aligned} \mathbf{\Phi}_2 = diag\{&\exp(j\tau_{xy}(-\cos\alpha_1 + \cos\beta_1))\ldots \\ &\exp(j\tau_{xy}(-\cos\alpha_D + \cos\beta_D))\} \end{aligned} \quad (16)$$

The virtual array outputs $\mathbf{y}_2$ and $\mathbf{y}_3$ are combined in order to obtain the array output of another virtual array, $\mathbf{y}_4$ as,

$$\begin{aligned} \mathbf{y}_2 + \mathbf{y}_3 &= (\mathbf{A}_2 + \mathbf{A}_3)\mathbf{s} + (\mathbf{B}_{12} + \mathbf{B}_{13})\mathbf{n}_1 \\ \mathbf{y}_4 &= \mathbf{A}_1\underbrace{(\mathbf{\Phi}_1 + \mathbf{\Phi}_2)}_{\mathbf{\Phi}}\mathbf{s} + (\mathbf{B}_{12} + \mathbf{B}_{13})\mathbf{n}_1 \end{aligned} \quad (17)$$

where

$$\begin{aligned} \mathbf{\Phi} = diag\Big\{&2\cos(\tau_{xy}\cos\alpha_1)e^{j\tau_{xy}(\cos\beta_1)}\ldots \\ &2\cos(\tau_{xy}\cos\alpha_D)e^{j\tau_{xy}(\cos\beta_D)}\Big\} \end{aligned} \quad (18)$$

Note that the eigenvalues of $\mathbf{\Phi}$ have DOA information at the magnitude and phase components. In fact, the proposed technique is based on this observation. The automatically paired azimuth and elevation DOA angles are obtained by using the magnitude and phase relations.

In the following subsection, the steps of the ESPRIT based automatically paired 2-D DOA estimation algorithm are presented. The proposed technique uses the real $\mathbf{y}_1$ and interpolated virtual measurements, $\mathbf{y}_4$.

3.2.2 Interpolated 2-D ESPRIT Algorithm

The procedure for the proposed algorithm is as follows.
Step 1: Construct $\mathbf{B}_{12}$ and $\mathbf{B}_{13}$ from (8) for the angular sector $[\phi_s, \phi_f]$ and $[\theta_s, \theta_f]$.
Step 2: Given $\mathbf{B}_{12}$ and $\mathbf{B}_{13}$, compute the virtual sensor outputs $\hat{\mathbf{y}}_2$ and $\hat{\mathbf{y}}_3$ using the output samples of the real planar array $\mathbf{y}_1$.

$$\hat{\mathbf{y}}_4 = \frac{1}{2}(\mathbf{B}_{12}\mathbf{y}_1 + \mathbf{B}_{13}\mathbf{y}_1) \quad (19)$$

Combine the measurements as

$$\hat{\mathbf{y}}_5 = \begin{bmatrix} \mathbf{y}_1 \\ \hat{\mathbf{y}}_4 \end{bmatrix} \quad (20)$$

Step 3: Apply ESPRIT algorithm to the combined measurement (20) to find $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_4$. The relation between $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_4$ is

$$\hat{\mathbf{S}}_4 = \hat{\mathbf{S}}_1\hat{\mathbf{\Phi}} \quad (21)$$

where $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_4$ are the signal spaces of the real, $\mathbf{y}_1$, and the combined virtual, $\hat{\mathbf{y}}_4$, array respectively. Find the least squares (LS) optimum solution for $\hat{\mathbf{\Phi}}$ which is related with the source azimuth and elevation angles as in (18).
Step 4: Compute the eigenvalues of $\hat{\mathbf{\Phi}}$ as

$$\hat{\mathbf{E}} = eig\{\hat{\mathbf{\Phi}}\} \quad (22)$$

where $\hat{\mathbf{E}}$ is a vector with D $\times$ 1 dimension. Compute the azimuth and elevation angles using the magnitude and phase components of the each element in (22) as,

$$\phi_i = \arctan\left(\frac{\arg(E_i)}{\arccos(|E_i|)}\right) \quad (23)$$

$$\theta_i = \arcsin\left(\sqrt{\left(\frac{\arg(E_i)}{\tau_{xy}}\right)^2 + \left(\frac{\arccos(|E_i|)}{\tau_{xy}}\right)^2}\right). \quad (24)$$

for $i = 1, \ldots, D$. For the iterative improvement, we can add the following additional step.
Step 5: (For iterative improvements) Use estimated angles in Step 4 and repeat Step 1, 2, 3 and 4 K times to improve the DOA estimates.

4. SIMULATION RESULTS

In this section, we apply the proposed method to some numerical examples in order to show 2-D DOA performance. The proposed method is applicable to the arbitrary planar array geometries. In these examples we used UCA as the base array. The distance between each antenna in the UCA is half a wavelength, i.e., $\lambda/2$.

In simulations, source angles are considered in degrees where azimuth angle is between 0 and 360 degrees and elevation angle is between 0 and 90 degrees. There are 1000 trials for each experiment and the number of snapshots is 256. The ϕ_ε and θ_ε defined in section 3.1.1, are chosen as 1 degrees. The displacements in (14) are taken as $d_x = d_y = \lambda/2$ in simulations.

In the first example M=13 element UCA is used as the base (real) array. As it is seen from the Figure 3, the virtual sub-arrays are the shifted versions of the real array. There are two sources at the (ϕ_1=44,θ_1=24) and (ϕ_2=56,θ_2=35) where

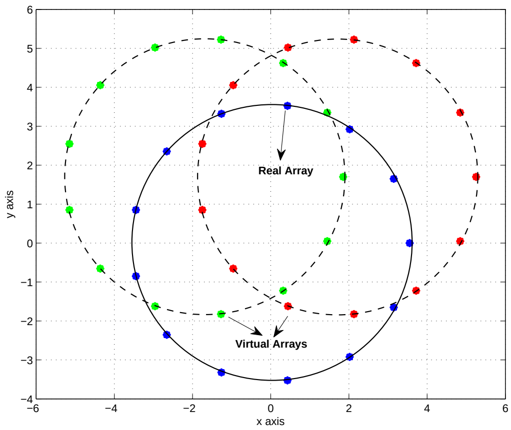

Figure 3: Base array and shifted sub arrays for UCA.

ϕ and θ angles are the azimuth and elevation angles respectively. In this example source signals are uncorrelated and 2-D array interpolation sectors are chosen as for azimuth $[\phi_s = 35, \phi_f = 65]$, for elevation $[\theta_s = 15, \theta_f = 45]$. For iterative improvements we chose K as three. The performance of the proposed method and the CRB is given in Figure 4. As it is seen from the Figure 4, the proposed method azimuth and elevation angle estimation performance is good and follows the CRB for azimuth and elevation especially when the iterative approach is employed. At high SNR, there is a tendency for flooring effect due to the array interpolation error.

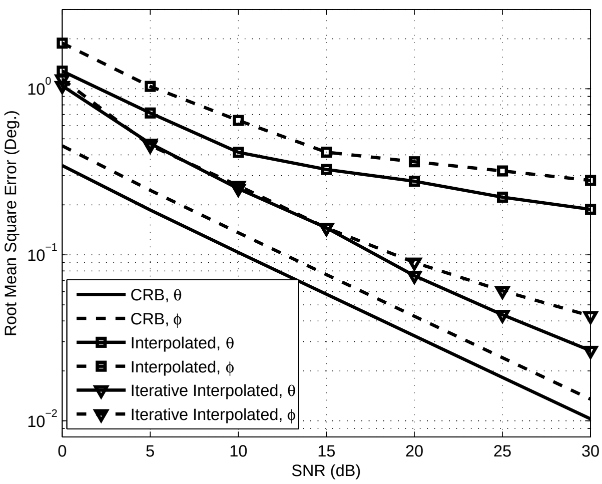

Figure 4: 2-D DOA performance for two sources at (44, 24) and (56, 35) when UCA is choosen as the base array.

Figure 5 shows the 2-D DOA performance when there are two sources; one is fixed at (50°, 36°) and the other source azimuth is swept between 25° and 75° in one degree resolution when the elevation angle is fixed at 25°. SNR is at 15 dB. The base array is the 13 element UCA. 2-D array interpolation sectors are chosen as $[\phi_s = 35, \phi_f = 65]$, $[\theta_s = 15, \theta_f = 45]$. In this simulation, the iterative approach uses three iterations. As it is seen from this figure, small number of iterations is sufficient to improve the DOA performance significantly. Furthermore, iterations also improve the DOA estimation for the out-of-sector sources.

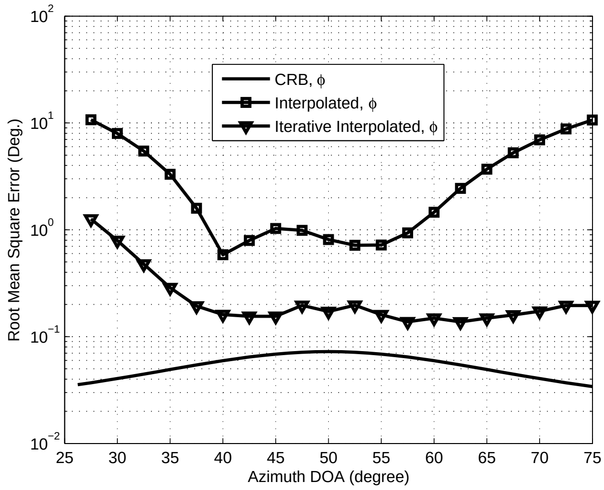

Figure 5: 2-D (Azimuth) DOA performance for two sources when one source is swept between (25°,25°)–(75°,25°) while the other source is at (50°, 36°). SNR is 15 dB.

A similar experiment is done in order to observe the elevation performance. Figure 6 shows the DOA performance for the elevation angle when there are two sources; one is fixed at (56°, 25°) and the other source elevation is swept between 10° and 50° in one degree resolution when the azimuth angle is fixed at 56°. SNR is at 15 dB. 2-D array interpolation sectors are chosen as $[\phi_s = 40, \phi_f = 70]$, $[\theta_s = 15, \theta_f = 45]$.

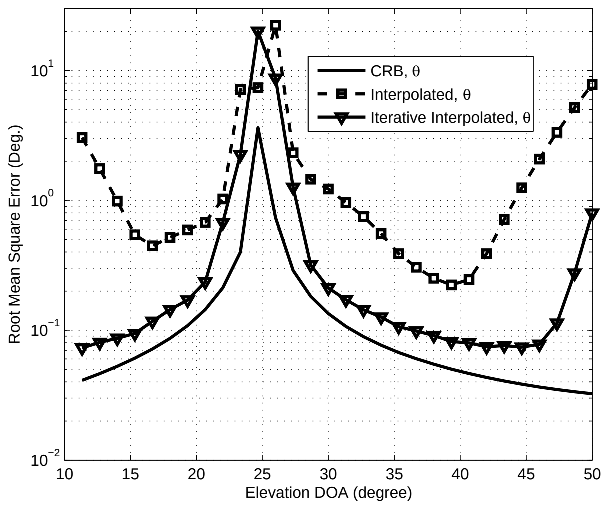

Figure 6: 2-D (Elevation) DOA performance for two sources when one source is swept between (56°,10°)–(56°,50°) while the other source is at (56°, 25°). SNR is 15 dB.

5. CONCLUSIONS

In this paper, we have proposed a fast and automatically paired 2-D DOA estimation method which can be applied to arbitrary array geometries. It is shown that if an array is shifted and combined appropriately, it is possible to estimate both azimuth and elevation angle by solving a single root. The shifted versions of the arbitrary array are found with 2-D array interpolation which provides certain advantages. It is possible to resolve D sources with $D+1$ sensors, while the full array requires $3 \times D$ sensors. The performance of the conventional array interpolation is improved by using a Wiener formulation [7]. This formulation also allows choosing wider 2-D angular sectors for array interpolation. Several experiments are performed and it is shown that the proposed method performs well in 2-D DOA estimation.

REFERENCES

[1] A. J. Barabell, "Improving the resolution performance of eigenstructure-based direction finding algorithms", in *IEEE ICASSP-83.*, pp. 336–339.

[2] R. H. Roy, "ESPRIT - Estimation of signal parameters via rotational invariance techniques", *Ph.D. dissertation*, Standford Univ., 1987.

[3] S. Marcos, A. Marsal, and M. Benidir, "The propagator method for source bearing estimation", *Signal Processing*, vol. 42, pp. 121–138, 1995

[4] S. Kikuchi, H. Tsuji, A. Sano, "Pair-Matching Method for Estimating 2-D Angle of Arrival With a Cross-Correlation Matrix", *IEEE Antennas and Wireless Propagation Letters*, vol. 5, pp. 35–40, 2006

[5] M. D. Zoltowski, M. Haardt, C. P. Mathews, "Closed-Form 2-D Angle Estimation with Rectangular Arrays in Element Space or Beamspace via Unitary ESPRIT", *IEEE Transactions on Signal Processing*, *44*(3), pp. 316–328, 1996

[6] T. Xia, Y. Zheng, Q. Wan, X. Wang, "Decoupled Estimation of 2-D Angles of Arrival Using Two Parallel Uniform Linear Arrays", *IEEE Transactions on Antenn. Propogat.*, *55*(9), pp. 2627–2632, 2007

[7] T. E. Tuncer, T. K. Yasar, and B. Friedlander, "Direction of arrival estimation for nonuniform linear arrays by using array interpolation", *Radio Sci.*, 42, 2007

[8] A. J. Weiss, M. Gavish, "The Interpolated Esprit Algorithm", in *International Conference on Acoustics, Speech, and Signal Processing*, ICASSP-1991, Canada, Vol.5, pp. 3725–3728

[9] T. P. Bronez, "Sector interpolation of non-uniform arrays for efficient high resolution bearing estimation", in *Acoustics, Speech, and Signal Processing, 1988. ICASSP-88., 1988 International Conference on*, New York, pp. 2885–2888

[10] P. Hyberg, M. Jansson, and B. Ottersten, "Array interpolation and DOA MSE reduction", *Signal Processing, IEEE Transactions on*, *53*(12), pp. 4464–4471, 2005