A POSITIVE PARTIAL REALIZATION OF TIME SERIES

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ABSTRACT

For a given partial covariance sequence $(C_0, C_1, \ldots, C_n)$ and for each MA part of the ARMA modeling filter of degree n, an AR part of the ARMA modeling filter of degree n for the solution to the rational covariance extension problem is obtained by solving a nonlinear equation, which is homotopic to a nonlinear equation determining the maximum entropy AR filter.

Index Terms-Covariance extension, ARMA modeling filter, McMillan degree constraint.

1. INTRODUCTION

It is well-known that the spectral density $\Phi(z)$ of a second-order, m-dimensional stationary stochastic process $\{y(t)\}$ with zero mean is given by the Fourier expansion

$$\Phi(e^{i\theta}) = \sum_{k=-\infty}^{\infty} C_k e^{-ik\theta}$$

on the unit circle, where the covariance matrices are given by

$$C_k = \mathrm{E}\{y(t)y(t-k)^T\}, \quad k = 0, 1, 2, \ldots.$$

For each spectral density, there exists a unique modeling filter, which shapes a white noise process into the stochastic process.

For a given partial covariance sequence $(C_0, C_1, \ldots, C_n)$, we want to determine the spectral density, which is consistent with the given partial covariance sequence [7, 4, 5, 16, 11, 3, 1, 6, 14, 10]. This is known as the covariance extension problem and it is common that the estimated spectral density is a rational function, and that the modeling filter is also a rational model, i.e., autoregressive moving average (ARMA) type [13, 9].

It is well-known that there exists a solution to the covariance extension problem, so-called maximum entropy solution [13]. The spectral density maximizes the entropy rate of the spectral density, and its first $n+1$ covariance matrices matches the given partial covariance sequence. The modeling filter of the maximum entropy spectral density is an autoregressive (AR) type [13].

In this paper, we show that, for a given partial covariance sequence $(C_0, C_1, \ldots, C_n)$ and for each MA part of the ARMA modeling filter of degree n, an AR part of the ARMA modeling filter of degree n for the solution to the covariance extension problem is determined by solving a nonlinear equation, which is homotopic to a nonlinear equation to determine the maximum entropy filter of AR type.

Notations

Real numbers are represented by $\mathbb{R}$ and complex numbers are represented by $\mathbb{C}$. $\bar{c}$ denotes the conjugate of the complex number c. Denote by $\mathbb{R}^{j\times k}$ $j \times k$ real matrices. I denotes $m \times m$ identity matrix, and 0 denotes $m \times m$ zero matrix. A^T denotes the transpose of a matrix A. $\det A$ denotes the determinant of a matrix A. We use the notations $A > 0$ to denote that a matrix A is positive definite.

Let us define

$$f(z)^* := \overline{f(\bar{z}^{-1})^T}.$$

We denote the unit disc by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. An $m \times m$ matrix-valued rational function $f(z)$ is called strictly positive real if it is analytic in the outside of the unit disc $\mathbb{D}^c := \{z \in \mathbb{C} : |z| \geq 1\}$ and

$$f(z) + f(z)^* > 0.$$

is positive definite on $\mathbb{T}$. The function $f(z)$ is strictly positive real if and only if $f(z)^{-1}$ is strictly positive real. Thus, it is necessary that all zeros of strictly positive real function lie in $\mathbb{D}$.

The state-space realization of transfer function $G(z) = C(zI - A)^{-1}B + D$ is denoted by

$$G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

2. PRELIMINARY

2.1 Covariance Extension Problem

For a given partial covariance sequence $(C_0, C_n, \ldots, C_n)$, which is positive in the sense that the Toeplitz matrix

$$T = \begin{bmatrix} C_0 & C_1 & \cdots & C_n \\ C_1^T & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ C_n^T & \cdots & \cdots & C_0 \end{bmatrix} \tag{1}$$

is positive definite, we want to parameterize $m \times m$ strictly positive real functions $f(z)$ such that the Fourier expansion of $f(z)$ begins with

$$\frac{1}{2}C_0 + C_1 z^{-1} + \cdots + C_n z^{-n}. \tag{2}$$

The solvability condition of this covariance extension problem is given by the positive definiteness of T, given by (1).

For a strictly positive real function $f(z)$, the spectral density is defined by

$$\Phi(z) := f(z) + f(z)^*. \tag{3}$$

It is well-known that there exists a unique outer spectral factor $W(z)$ such that

$$\Phi(z) = W(z)W(z)^*.$$

$W(z)$ is called a modeling filter of stochastic process, and it is assumed that it is an ARMA type

$$W(z) = A(z)^{-1}\Sigma(z), \tag{4}$$

where

$$\begin{aligned} A(z) &:= A_0 + A_1 z^{-1} + \cdots + A_n z^{-n} \\ \Sigma(z) &:= \Sigma_0 + \Sigma_1 z^{-1} + \cdots + \Sigma_n z^{-n}, \end{aligned}$$

$A_k \in \mathbb{R}^{m\times m}$ and $\Sigma_k \in \mathbb{R}^{m\times m}$, $k = 0,\dots,n$. We observe that

$$W(z)W(z)^* \quad = \quad f(z) + f(z)^*.$$

Thus, we can formulate the covariance extension problem in terms of $W(z)$. Namely, we seek modeling filters $W(z)$ such that

$$W(z)W(z)^* = \hat{C}_0 + \sum_{k=1}^{\infty} \hat{C}_k(z^k + z^{-k})$$
$$\hat{C}_k = C_k \quad \text{for} \quad k = 0,1,\dots,n.$$

Let us introduce notations,

$$A(z) = AG, \quad \Sigma(z) = \Sigma G,$$

where

$$\begin{aligned} A &:= \begin{bmatrix} A_0 & A_1 & \cdots & A_n \end{bmatrix} \\ \Sigma &:= \begin{bmatrix} \Sigma_0 & \Sigma_1 & \cdots & \Sigma_n \end{bmatrix} \\ G &:= \begin{bmatrix} I & z^{-1}I & \cdots & z^{-n}I \end{bmatrix}^T. \end{aligned}$$

Let us define

$$\mathscr{A} := \left\{ A \in \mathbb{R}^{m\times m(n+1)} : A(z)^{-1} \text{ is analytic in } \mathbb{D}^c \right\}. \tag{5}$$

Note that $A(\infty) = A_0$ is invertible. $\mathscr{A}$ is identified with the set of $m\times m$ outer matricial pseudopolynomials of degree at most n. With a slight abuse of notations, we denote by $A \in \mathscr{A}$ and $A(z) \in \mathscr{A}$ that $A(z)$ is outer.

The minimal state-space realizations of $A(z)$ and $\Sigma(z)$ are given by

$$A(z) = \left[\begin{array}{c|c} Z & A_1^v \\ \hline e_1^T & A_0 \end{array}\right], \quad \Sigma(z) = \left[\begin{array}{c|c} Z & \Sigma_1^v \\ \hline e_1^T & \Sigma_0 \end{array}\right],$$

where

$$\begin{aligned} Z &:= \begin{bmatrix} 0 & I & \cdots & 0 \\ 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & I \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \\ e_1 &:= \begin{bmatrix} I & 0 & \dots & 0 \end{bmatrix}^T \\ A_1^v &:= \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}, \quad \Sigma_1^v := \begin{bmatrix} \Sigma_1 \\ \vdots \\ \Sigma_n \end{bmatrix}. \end{aligned}$$

The minimal state-space realization of (4) is given by

$$W(z) = \left[\begin{array}{c|c} Z - A_1^v A_0^{-1} e_1^T & \Sigma_1^v - A_1^v A_0^{-1}\Sigma_0 \\ \hline A_0^{-1} e_1^T & A_0^{-1}\Sigma_0 \end{array}\right]. \tag{6}$$

2.2 Maximum Entropy Solution

It is well-known that there exists a unique spectral density, which is a solution to the covariance extension problem and maximizes the entropy rate of spectral density [13], defined by

$$\mathbb{I}(\Phi) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\det\Phi(e^{i\theta})d\theta. \tag{7}$$

It is well-known that the modeling filter of the maximum entropy spectral density is an AR type

$$\Phi(z) = A(z)^{-1}A(z)^{-*}. \tag{8}$$

It turns out that the maximum entropy filter of AR type is obtained by solving a nonlinear equation [8], $G(A) = 0$, where

$$G(A) := AT - \frac{1}{2\pi}\int_{-\pi}^{\pi} A(e^{i\theta})^{-*}G^* d\theta. \tag{9}$$

We shall show that the solution to the covariance extension problem is obtained by solving a nonlinear equation $F(A) = 0$, defined by (11) below, where the nonlinear map $F(A)$ is homotopic to the nonlinear map $G(A)$.

2.3 Topological Degree Theory

The AR part of the ARMA modeling filter is determined by solving the nonlinear equation $F(A) = 0$. The existence of the solutions to the nonlinear equation is shown in terms of topological degree theory. We briefly review the topological degree theory [12, 2].

Suppose that U, $V \subset \mathbb{R}^n$ are open subsets, and that U is not necessarily bounded and V is connected. Let

$$F : U \to V$$

be a continuously differentiable map on U, which is also proper, i.e., the property that the inverse image $F^{-1}(K)$ is compact for all compact K [17]. It is frequently of considerable interest to know in advance the number of solutions of the nonlinear equation

$$y = F(x)$$

in some specified sets.

Denote by ∂U the boundary of the set U. For a given

$$y \notin F(\partial U),$$

denote by $\deg_y(F,U)$ the topological degree of the map f at the point y relative to the set U. Since F is proper, $F^{-1}(y)$ is compact.

Suppose that the Jacobian matrix of F at x, denoted by $\mathrm{Jac}_x(F)$, is nonsingular for all

$$x \in U_s := \{x \in U | y = F(x)\}.$$

Then, the degree of $\deg_y(F,U)$ is given by

$$\deg_y(F,U) = \sum_{U_s} \mathrm{sign}\det\mathrm{Jac}_x(F).$$

The definition of the topological degree for the singular Jacobian is found in [12, 2].

A consequence of the topological degree theory is that the nonzero degree of the map F with respect to zero guarantees the existence of the solutions to the nonlinear equation $F(x) = 0$. This is known as the Kronecker Theorem [12].

One of the important property of the degree of map is the homotopy invariance of the degree. Let H be a jointly continuous map from $U \times [0,1] \to V$ such that $H(x,0) = G(x)$ and $H(x,1) = F(x)$. Suppose that y satisfies $H(x,\lambda) \neq y$ for all $(x,\lambda) \in \partial U \times [0,1]$. Then,

$$\deg_y(F,U) = \deg_y(G,U)$$

holds.

We only consider the topological degree of map F at the point zero relative to the set $\mathscr{A}$. Thus, we simply denote $\deg_0(F,\mathscr{A})$ as $\deg(F)$.

3. MAIN RESULT

3.1 ARMA Filter Design via A Nonlinear Equation

We present the nonlinear equation to determine the ARMA modeling filter for the solution to the covariance extension problem.

Theorem 1. *Suppose that the Toeplitz matrix T is positive definite. Then, for a given partial covariance sequence $(C_0, C_1, \ldots, C_n)$ and for each MA part $\Sigma \in \mathscr{A}$ of the ARMA modeling filter, there exists an AR part $A \in \mathscr{A}$ of the ARMA modeling filter such that*

$$W(z) = A(z)^{-1}\Sigma(z)$$

satisfies

$$\begin{aligned} W(z)W(z)^* &= \hat{C}_0 + \sum_{k=1}^{\infty} \hat{C}_k (z^k + z^{-k}) \\ \hat{C}_k &= C_k \quad \text{for} \quad k = 0, 1, \ldots, n. \end{aligned} \tag{10}$$

The solution A is determined by solving the nonlinear equation $F(A) = 0$, where the nonlinear map $F(A) : \mathscr{A} \to \mathbb{R}^{m \times m(n+1)}$ is defined by

$$F(A) := AT - \frac{1}{2\pi}\int_{-\pi}^{\pi} \Sigma(e^{i\theta})\Sigma(e^{i\theta})^* A(e^{i\theta})^{-*} G^* d\theta. \tag{11}$$

Proof. First, we show the existence of the solution to the nonlinear equation $F(A) = 0$. We observe that the solution to $F(A) = 0$ cannot be on the boundary of $\mathscr{A}$. This is a corollary of *Lemma* 1.

Lemma 1. *The nonlinear map $F : \mathscr{A} \to \mathbb{R}^{m \times m(n+1)}$, defined by* (11)*, is proper, i.e., the inverse image $F^{-1}(K)$ is compact for all compact K.*

If $A(z)$ has a zero on $\mathbb{T}$, then, $F(A) = \infty$ since the integral term of $F(A)$ diverges. Moreover, $\Sigma(z) \in \mathscr{A}$ implies that the zeros of $\Sigma(z)$ lie in $\mathbb{D}$. Thus, there is no possible cancellation of the zero of $A(z)$ on $\mathbb{T}$ by a zero of $\Sigma(z)$. Note that *Lemma* 1 implies that $F^{-1}(0)$ is compact.

We use topological degree theory to prove the existence of the solution to the nonlinear equation $F(A) = 0$. Denote by $\partial\mathscr{A}$ the boundary of $\mathscr{A}$. It turned out that

$$0 \notin F(\partial\mathscr{A}).$$

Thus, the degree of F with respect to zero is computed. It is shown that there is the unique maximum entropy solution, which is determined by solving the nonlinear equation $G(A) = 0$, defined by (9). The uniqueness of the maximum entropy solution implies that $\deg G = 1$. We construct a homotopy from $G(A)$ to $F(A)$. Consider a jointly continuous map $H : A \times [0,1] \to \mathbb{R}^{m \times m(n+1)}$, which is defined by

$$H(A, \lambda) := (1-\lambda)G(A) + \lambda F(A), \quad \lambda \in [0,1].$$

This map satisfies $H(A,0) = G(A)$, $H(A,1) = F(A)$, and

$$0 \notin H(\partial\mathscr{A}, \lambda)$$

since

$$(1-\lambda)I + \lambda\Sigma(e^{i\theta})\Sigma(e^{i\theta})^* > 0, \quad \lambda \in [0,1],$$

on $\mathbb{T}$. The homotopy invariance of the degree of map guarantees that $\deg F = 1$. The nonzero degree implies the existence of the solution to $F(A) = 0$.

We show that the ARMA modeling filter $W(z)$, which is determined by $F(A) = 0$, satisfies (10). The second term of $F(A)$ is

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} \Sigma(e^{i\theta})\Sigma(e^{i\theta})^* A(e^{i\theta})^{-*} G^* d\theta \\ &= A\frac{1}{2\pi}\int_{-\pi}^{\pi} GW(e^{i\theta})W(e^{i\theta})^* G^* d\theta. \end{aligned}$$

Let $w(z)$ be an analytic function in $\mathbb{D}^c$, which satisfies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}[w(e^{i\theta}) + w(e^{i\theta})^*]d\theta = C_k, \quad k = 0, \ldots, n.$$

Then, the Toeplitz matrix T of the first term of $F(A)$ is written by

$$T = \frac{1}{2\pi}\int_{-\pi}^{\pi} G[w(e^{i\theta}) + w(e^{i\theta})^*]G^* d\theta.$$

Let us define an analytic function $M(z)$ in $\mathbb{D}^c$ by

$$M(e^{i\theta}) + M(e^{i\theta})^* = w(e^{i\theta}) + w(e^{i\theta})^* - W(e^{i\theta})W(e^{i\theta})^*.$$

Denote the Fourier expansion of $M(e^{i\theta}) + M(e^{i\theta})^*$ by

$$M(e^{i\theta}) + M(e^{i\theta})^* \quad = \quad \sum_{k=-\infty}^{\infty} M_k e^{-ik\theta},$$

where $M_0 = M_0^T$. Then,

$$\begin{aligned} F(A) &= A\frac{1}{2\pi}\int_{-\pi}^{\pi} G[M(e^{i\theta}) + M(e^{i\theta})^*]G^* d\theta \\ &= AT_M, \end{aligned}$$

where

$$T_M := \begin{bmatrix} M_0 & M_1 & \cdots & M_n \\ M_1^T & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ M_n^T & \cdots & \cdots & M_0 \end{bmatrix}.$$

The Toeplitz matrix T_M has nothing to do with the positivity, and $T_M = 0$ implies that (10) holds. We shall see $T_M = 0$ by using the structure of the Toeplitz matrix and $A \in \mathscr{A}$.

We write the matrix A as

$$\begin{aligned} A &= \begin{bmatrix} A_0 & A_1^h \end{bmatrix} \\ A_1^h &:= \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}. \end{aligned}$$

Similarly, we write the Toeplitz matrix

$$T_M = \begin{bmatrix} M_0 & S^T \\ S & T_1 \end{bmatrix}.$$

It is clear that $F(A) = 0$ gives

$$\begin{aligned} M_0 &= -A_0^{-1}A_1^h S \\ S^T &= -A_0^{-1}A_1^h T_1. \end{aligned} \tag{12}$$

The structure of the Toeplitz matrix T_M and (12) imply that

$$T_1 = A_c T_1 A_c^T \tag{13}$$

holds, where

$$A_c := \begin{bmatrix} -A_0^{-1}A_1 & \cdots & -A_0^{-1}A_{n-1} & -A_0^{-1}A_n \\ I & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}.$$

$\det z^n A_0^{-1}A(z)$ is the characteristic polynomial of A_c. Moreover, $\det z^n A_0^{-1}A(z)$ has all zeros in $\mathbb{D}$ since $A(z)$ is outer, which implies that all eigenvalues of A_c lie in $\mathbb{D}$. Hence, we obtain $T_1 = 0$ for the unique solution to the Lyapunov equation (13), and it implies $M_0 = 0$ due to the Toeplitz structure of T_M. We also obtain $S = 0$ by (12). Thus, $T_M = 0$. □

3.2 A Uniqueness of A to $f(z)$

For a given solution to the nonlinear equation $F(A)=0$, we obtain an ARMA modeling filter, and we also obtain a strictly positive real function $f(z)$ by

$$W(z)W(z)^* = f(z)+f(z)^*, \tag{14}$$

where the Fourier expansion of $f(z)$ begins with (2). For each $f(z)$, there exists a unique outer spectral factor. However, this uniqueness of the outer spectral factor of $f(z)$ does not imply the uniqueness of A, for which $W(z)$ of (14) is outer. Let A_k, $k=1,2$ be solutions to $F(A)=0$. Denote by $W_k(z)$, $k=1,2$, the corresponding ARMA modeling filters. Then, for the fixed $f(z)$, we consider whether

$$W_k(z)W_k(z)^* = f(z)+f(z)^*, \quad k=1,2$$

is possible or not.

Theorem 2. *Let A be a solution to the nonlinear equation $F(A)=0$. Then, it determines $f(z)$ via* (14)*, and there is no other A to give the same $f(z)$.*

Proof. Suppose that there exist two solutions to the nonlinear equation $F(A)=0$. They are denoted by

$$\begin{aligned} A_k &= \begin{bmatrix} A_{k,0} & A_{k,1}^h \end{bmatrix} \\ A_{k,1}^h &:= \begin{bmatrix} A_{k,1} & \cdots & A_{k,n} \end{bmatrix}, \quad k=1,2. \end{aligned}$$

Let us define

$$A_{k,1}^v := \begin{bmatrix} A_{k,1} \\ \vdots \\ A_{k,n} \end{bmatrix}, \quad k=1,2.$$

Each A_k determines $W_k(z), k=1,2$, of which minimal state-space realizations are given by

$$\begin{aligned} W_k(z) &= \left[\begin{array}{c|c} Z - A_{k,1}^v A_{k,0}^{-1} e_1^T & \Sigma_1 - A_{k,1}^v A_{k,0}^{-1}\Sigma_0 \\ \hline A_{k,0}^{-1} e_1^T & A_{k,0}^{-1}\Sigma_0 \end{array}\right] \\ &= \left[\begin{array}{c|c} A_{W,k} & B_{W,k} \\ \hline C_{W,k} & D_{W,k} \end{array}\right], \quad k=1,2, \end{aligned} \tag{15}$$

see (6).

Consider

$$W_1(z)W_1(z)^* = W_2(z)W_2(z)^*.$$

$A_k(z) \in \mathscr{A}, k=1,2$, implies that all poles of $W_k(z), k=1,2$, lie in $\mathbb{D}$, and $\Sigma(z) \in \mathscr{A}$ implies that all zeros of $W_k(z), k=1,2$ lie in $\mathbb{D}$. Thus, $W_k(z), k=1,2$, are outer. Then, modulo constant unitary matrix multiplication from the right, we obtain

$$W_1(z) = W_2(z).$$

However, this does not imply the two state-space realizations are identical since there exists a freedom of similar transformations of the state-space realizations of $W_k(z), k=1,2$. By evaluating $W_k(z), k=1,2$ at ∞, we obtain

$$A_{1,0}^{-1}\Sigma_0 = A_{2,0}^{-1}\Sigma_0 \Longrightarrow A_{1,0} = A_{2,0},$$

where Σ_0 is invertible since $\Sigma(z) \in \mathscr{A}$. It implies $C_{W,1} = C_{W,2}$ due to the state-space realizations (15). Thus, there is no freedom of similar transformations of the state-space realizations. Hence, $A_{W,1} = A_{W,2}$ and $B_{W,1} = B_{W,2}$. $B_{W,1} = B_{W,2}$ implies

$$A_{1,1}^v = A_{2,1}^v.$$

□

Remark 1. *We showed the existence of an AR part in Theorem 1 in terms of the nonlinear map $F(A)$. This existence of an AR part for the solution to the covariance extension problem was also shown in [4], while a different map is used in [4]. We discussed a uniqueness of $f(z)$ in Theorem 2, and an alternative approach to the uniqueness in terms of a different map was discussed in [15].*

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