POSTERIOR CRAMER RAO LOWER BOUNDS FOR THE RESPIRATORY MODEL PARAMETER ESTIMATION

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ABSTRACT

In this paper, we introduce a new approach for the evaluation of dual Posterior Cramer-Rao Lower Bounds (PCRLBs) where the estimation procedure involves time-invariant, stationary system parameters and system states. Dual estimation may be required in respiratory system modeling where the parameters are usually physiological model settings. Bayesian solution of the parameter estimation lets us derive the dual PCRLBs with the help of the block matrix algebra. For the state estimation bound, our results give the same expressions as in the previous studies. In addition, we have obtained the iterative PCRLB expressions for the parameter estimation in the Mead respiratory model. Dual UKF and EKF error variances that were obtained in our previous work are demonstrated with respect to these bounds. Results show that UKF performs better than the EKF for the dual estimation in the Mead model.

1. GENERAL INFORMATION

One of the widely accepted ways to demonstrate the performance of the unbiased estimator is to give the lower bounds for the error covariance that can be achieved for the state-observation model of interest:

$$\mathscr{P} = E\left\{\left[\hat{\mathbf{X}} - \mathbf{X}\right]\left[\hat{\mathbf{X}} - \mathbf{X}\right]^T\right\} \geq J^{-1} \qquad (1)$$

where $\mathscr{P}$ is the estimator's error covariance matrix, J is the *Fisher Information Matrix* (FIM), and $\hat{\mathbf{X}}$ is the estimate of the random process $\mathbf{X}$. Posterior Cramer Rao Lower Bound (PCRLB) developed by [1] serves an important tool for time-varying random parameter and state estimation. PCRLB was studied in the literature for many tracking, detection and estimation problems [1],[2].

Mead respiratory model is used for respiratory mechanics determination from measured pressure and flow in the respiratory system [3]. Estimation of the parameters for the Mead model is not a straight-forward task due to noise and error involved in the measurements. It is previously shown that Kalman Filtering may be used in order to estimate the parameters as well as the states in respiratory system models [4]. In this work, we present the closed-form expression of the PCRLB for the dual estimation in the Mead model and illustrate the bounds for each parameters and states. In the simulations, system parameters are assumed to be Gaussian Distributed. Moreover, previously calculated Unscented Kalman Filter (UKF) and Extended Kalman Filter (EKF) estimators' error covariances [4] were demonstrated with respect to calculated PCRLB to compare the performance of the estimators for the Mead model.

2. PCRLB FOR DUAL ESTIMATION

A classical Fisher information matrix [1] is used for the lower bound determination in the state or parameter estimation. However, the problem rises if the parameters are considered as a time-invariant unknown random variables and the uncertainty on the parameter transition is set to null. That is referred as a singular state covariance matrix, $\mathbf{Q}_k$, where k is the time index. The difficulties of singular $\mathbf{Q}_k$ matrix are handled by PCRLB [1], but direct utilization of the general PCRLB recursions do not necessarily give closed-form expressions for all models. Thus, in this work we present the PCRLB for both states and parameter simultaneously, where the parameters are assumed to be time-invariant and stationary random variables. PCRLB recursions are driven for finite observation time, and the relations between PCRLB, system states, and parameters are established.

The Mead model of the respiratory system can be generally represented as:

$$\mathbf{x}_k = f_k\left(\mathbf{x}_{k-1}, \mathbf{\Theta}, \mathbf{u}_k\right) + \mathbf{q}_k \qquad (2)$$

$$\mathbf{z}_k = h_k\left(\mathbf{x}_k, \mathbf{\Theta}, \mathbf{u}_k\right) + \mathbf{r}_k \qquad (3)$$

where $\mathbf{x_k}$ is the system state vector, $\mathbf{z_k}$ is the measurements in the respiratory system, $f(\bullet)$ and $h(\bullet)$ are the nonlinear system functions. $\mathbf{u}_k$ is devoted to the known inputs and $\mathbf{\Theta}$ is the unknown parameter vector of interest. We assume that $\mathbf{q}_k \sim \mathscr{N}[0, Q_k]$ and $\mathbf{r}_k \sim \mathscr{N}[0, R_k]$ are the Gaussian type additive state and observation noises respectively. Then it is driven in Appendix that PCRLB for the state and parameter estimation respectively are:

$$\mathscr{P}_k^x = \mathbf{Q}_{k-1} + F_{k-1}^x\left(\left(H_{k-1}^x\right)^T \mathbf{R}_{k-1}^{-1} H_{k-1}^x + \mathscr{P}_{k-1}^{-1}\right)^{-1}\left(F_{k-1}^x\right)^T \qquad (4)$$

$$\mathscr{P}_k^\theta = \left(T_{k-1}^{\theta\theta}\right)^{-1} + \left(\left(H_{k-1}^\theta\right)^T \mathbf{R}_{k-1}^{-1} H_{k-1}^\theta + \left(F_{k-1}^\theta\right)^T \mathbf{Q}_{k-1}^{-1} F_{k-1}^\theta + \mathscr{P}_{k-1}^{-1}\right)^{-1} \qquad (5)$$

where $\mathbf{Q}_{k-1}$ and $\mathbf{R}_{k-1}$ are the state noise error covariance matrix and measurement noise error covariance matrix respectively. In order to generate PCRLB, we utilize the below definitions:

$H_k^x = \frac{\partial h_k(\mathbf{x}_k,\theta)}{\partial \mathbf{x}_k}$, $H_k^\theta = \frac{\partial h_k(\mathbf{x}_k,\theta)}{\partial \theta}$, $F_k^x = \frac{\partial f_k(\mathbf{x}_k,\theta)}{\partial \mathbf{x}_k}$, $F_k^\theta = \frac{\partial f_k(\mathbf{x}_k,\theta)}{\partial \theta}$

$M^{\theta\theta} = E\left\{\frac{\partial^2}{\partial \theta^2}\log \Pr_\theta(\theta)\right\}$ and

$T_k^{\theta\theta} = -E\left\{\frac{\partial T_k^\theta}{\partial \theta}\right\} =$

$= \sum_{n=0}^{k} \left(H_n^\theta\right)^T \mathbf{R}_n^{-1} H_n^\theta + \left(F_n^\theta\right)^T \mathbf{Q}_n^{-1} F_n^\theta + M^{\theta\theta}$

3. MEAD RESPIRATORY MODEL

In this section expressions derived for the PCRLB of the states and the parameters are applied to the linear respiratory system model, called the Mead model [3]. The model state and the observation equations are obtained in the discrete-time as follows:

$$\mathbf{x}_{k+1} = \begin{bmatrix} 1-R_c/L & 0 & -1/L & -1/L & 1/L \\ 0 & 1-1/R_pC_l & 1/R_pC_l & 0 & 0 \\ 1/C_b & 1/R_pC_b & 1-1/R_pC_b & 0 & 0 \\ 1/C_w & 0 & 0 & 1 & 0 \\ -1/C_e & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/C_e \end{bmatrix} \dot{V}_k + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} P_k^{mus} + \mathbf{q}_k \tag{6}$$

$$\mathbf{z}_k = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_k + P_k^{ven} + \mathbf{r}_k \tag{7}$$

where measurement noise and the state noise are zero-mean, white Gaussian, $\mathbf{q}_k \sim \mathcal{N}[0, Q_k]$ and $\mathbf{r}_k \sim \mathcal{N}[0, R_k]$. $\dot{V}_k$ is the measured airway flow sequence. Parameter vector and state vector are defined respectively as:
$\theta = \begin{bmatrix} R_c & L & C_l & C_b & R_p & C_w & C_e & P_{mus\,\max} \end{bmatrix}^T$ and
$\mathbf{x}_k = \begin{bmatrix} \dot{V}_k^L & P_k^{C_l} & P_k^{C_b} & P_k^{C_w} & P_k^{C_e} \end{bmatrix}^T$.
In (6) and (7) the muscular pressure and the non-invasive ventilatory effects are simulated as exponential functions:

$$P_k^{mus} = \begin{cases} -P_{mus\,\max}\left(1-\frac{k}{T_I}\right) + P_{mus\,\max} & 0 \le k \le T \\ P_{mus\,\max} e^{-k/\tau_m} & T_I \le k \le T \end{cases} \tag{8}$$

$$P_k^{ven} = \begin{cases} PEEP & 0 \le k \le t_{trig} \\ P_{ps}\left(1-e^{-k/\tau_{vi}}\right) & t_{trig} < k \le T_I \\ P_{ps}\left(e^{-k/\tau_{ve}}\right) & T_I < k \le T \end{cases} \tag{9}$$

where the constant model parameters P_{ps}, T_I, T, τ_m, τ_{vi}, and τ_{ve} were set to constant values to resemble the corresponding respiratory pressures [?]. Then, the associated Jakobian matrices can be obtained:

$$H_k^x = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

$$H_k^\theta = 0 \tag{11}$$

$$F_k^x = \begin{bmatrix} 1-R_c/L & 0 & -1/L & -1/L & 1/L \\ 0 & 1-1/R_pC_l & 1/R_pC_l & 0 & 0 \\ 1/C_b & 1/R_pC_b & 1-1/R_pC_b & 0 & 0 \\ 1/C_w & 0 & 0 & 1 & 0 \\ -1/C_e & 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

$$F_k^\theta = g_k(\mathbf{X}, \theta) \tag{13}$$

We assume the state and measurement noises are Gaussian with constant covariances $\mathbf{Q} = 10^{-4} I_{5\times5}$ and $R = 1$. Furthermore, since *a prior* distribution of the parameters effects the performance of the estimator, we require to define the probability model of the parameters. Gaussian assumption for the parameter estimation in Kalman Filters oblige us to use Gaussian Distribution for the probability model $\Pr_\theta(\theta) = 1\Big/\sqrt{2\pi|\mathbf{Q}_\theta|}\exp\left\{-0.5\theta\mathbf{Q}_\theta^{-1}\theta^T\right\}$. Thus, it is found that:

$$M^{\theta\theta} = \mathbf{Q}_\theta^{-1} \tag{14}$$

In order to illustrate the PCRLBs, artificial respiratory signals are produced with (6) and (7), and bounds are computed for one breathing cycle. The specified parameter model covariance matrix was set to $\mathbf{Q}_\theta = 10^{-3} I_{8\times8}$, and the PCRLB initials are $\mathscr{P}_0^x = 2\times10^{-1} I_{5\times5}$, $\mathscr{P}_0^\theta = 10^{-1} I_{8\times8}$ for the Mead model.

4. RESULTS AND DISCUSSIONS

UKF and EKF estimators were previously applied to the respiratory parameter estimation problem where the real signals acquired from the patients and healthy subjects were used for the observed signals. In this section UKF and EKF estimators won't be explained in details due to the physical limitations. Comprehensive explanations and estimators' algorithm can be found in [6, 7] and more specifically in [4]. In this work we used dual UKF and EKF estimators which enable simultaneously estimate both the states and the parameters from the noisy observations. In this respect, the parameters are assumed to be the time-invariant random processes which are modeled as:

$$\theta_{k+1} = \theta_k \tag{15}$$

Dual UKF and EKF estimators were applied to artificially produced respiratory signals by using Mead model equations in (6), (7), and (15). Constraints due to the physical limitations on the parameters ($0 \le \theta \le \infty$) are also applied to the dual estimation problem. In the algorithm, after the sigma points were calculated and after the Prediction step, constrained sigma points were obtained with the projection method explained in [4]. Artificial output signal was computed with the parameters given in [4]. Furthermore, UKF algorithm parameters were set according to the minimum Mean Squared Error (MSE) computed in the artificial data run. Monte Carlo simulations were performed with $MC = 100$ run by artificial data series. $\alpha = 0.1$ for the parameters whereas $\alpha = 0.5$ for the states. κ, the secondary adjustment parameter was set to $\kappa = 1$ for the minimum MSE. Finally, set $\beta = 2$ indicates that *a prior* distribution of the output signal was Gaussian distributed.

Fig.1a and Fig.1b illustrate the relation between the parameter and state PCRLBs and the estimators' error variance evolution throughout the breathing period. It is notable that error variance of UKF estimator follow more closely to the PCRLB than error variance of EKF estimator does. It is the fundamental indicator that UKF perform better than EKF for our models. However, the degree of the performance difference between the estimators vary among the parameters and states. On the other hand, parameter PCRLBs alone can be seen on Fig.2. It can be noticed that each parameter has its own PCRLB that presumedly depends on the relation between the parameter and the states. $P_{mus\,\max}$ has the lowest bound due to the direct proportion between the parameter and the states. On the other hand C_w has the worst bound. We observed that bounds on the parameters nicely coincides with the estimation performance demonstrated in [4]. It was noticed that nonlinear dependence and inverse ratio are the contributions to the high PCRLB values. It is also observed that the transition from the inspiration to expiration results in the knees in the PCRLB curves (see Fig.2). The effect of sudden decrease in the P_k^{ven} and P_k^{mus} results in the reduction on the resistive parameters and inductance element, L.

Moreover, we investigated the signal-to-noise ratio (SNR) effects on the state PCRLBs. In the respiratory signal measurements SNR is quite high due to the accurate airflow and the pressure transducers. Thus, we range SNR between $25-75dB$ which corresponds to the measurement noise variances of $R=2-0.02$ (Fig.3). As it is expected the PCRLB of the state P_{C_e} decreases with increasing SNR. The direct effects of the measurement equation on the state bound is obvious in Fig.3. The use of the measured airflow on the determination of the state $\dot{V}_l$ results in more than $10th$ times lower PCRLB, which shows the importance of the different measurement locations on the respiratory system modeling.

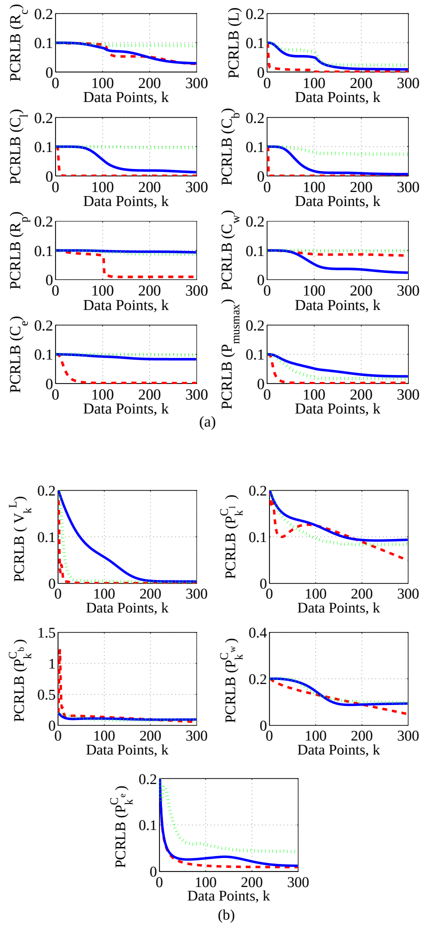

Figure 1: Posterior Cramer-Rao Lower Bound (PCRLB) evolution in the Mead model throughout one breathing cycle (a) for the parameter estimation, and (b) for the state estimation. Dashed red lines represent the PCRLB while solid blue and doted green lines are the estimator error variances of Unscented Kalman Filter (UKF) and Extended Kalman Filter (EKF) respectively.

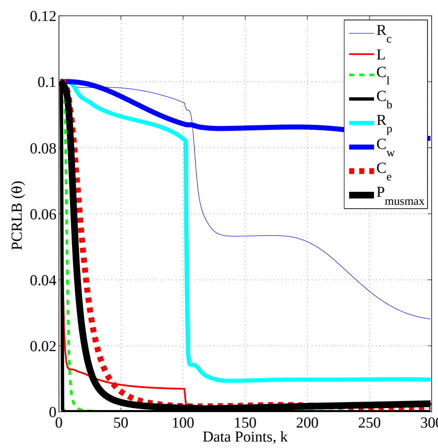

Figure 2: Posterior Cramer-Rao Lower Bound evaluation through one breathing cycle for the parameter estimation.

5. APPENDIX

FIM is redefined for the dual estimation where the parameters are also assumed to be the random variables:

$$J_{ij}=E\left\{-\frac{\partial^2 \log \Pr_{z,x,\theta}(\mathbf{Z},\mathbf{X},\theta)}{\partial \mathbf{w}_i \partial \mathbf{w}_j}\right\} \quad i,j=1,\cdots N_x \tag{16}$$

where joint probability distribution is given below:

$$\begin{aligned}
&\Pr_{z,x,\theta}(\mathbf{Z},\mathbf{X},\theta)\\
&=\Pr_{z|x,\theta}(\mathbf{Z}|\mathbf{X},\theta)\Pr_{x|\theta}(\mathbf{X}|\theta)\Pr(\theta)\\
&=\prod_{i=0}^{k-1}\Pr_{z|x,\theta}(\mathbf{z}_i|\mathbf{x}_i,\theta)\Pr_{x|\theta}(\mathbf{x}_0|\theta)\\
&\times\prod_{i=1}^{k}\Pr_{x|\theta}(\mathbf{x}_i|\mathbf{x}_{i-1},\theta)\Pr(\theta)
\end{aligned} \tag{17}$$

In order to generate the FIM, we utilize the score function $s(\mathbf{Z}_k,\mathbf{X}_k,\theta)=\partial \log \Pr_{z,x,\theta}(\mathbf{Z}_k,\mathbf{X}_k,\theta)/\partial \mathbf{w}_k$ where $\mathbf{w}_k = \begin{bmatrix} \mathbf{x}_k^T & \theta^T \end{bmatrix}^T$ is the joint parameter vector. Then the score function can be given as:

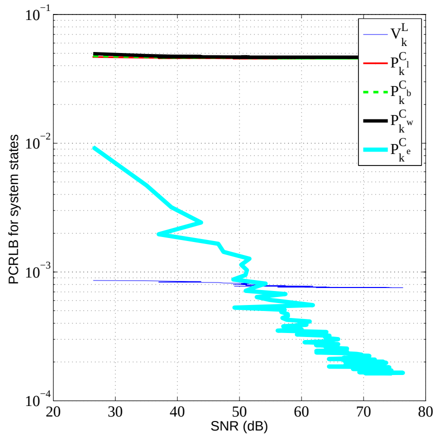

Figure 3: Posterior Cramer-Rao Lower Bound evaluation versus signal-to-noise ratio for the state estimation.

$$s(\mathbf{Z}_k,\mathbf{X}_k,\theta)=\begin{bmatrix} s_0(\mathbf{Z}_0,\mathbf{X}_0,\theta)\\ \vdots \\ s_i(\mathbf{Z}_i,\mathbf{X}_i,\theta)\\ \vdots \\ s_k(\mathbf{Z}_k,\mathbf{X}_k,\theta)\end{bmatrix}^T_{1\times(k+1)(N_x+N_\theta)} \tag{18}$$

where N_x and N_θ are the length of the state and parameter vectors respectively. The elements of the score function from the observations up to and including k are defined as follows:

$$s_i(\mathbf{Z}_i,\mathbf{X}_i,\theta)=\begin{bmatrix} H_i^x\mathbf{R}_i^{-1}(\mathbf{z}_i-h_i(\mathbf{x}_i,\theta))\\ +F_i^x\mathbf{Q}_i^{-1}(\mathbf{x}_{i+1}-f_i(\mathbf{x}_i,\theta))\\ -\mathbf{Q}_{i-1}^{-1}(\mathbf{x}_i-f_{i-1}(\mathbf{x}_{i-1},\theta))\\ T_i^\theta\end{bmatrix}^T_{1\times(N_x+N_\theta)} \tag{19}$$

$$s_0(\mathbf{Z}_0,\mathbf{X}_0,\theta)=\begin{bmatrix} H_0^x\mathbf{R}_0^{-1}(\mathbf{z}_0-h_0(\mathbf{x}_0,\theta))\\ +F_0^x\mathbf{Q}_0^{-1}(\mathbf{x}_1-f_0(\mathbf{x}_0,\theta))\\ -\mathbf{P}_0^{-1}(\hat{\mathbf{x}}_0-\mathbf{x}_0)\\ T_0^\theta\end{bmatrix}^T_{1\times(N_x+N_\theta)} \tag{20}$$

and

$$s_k(\mathbf{Z}_k,\mathbf{X}_k,\theta)=\begin{bmatrix} -\mathbf{Q}_{k-1}^{-1}(\mathbf{x}_k-f_{k-1}(\mathbf{x}_{k-1},\theta))\\ T_k^\theta\end{bmatrix}^T_{1\times(N_x+N_\theta)} \tag{21}$$

FIM is obtained as the gradient of the score function. Thus, by using the block matrix differentiation, FIM can be written as:

$$J_k=-E\left\{\frac{\partial s(\mathbf{Z}_k,\mathbf{X}_k,\theta)^T}{\partial\mathbf{w}_k}\right\} = \begin{bmatrix} \frac{\partial s_0}{\partial\mathbf{w}_0} & \frac{\partial s_0}{\partial\mathbf{w}_1} & \cdots & \frac{\partial s_0}{\partial\mathbf{w}_k}\\ \frac{\partial s_1}{\partial\mathbf{w}_0} & \frac{\partial s_1}{\partial\mathbf{w}_1} & \cdots & \frac{\partial s_1}{\partial\mathbf{w}_k}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial s_k}{\partial\mathbf{w}_0} & \frac{\partial s_k}{\partial\mathbf{w}_1} & \cdots & \frac{\partial s_k}{\partial\mathbf{w}_k}\end{bmatrix}_{(k+1)(N_x+N_\theta)\times(k+1)(N_x+N_\theta)} \tag{22}$$

Using the score function (19), (20), and (21) FIM takes the following form:

$$J_k=\begin{bmatrix} J_{k(11)} & J_{k(12)} & \cdots & J_{k(1(k+1))}\\ J_{k(21)} & J_{k(22)} & \cdots & J_{k(2(k+1))}\\ \vdots & \vdots & \ddots & \vdots\\ J_{k((k+1)1)} & J_{k((k+1)2)} & \cdots & J_{k((k+1)(k+1))}\end{bmatrix} \tag{23}$$

where each element of the matrix can be defined as:

$$J_{k(ij)}=\left[\begin{array}{c|c} J_{ij}^{11} & J_{ij}^{12}\\ \hline J_{ij}^{21} & J_{ij}^{22}\end{array}\right] \tag{24}$$

$\forall i=j\neq 1$

$$J_{k(ij)}=\left[\begin{array}{c|c} \begin{array}{c}(H_{i-1}^x)^T\mathbf{R}_{i-1}^{-1}H_{i-1}^x\\ +(F_{i-1}^x)^T\mathbf{Q}_{i-1}^{-1}F_{i-1}^x\\ +\mathbf{Q}_{i-2}^{-1}\end{array} & \begin{array}{c}(H_{i-1}^x)^T\mathbf{R}_{i-1}^{-1}H_{i-1}^\theta\\ +(F_{i-1}^x)^T\mathbf{Q}_{i-1}^{-1}F_{i-1}^\theta\\ -\mathbf{Q}_{i-2}^{-1}F_{i-2}^\theta\end{array}\\ \hline (J_{ij}^{12})^T & T_i^{\theta\theta}\end{array}\right] \tag{25}$$

$\forall i, j=i+1$

$$J_{k(ij)}=\left[\begin{array}{c|c} -(F_{i-1}^x)^T\mathbf{Q}_{i-1}^{-1} & \begin{array}{c}(H_{i-1}^x)^T\mathbf{R}_{i-1}^{-1}H_{i-1}^\theta\\ +(F_{i-1}^x)^T\mathbf{Q}_{i-1}^{-1}F_{i-1}^\theta\\ -\mathbf{Q}_{i-2}^{-1}F_{i-2}^\theta\end{array}\\ \hline \begin{array}{c}(H_i^\theta)^T\mathbf{R}_i^{-1}H_i^x\\ +(F_i^\theta)^T\mathbf{Q}_i^{-1}F_i^x\\ -(F_{i-1}^\theta)^T\mathbf{Q}_{i-1}^{-1}\end{array} & T_i^{\theta\theta}\end{array}\right]$$

$$J_{k(ji)}=(J_{k(ij)})^T \tag{26}$$

$\forall i, j>i+1$

$$J_{k(ij)}=\left[\begin{array}{c|c} \emptyset & \begin{array}{c}(H_{i-1}^x)^T\mathbf{R}_{i-1}^{-1}H_{i-1}^\theta\\ +(F_{i-1}^x)^T\mathbf{Q}_{i-1}^{-1}F_{i-1}^\theta\\ -\mathbf{Q}_{i-2}^{-1}F_{i-2}^\theta\end{array}\\ \hline \begin{array}{c}(H_i^\theta)^T\mathbf{R}_i^{-1}H_i^x\\ +(F_i^\theta)^T\mathbf{Q}_i^{-1}F_i^x\\ -(F_{i-1}^\theta)^T\mathbf{Q}_{i-1}^{-1}\end{array} & T_i^{\theta\theta}\end{array}\right]$$

$$J_{k(ji)}=(J_{k(ij)})^T \tag{27}$$

$\forall i=j=1$

$$J_{k(11)}=\left[\begin{array}{c|c} \begin{array}{c}(H_0^x)^T\mathbf{R}_0^{-1}H_0^x\\ +(F_0^x)^T\mathbf{Q}_0^{-1}F_0^x\\ +\mathbf{P}_0^{-1}\end{array} & \begin{array}{c}(H_0^x)^T\mathbf{R}_0^{-1}H_0^\theta\\ +(F_0^x)^T\mathbf{Q}_0^{-1}F_0^\theta\end{array}\\ \hline (J_{ij}^{12})^T & T_0^{\theta\theta}\end{array}\right] \tag{28}$$

J is not a sparse matrix due to the correlations and dependencies between the states and the parameters; therefore the

inverse of the FIM can be calculated recursively. Thus, a two-step recursion of the FIM can be written as follows:
1) Time Update

$$J_{k|k-1} = \left[\begin{array}{c|c} J^{11}_{k|k-1} & \begin{array}{c}\mathbf{A}_k \\ \mathbf{B}_k\end{array} \\ \hline \mathbf{C}_k \quad \mathbf{D}_k & \mathbf{F}_k \end{array}\right] \tag{29}$$

where

$$J^{11}_{k|k-1} = \left[\begin{array}{cc} J^{11}_{k-1|k-1} & J^{12}_{k-1|k-1} \\ J^{21}_{k-1|k-1} & J^{22}_{k-1|k-1} + \left(F^x_{k-1}\right)^T \mathbf{Q}^{-1}_{k-1} F^x_{k-1} \left[\begin{array}{c|c} \mathbf{I} & \emptyset \\ \hline \emptyset & \emptyset \end{array}\right] \end{array}\right]$$

$$\mathbf{A}_k = \begin{bmatrix} \mathbf{A}_0 \\ \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_{k-2} \end{bmatrix}$$

$$\mathbf{A}_i = \left[\begin{array}{c|c} \emptyset & \left(F^x_i\right)^T \mathbf{Q}^{-1}_i F^\theta_i - \mathbf{Q}^{-1}_{i-1} F^\theta_{i-1} \\ \hline \left(F^\theta_{k-1}\right)^T \mathbf{Q}^{-1}_{k-1} F^x_{k-1} - \left(F^\theta_{k-2}\right)^T \mathbf{Q}^{-1}_{k-2} & T^{\theta\theta}_{k-1} \end{array}\right]$$

$$\mathbf{B}_k = \left[\begin{array}{c|c} -\left(F^x_{k-1}\right)^T \mathbf{Q}^{-1}_{k-1} & \begin{array}{c}\left(F^x_{k-1}\right)^T \mathbf{Q}^{-1}_{k-1} F^\theta_{k-1} \\ -\mathbf{Q}^{-1}_{k-2} F^\theta_{k-2}\end{array} \\ \hline \begin{array}{c}\left(F^\theta_k\right)^T \mathbf{Q}^{-1}_k F^x_k \\ -\left(F^\theta_{k-1}\right)^T \mathbf{Q}^{-1}_{k-1}\end{array} & T^{\theta\theta}_{k-1} \end{array}\right]$$

$$\mathbf{C}_k = \begin{bmatrix} \mathbf{C}_0 \\ \mathbf{C}_1 \\ \vdots \\ \mathbf{C}_{k-2} \end{bmatrix}^T$$

$$\mathbf{C}_i = \left[\begin{array}{c|c} \emptyset & \begin{array}{c}\left(F^x_{k-1}\right)^T \mathbf{Q}^{-1}_{k-1} F^\theta_{k-1} \\ -\mathbf{Q}^{-1}_{k-2} F^\theta_{k-2}\end{array} \\ \hline \begin{array}{c}\left(F^\theta_i\right)^T \mathbf{Q}^{-1}_i F^x_i \\ -\left(F^\theta_{i-1}\right)^T \mathbf{Q}^{-1}_{i-1}\end{array} & T^{\theta\theta}_{k-1} \end{array}\right]$$

$$\mathbf{D}_k = \left(\mathbf{B}_k\right)^T$$

$$\mathbf{F}_k = \left[\begin{array}{c|c} \mathbf{Q}^{-1}_{k-1} & \left(F^x_k\right)^T \mathbf{Q}^{-1}_k F^\theta_k - \mathbf{Q}^{-1}_{k-1} F^\theta_{k-1} \\ \hline F^\theta_k \mathbf{Q}^{-1}_k \left(F^x_k\right)^T - F^\theta_{k-1} \mathbf{Q}^{-1}_{k-1} & T^{\theta\theta}_{k-1} \end{array}\right]$$

2) Measurement Update

$$J_{k|k} = \left[\begin{array}{c|c} J^{11}_{k|k} & J^{12}_{k|k} \\ \hline J^{21}_{k|k} & J^{22}_{k|k} \end{array}\right] \tag{30}$$

where

$$\begin{aligned} J^{11}_{k|k} &= J^{11}_{k|k-1} \\ J^{12}_{k|k} &= J^{12}_{k|k-1} + \left[\begin{array}{c|c} \emptyset & \left(H^x_{k-1}\right)^T \mathbf{R}^{-1}_{k-1} H^\theta_{k-1} \\ \hline \left(H^\theta_k\right)^T \mathbf{R}^{-1}_k H^x_k & \left(H^\theta_{k-1}\right)^T \mathbf{R}^{-1}_{k-1} H^\theta_{k-1} \end{array}\right] \\ J^{21}_{k|k} &= \left(J^{12}_{k|k}\right)^T \\ J^{22}_{k|k} &= J^{22}_{k|k-1} + \left[\begin{array}{c|c} \left(H^x_k\right)^T \mathbf{R}^{-1}_k H^x_k & \left(H^x_k\right)^T \mathbf{R}^{-1}_k H^\theta_k \\ \hline \left(H^\theta_k\right)^T \mathbf{R}^{-1}_k H^x_k & \left(H^\theta_k\right)^T \mathbf{R}^{-1}_k H^\theta_k \end{array}\right] \end{aligned}$$

The error covariance matrix for the joint bound is:

$$\mathscr{P}_k = E\left\{\left[\hat{\mathbf{w}}_k - \mathbf{w}_k\right]\left[\hat{\mathbf{w}}_k - \mathbf{w}_k\right]^T\right\} \geq J^{-1}_k \tag{31}$$

Thus, from (29), (30) and block matrix inversion [5] $\mathscr{P}_k$ is:

$$\begin{aligned} \mathscr{P}^{-1}_k &\geq \mathbf{F}_k - \left[\begin{array}{cc} \mathbf{C}_k & \mathbf{D}_k \end{array}\right] \left(J^{11}_{k|k-1}\right)^{-1} \begin{bmatrix} \mathbf{A}_k \\ \mathbf{B}_k \end{bmatrix} \\ &= \left\{\begin{array}{l} \mathbf{F}_k - \left[\begin{array}{cc} \mathbf{C}_k & \mathbf{D}_k \end{array}\right] \\ \times \left\{\mathbf{F}_{k-1} + \left(H^x_{k-1}\right)^T \mathbf{R}^{-1}_{k-1} H^x_{k-1} + \mathscr{P}^{-1}_{k-1}\right\}^{-1} \\ \times \begin{bmatrix} \mathbf{A}_k \\ \mathbf{B}_k \end{bmatrix} \end{array}\right\} \end{aligned} \tag{32}$$

In this respect, the state error covariance matrix is computed by the the upper left quarter of the FIM, J^{11}_{ij} and the classical Kalman Filter recursive solution for the error covariance matrix is achieved [2]:

$$\begin{aligned} \mathscr{P}^{-1}_k &\geq \mathbf{Q}^{-1}_{k-1} \\ &- \left\{\begin{array}{l} \left[\begin{array}{cc} \emptyset & -\mathbf{Q}^{-1}_{k-1} F^x_{k-1} \end{array}\right] \\ \times \left(\left(F^x_{k-1}\right)^T \mathbf{Q}^{-1}_{k-1} F^x_{k-1} + \left(H^x_{k-1}\right)^T \mathbf{R}^{-1}_{k-1} H^x_{k-1} + \mathscr{P}^{-1}_{k-1}\right)^{-1} \\ \times \begin{bmatrix} \emptyset \\ -\left(F^x_{k-1}\right)^T \mathbf{Q}^{-1}_{k-1} \end{bmatrix} \end{array}\right\} \\ &= \left(\mathbf{Q}_{k-1} + F^x_{k-1} \left(\left(H^x_{k-1}\right)^T \mathbf{R}^{-1}_{k-1} H^x_{k-1} + \mathscr{P}^{-1}_{k-1}\right)^{-1} \left(F^x_{k-1}\right)^T\right)^{-1} \end{aligned} \tag{33}$$

In the case of parameter estimation, the lower bound of the error covariance matrix can be obtained with the same approach using the lower right quarter of the FIM, J^{11}_{ij}:

$$\begin{aligned} \mathscr{P}^{-1}_k &\geq T^{\theta\theta}_{k-1} \\ &- \left\{\begin{array}{l} \left[T^{\theta\theta}_{k-1}\right] \\ \times \left(T^{\theta\theta}_{k-1} + \left(H^\theta_{k-1}\right)^T \mathbf{R}^{-1}_{k-1} H^\theta_{k-1} + \left(F^\theta_{k-1}\right)^T \mathbf{Q}^{-1}_{k-1} F^\theta_{k-1} + \mathscr{P}^{-1}_{k-1}\right)^{-1} \\ \times \left[T^{\theta\theta}_{k-1}\right] \end{array}\right\} \\ &= \left(\left(T^{\theta\theta}_{k-1}\right)^{-1} + \left(\left(H^\theta_{k-1}\right)^T \mathbf{R}^{-1}_{k-1} H^\theta_{k-1} + \left(F^\theta_{k-1}\right)^T \mathbf{Q}^{-1}_{k-1} F^\theta_{k-1} + \mathscr{P}^{-1}_{k-1}\right)^{-1}\right)^{-1} \end{aligned} \tag{34}$$

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