STATISTICAL NONIDENTIFIABILITY OF CLOSE EMITTERS: MAXIMUM-LIKELIHOOD ESTIMATION BREAKDOWN

Yuri I. Abramovich, Ben A. Johnson, Nicholas K. Spencer

Intelligence, Surveillance and Reconnaissance Division,
Defence Science and Technology Organisation (DSTO)
PO Box 1500, Edinburgh SA 5111, Australia
Yuri.Abramovich@dsto.defence.gov.au

Lockheed Martin Australia (LMA)
82–86 Woomera Ave, Edinburgh SA 5111, Australia
Ben.A.Johnson@ieee.org

Adelaide Research & Innovation Pty Ltd (ARI),
c/- 200 Labs, ISRD, DSTO
PO Box 1500, Edinburgh SA 5111, Australia
Nick.Spencer@adelaide.edu.au

ABSTRACT

For closely spaced signals, we demonstrate that the minimum signal-to-noise ratio (SNR) at which they can be resolved (and their parameters reliably estimated) strongly depends on the *a priori* source-power model. If no restrictions are made on the admissible powers, then there will exist an "ambiguity region" that encompasses scenarios with completely erroneous direction-of-arrival (DOA) estimates in addition to the true scenario with closely spaced emitters. We implement a global maximum-likelihood (ML) search to support the Cramér–Rao bound (CRB) predictions.

1. INTRODUCTION AND BACKGROUND

In a number of classic papers, such as [1], it has been argued that an M-sensor antenna array can correctly detect m closely spaced sources down to a SNR ε_{ITC} that is significantly lower than the minimum SNR at which ML techniques can properly resolve the signals and estimate their parameters (such as DOA and power), ε_{ML}. To put it crudely, "estimation breaks before detection":

$$\varepsilon_{ITC} \ll \varepsilon_{ML}. \tag{1}$$

Yet most of the empirical evidence that supports this argument has come from practical high-resolution techniques, such as MUSIC, which approximate maximum-likelihood estimation (MLE) but are not precisely ML. Such "ML proxy" techniques are asymptotically equivalent to MLE, i.e. as the number of training samples T and/or the SNR tends to infinity. However, their behaviour in the practical domain of few snapshots and low SNR can be completely different. It is well-known that at sufficiently small T and/or SNR (the "threshold region"), these techniques suffer from a dramatic failure rate ("performance breakdown"); moreover, MUSIC performance breakdown often occurs at T and SNR values where MLE still gives proper estimates [2]. We recently demonstrated [3,4] that MUSIC breakdown could be "cured" in this threshold region.

Another signal-processing maxim is that the resolution limit of MLE is equal to the sum of the standard deviations of the two DOA estimates, as predicted by the CRB [5,6]. Thus we could not expect to accurately estimate DOAs below the CRB limit, and yet this approach still does not explicitly describe MLE performance in the threshold region. In particular, we may expect MLE to breakdown far above the CRB limit [7–11].

Apart from obvious theoretical interest, this issue is also of practical importance. Techniques developed in [3,4] for MUSIC performance breakdown "prediction and cure" have been proven capable of rectifying erroneous MUSIC estimates "up to ML quality". More specifically, those methods give estimates that are statistically as likely as the true (exact) scenario, and still may contain an outlier. In this regard, it is important to understand the conditions under which we can get an ML estimate that nevertheless is an outlier.

The ultimate performance of ML DOA estimation must be investigated directly in the threshold region, where the intersource separation is comparable with the CRB limit. In this paper, we investigate the threshold behaviour of the accurate ML DOA estimate for two close emitters, for different assumptions on the admissible source powers. In a related paper [12], we demonstrated that the random matrix theory (RMT) [also known as general statistical analysis (GSA)] technique can be used to accurately predict statistical nonidentifiability.

2. MLE BREAKDOWN FOR DIFFERENT POWER MODELS

We consider the standard detection-estimation problem of $m < M$ independent Gaussian sources in white noise, where the M-variate vector $\boldsymbol{y}$ that represents the output of the M-sensor antenna array can be expressed as

$$\boldsymbol{y}(t) = \sum_{j=1}^{m} \boldsymbol{s}(\theta_j)\boldsymbol{x}_j(t) + \boldsymbol{\eta}(t), \qquad t = 1, \ldots, T \tag{2}$$

where $\boldsymbol{x}_j$ is the j^{th} independent signal complex amplitude with power p_j, and

$$\mathcal{E}\{\boldsymbol{\eta}(t)\boldsymbol{\eta}^H(t)\} = p_0 I_M \tag{3}$$

with $\boldsymbol{s}(\theta_j) \in \mathcal{C}^{M\times 1}$ being the antenna array manifold (steering) vector for the DOA (azimuthal angle) θ_j, and p_0 being the white-noise power (which we assume to be known *a priori*). Here we consider a uniform linear array (ULA), so that

$$\boldsymbol{s}(\theta_j) = \left[1,\, \exp(2\pi i \frac{d}{\lambda}\sin\theta_j), \ldots, \exp(2\pi i (M-1)\frac{d}{\lambda}\sin\theta_j)\right]^T \tag{4}$$

even though the properties of the array geometry are not important for this analysis.

In this model, the true covariance matrix is

$$R = SPS^H + p_0 I_M \tag{5}$$

where

$$S = [\boldsymbol{s}(\theta_1),\, \ldots, \boldsymbol{s}(\theta_m)], \qquad P = \operatorname{diag}[p_1,\, \ldots, p_m] \tag{6}$$

with the source powers p_j ($j = 1, \ldots, m$). For MLE with $T > M$, we adopt the likelihood ratio (LR) as our (normalised) likelihood function [13], which is defined as

$$LR(R_\mu) = \frac{\det(R_\mu^{-1}\hat{R})\exp M}{\exp \operatorname{tr}(R_\mu^{-1}\hat{R})} \le 1 \tag{7}$$

where the estimated covariance matrix is, as usual,

$$\hat{R} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{y}(t)\boldsymbol{y}^H(t) \tag{8}$$

and the model covariance matrix is

$$R_\mu = S_\mu P_\mu S_\mu^H + p_0 I_M \tag{9}$$

where

$$S_\mu = [\boldsymbol{s}(\breve{\theta}_1),\, \ldots, \boldsymbol{s}(\breve{\theta}_\mu)], \quad P_\mu = \operatorname{diag}[\breve{p}_1,\, \ldots, \breve{p}_\mu]. \tag{10}$$

Thus μ, $[\breve{\theta}_1, \ldots, \breve{\theta}_\mu]$ and $[\breve{p}_1, \ldots, \breve{p}_\mu]$ are the modelled number of sources with their DOAs and powers, respectively.

The estimated number of sources μ can be found either by information-theoretic criteria (ITC) [14], or by the generalised likelihood-ratio test (GLRT) method [15] whereby

$$\mu = \arg\min_j \left\{ \max_{\theta_1,\ldots,\theta_j, p_1,\ldots,p_j} LR(R_j) \ge \gamma_{LR}(P_{FA}) \right\} \tag{11}$$

where the false-alarm threshold is

$$\gamma_{LR}(P_{FA}) = \arg_\gamma \left\{ \int_0^\gamma w(x)\, dx = P_{FA} \right\} \tag{12}$$

and $w(x)$ is the probability density function (pdf) for $LR(R)$. Note that $w(x)$ does not depend on R, but is fully specified by the known parameters T and M in a complex Wishart distribution:

$$LR(R) = \frac{\det(\hat{C}/T)\exp M}{\exp \operatorname{tr}(\hat{C}/T)} \tag{13}$$

where

$$\hat{C} = R^{-\frac{1}{2}}\hat{R}R^{-\frac{1}{2}} \sim \mathcal{CW}(T \ge M,\, M,\, I_M). \tag{14}$$

In this study, we consider the same scenario as in Lee and Li [1], that was chosen to demonstrate $\varepsilon_{ITC} \ll \varepsilon_{ML}$:

$$M = 3, \quad T = 100, \quad m = 2,$$
$$p_1 = p_2, \quad p_0 = 1, \quad \{\theta_1, \theta_2\} = \{0^\circ, 1.08^\circ\} \tag{15}$$

with half-wavelength spacing in the antenna array. As demonstrated in [1], reliable detection of $\mu = 2$ sources in this closely spaced scenario by the minimum description length (MDL) ITC method occurs down to $\varepsilon_{ITC} \simeq 22$ dB (where MDL's failure rate suddenly rises to 62%). Obviously it is desirable that ML DOA estimation also be accurate down to at least this SNR. (To put it crudely, we would like to have "estimation and detection break simultaneously".)

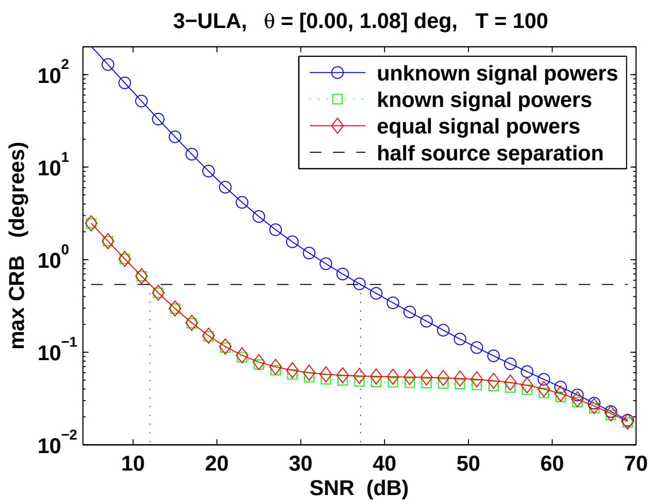


Figure 1: CRB analysis for the Lee and Li scenario.

Fig. 1 shows the behaviour of the CRB (equal for the two sources in this case) as a function of SNR for the following three source-power models in (9):

- arbitrary powers,
- *a priori* known powers, and
- equal (but *a priori* unknown) powers.

These CRBs are calculated for scenario (15) with equal powers using the standard technique [15] that involves, respectively, the $2m$-variate, m-variate and $(m+1)$-variate Fisher information matrices. This simple analysis shows the crucial dependence of MLE performance on the admissible source powers below about 60 dB, above which the CRB is practically the same for these three power models. For the arbitrary/unrestricted model (plotted with circles), the CRB limit of $(\theta_2 - \theta_1)/2 = 0.54^\circ$ occurs at the very high SNR of 37 dB. On the other hand, the CRB limit for the equi-power model (diamonds) is at 12 dB, and moreover this CRB is almost independent of SNR in the range 25–55 dB approximately. Meanwhile, the arbitrary model has its conventional CRB increasing so rapidly at low SNRs that by 12 dB it has reached 41°. Interestingly, the model with accurately known powers negligibly improves the CRB compared with equal (but *a priori* unknown) signal powers.

These dramatic differences in potential DOA estimation accuracy can be interpreted as the transition from statistical identifiability (under the equi-power model) to statistical nonidentifiability (if a source power could be arbitrarily small). This means that, with equal powers, an optimally high LR value (7) can be achieved only if two DOAs in the model covariance matrix R_μ (9) are sufficiently close to the true ones $\{\theta_1, \theta_2\}$. On the other hand, if we allow arbitrary powers, most of the total power $p_1 + p_2$ can be attributed to a source close to the midpoint DOA

$$\bar{\theta} \equiv (\theta_1 + \theta_2)/2 = 0.54°, \tag{16}$$

with the remaining insignificant power attributed to another source with a large DOA estimation error (an outlier). Such solutions are frequently produced by MUSIC (at much higher SNRs) when its performance starts to break down [4].

Now we shall demonstrate how the dramatic differences in CRB reflect the actual MLE behaviour at these "threshold" SNR values.

To investigate the true ML performance (rather than that of an ML-proxy algorithm), we have searched for the global LR maximum in two stages. First, on a finely discretised grid over the diagonal half-plane ($\theta_2 > \theta_1$), we find the maximum $LR(\theta_1, \theta_2)$. We calculate the powers for the two DOAs (in the arbitrary power model) following [15]:

$$[\breve{p}_1, \breve{p}_2] = \mathrm{diag}_+ \left\{ (\boldsymbol{s}^H \boldsymbol{s})^{-1} \boldsymbol{s}^H (\hat{R} - p_0 I_M) \boldsymbol{s} (\boldsymbol{s}^H \boldsymbol{s})^{-1} \right\} \tag{17}$$

where

$$\mathrm{diag}_+\{x\} \equiv \begin{cases} x & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \end{cases} \tag{18}$$

For the equi-power model, we use

$$\breve{p}_1 = \breve{p}_2 = \frac{1}{2M} \operatorname{tr}(\hat{R} - p_0 I_M). \tag{19}$$

Then we use (9) and (7) to calculate $LR(R_\mu)$ at each grid point.

The second stage is to use this grid global maximum as the initial solution to a numerical optimisation routine; we use the Matlab function `fmincon` to find the local LR maximum over either $\{\theta_1, \theta_2, p_1, p_2\}$ or $\{\theta_1, \theta_2, p_1 = p_2\}$, i.e. over all four or just three parameters.

Fig. 2 shows a plot of $LR(\theta_1, \theta_2)$ for an example realisation of our scenario at 22 dB SNR. The left plot is for the arbitrary power model, while the right (zoomed) plot is for the equi-power model (with the same set of training data). This realisation was chosen to illustrate how the completely different behaviour in CRB (depending on signal-power model) arises. For unrestricted powers, the global ML estimates are found to be

$$\{\hat{\theta}_1, \hat{\theta}_2\} = \{0.51°, 38.28°\},$$

$$\{\hat{p}_1 = 24.93\,\mathrm{dB}, \hat{p}_2 = -7.92\,\mathrm{dB}\}, \quad LR = 0.961. \tag{20}$$

As predicted, one source "grabs" most of the total power and lies near $\bar{\theta}$ (the midpoint of the true DOAs), while the other source has a ridiculous power and almost arbitrary DOA. This accounts for a typical outlier.

If instead we assume equal powers for this same random realisation, the ML estimates are

$$\{\hat{\theta}_1, \hat{\theta}_2\} = \{0.09°, 0.96°\},$$

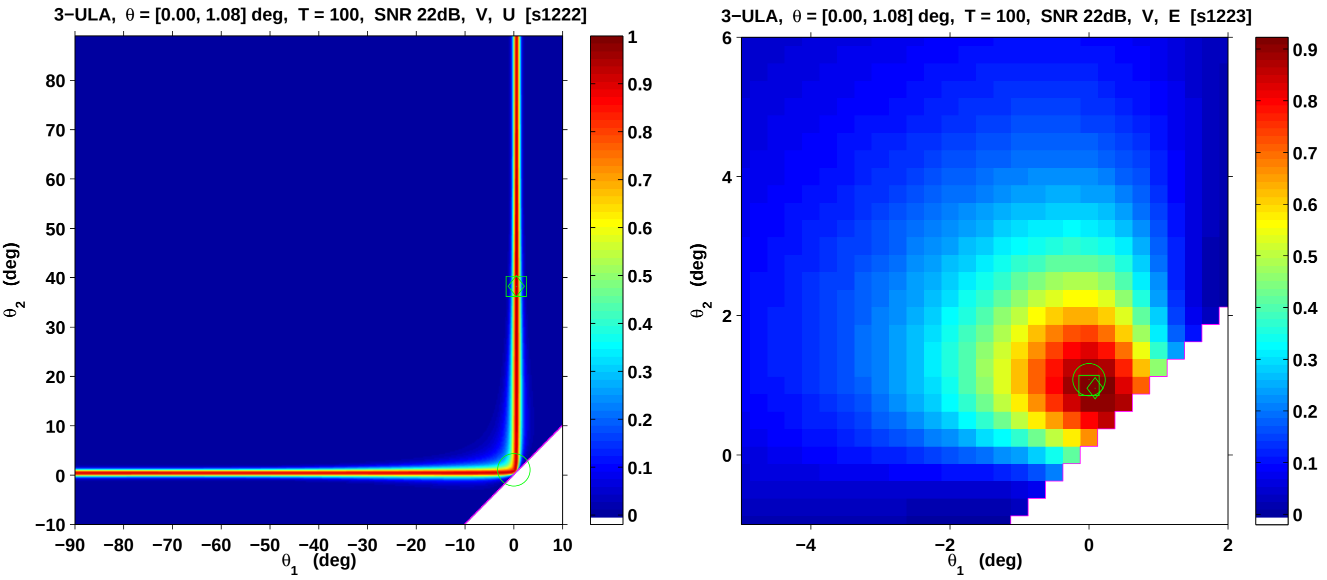
$$\hat{p}_1 = \hat{p}_2 = 21.93\,\mathrm{dB}, \qquad LR = 0.927. \tag{21}$$

We also see that ML optimisation for the equi-power (constrained) model results in the optimised LR of 0.927 which is closer to the LR of the exact covariance matrix (0.918) than is the LR for the unconstrained model (0.961).

Underlying this dramatic difference in MLE is the distinction between the two likelihood functions in Fig. 2. The equi-power model has a tiny region (in the (θ_1, θ_2) plane) of high LR values, whereas the arbitrary source-power model has a likelihood function that has two "arms" forming an L-shape centred on the midpoint DOA $\{\theta_1, \theta_2\} = \{\bar{\theta}, \bar{\theta}\}$, both of which have very high LR values in this case. In fact, these "arms" constitute an "ambiguity region" of DOAs with almost equal and maximal likelihood along them, so that MLE will choose a scenario comprising one source near the midpoint $\bar{\theta}$ and another source that can have almost any value; this also explains why MLE does not produce two outliers simultaneously. In the equi-power model, it seems that the "arms" in the likelihood surface have degenerated, leaving only a tiny "ambiguity region" so that MLE has little latitude to produce outliers.

It is worth mentioning that the GLRT detection-estimation routine (11), for a single source with $P_{FA} = 10^{-2}$ and $\gamma_{LR}(P_{FA}) = 0.896$, gives approximately the same probability of detection as the ITC criterion. Therefore both conventional (ITC) and more recent (GLRT-based) detection-estimation routines demonstrate the unavoidable failure of the entire detection-estimation procedure within the SNR range of about 22–37 dB [3,4,12]. This is because the ML single-source model is not "sufficiently likely", but on the other hand the "sufficiently likely" two-source model contains an outlier. A similar phenomenon was observed in [2] for more sources and a circular antenna array, but since they could not prove their solutions were globally optimal, it has been necessary for us to demonstrate that the "ML breakdown" encountered by our detection-estimation routine is an intrinsic property of ML estimation.

Apart from imposing source-power constraints, we can try to find DOAs from this ambiguity region that, in addition to the global ML estimated DOAs, will demonstrate the nonidentifiability of that particular scenario. Fig. 3 compares sample pdf's over 1000 trials, each of DOA estimates and arbitrary source powers, for (a) the



(a) (b)

Figure 2: Example LR plot for two different source-power models. (a) arbitrary source-power model (yielding an *outlier*): exact DOAs $[0, 1.08]^\circ$, [22, 22] dB, LR=0.918 (circle); stage 1 (maximum over grid, square): $[0.50, 38.25]^\circ$, [24.93, -7.90] dB, LR=0.960; stage 2 (fmincon optimised, diamond): $[0.51, 38.28]^\circ$, [24.93, -7.92] dB, LR=0.961 (b) equi-power model (*no outlier*): exact DOAs $[0, 1.08]^\circ$, [22, 22] dB, LR=0.918; stage 1: $[0, 1]^\circ$, [21.93, 21.93] dB, LR=0.923; stage 2: $[0.09, 0.96]^\circ$, [21.93, 21.93] dB, LR=0.927.

ML search described above, and (b) an "expected likelihood" (EL) [3] search for two DOAs with minimum intersource separation such that $LR(R_\mu)$ exceeds the precalculated LR threshold $\gamma_{LR}(P_{FA})$ (12). For $M = 3$, $T = 100$ and $P_{FA} = 10^{-2}$, we have $\gamma_{LR}(P_{FA}) = 0.896$. We see a significant difference in the two pdf's, as predicted by CRB equi-power analysis. The (unrestricted) global ML search gives a pdf for each DOA that looks like a heavy-tailed χ^2-distribution. While the maximum outlier is 38.3° (which we investigated in Fig. 2(a)), the standard deviations of the DOA estimates are both 5.7°, which is surprisingly close to the CRB of 5.0° (see Fig. 1 for 22 dB SNR). Of course, the severe non-Gaussianity of the pdf's together with the significant bias (2.7°) means that we cannot use the CRB to accurately describe the performance of MLE in this "threshold region" (with performance breakdown).

On the contrary, the EL DOA estimates at Fig. 3(b) form Gaussian-like pdfs, with a bias due to the relatively low EL threshold 0.896 compared with 99% of the optimised LRs. Yet the DOA standard deviation (0.09°) is now far below the CRB (and is comparable with the CRB for the equi-power model), while the worst DOA total error is 0.22°, which is significantly smaller than the intersource separation (1.08°).

3. SUMMARY AND CONCLUSIONS

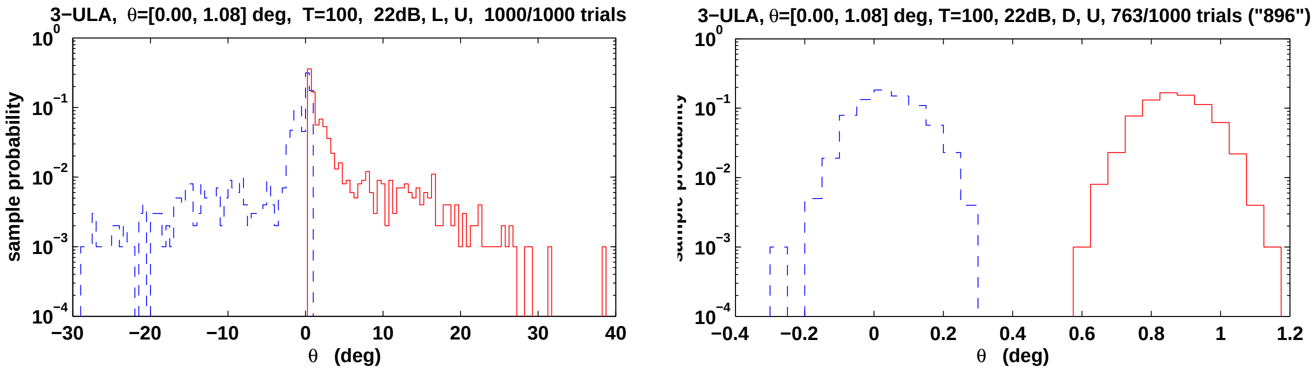
Both CRB analysis and Monte-Carlo simulations of the globally optimal ML estimator have demonstrated that accurate MLE of close sources strongly depends on any *a priori* assumptions about the source powers. Under the equi-power source model, MLE resolution is far below the SNR threshold for detection (where ITC ceases to reliably detect the proper number of sources). For the model with arbitrary source powers, DOA resolution suffers from *statistical nonidentifiability* that is poorly described by CRB analysis. The mechanism for this phenomenon is when MLE randomly selects one of a range of statistically equally likely models. This group of models has one source at approximately the midpoint of the two true DOAs with a power approximately the sum of the two true powers; the other source (an outlier) wanders over the "ambiguity region" (which depends on SNR and sample support). In a related paper [12], we demonstrated that the random matrix theory (RMT) [also known as general statistical analysis (GSA)] technique can be used to accurately predict the ambiguity region analytically.

We also introduced the *expected-likelihood* (EL) estimate which searches for a model with the smallest intersource separation that exceeds the precalculated LR. This EL estimate is much better than the unconstrained global ML estimate, but of course worse than the ML estimate under the equi-power source model. The EL estimate may be used in addition to the ML one, or even a MUSIC one, to establish the statistical nonidentifiability of a given scenario (where two different models have equally high LRs).

This study has investigated the global ML search for a very simple scenario to demonstrate "ML breakdown" that has been observed in a number of "ML proxy" routines with practically important scenarios.

.

.



(a) (b)

Figure 3: Statistical analysis of DOA estimation results for (a) conventional ML global search, and (b) proposed EL threshold method.

REFERENCES

[1] H. Lee and F. Li, "Quantification of the difference between detection and resolution thresholds for multiple closely spaced emitters," *IEEE Trans. Sig. Proc.*, vol. 41, no. 6, pp. 2274–2277, Jun. 1993.

[2] D. Tufts, A. Kot, and R. Vaccaro, "The threshold analysis of SVD-based algorithms," in *Proc. ICASSP-88*, New York, 1988, pp. 2416–2419.

[3] Y. Abramovich, N. Spencer, and A. Gorokhov, "GLRT-based threshold detection-estimation performance improvement and application to uniform circular antenna arrays," *IEEE Trans. Sig. Proc.*, vol. 55, no. 1, pp. 20–31, Jan. 2007.

[4] B. Johnson, Y. Abramovich, and X. Mestre, "MUSIC, G-MUSIC, and maximum-likelihood performance breakdown," *IEEE Trans. Sig. Proc.*, vol. 56, no. 8, pp. 3944–3958, Aug. 2008.

[5] A. Amar and A. Weiss, "Fundamental resolution limits of closely spaced random signals," *IET Radar Sonar Navig.*, vol. 2 (3), pp. 170–179, 2008.

[6] S. Smith, "Statistical resolution limits and the complexified Cramér–Rao bound," *IEEE Trans. Sig. Proc.*, vol. 53, no. 5, pp. 1597–1609, May 2005.

[7] H. Messer, Y. Rockah, and P. Schultheiss, "Localization in the presence of coherent interference," *IEEE Trans. Acoust. Sp. Sig. Proc.*, vol. 38, no. 12, pp. 2025–2032, 1990.

[8] H. Messer, "The potential performance gain in using spectral information in passive detection/localization of wideband sources," *IEEE Trans. Sig. Proc.*, vol. 43, no. 12, pp. 2964–2974, Dec. 1995.

[9] I. Reuven and H. Messer, "The use of the Barankin bound for determining the threshold SNR in estimating the bearing of a source in the presence of another," in *Proc. ICASSP-95*, vol. 3, Detroit, 1995, pp. 1645–1648.

[10] F. Athley, "Threshold region performance of maximum likelihood direction of arrival estimators," *IEEE Trans. Signal Processing*, vol. 53, no. 4, pp. 1359–1373, Apr 2005.

[11] P. Forster, P. Larzabal, and E. Boyer, "Threshold performance analysis of maximum likelihood DOA estimation," *IEEE Trans. Signal Processing*, vol. 52, no. 11, pp. 3183–3191, Nov 2004.

[12] Y. Abramovich, B. Johnson, and N. Spencer, "Statistical nonidentifiability of close emitters: Maximum-likelihood estimation breakdown and its GSA analysis," in *Proc. ICASSP-2009*, Taipei, Taiwan, 2009, pp. 2133–2136.

[13] R. Muirhead, *Aspects of Multivariate Statistical Theory*. New York: Wiley, 1982.

[14] M. Wax and T. Kailath, "Detection of signals by information theoretic criteria," *IEEE Trans. Acoust. Sp. Sig. Proc.*, vol. 33, no. 2, pp. 387–392, Apr. 1985.

[15] B. Ottersten, M. Viberg, P. Stoica, and A. Nehorai, "*Exact and large sample maximum likelihood techniques for parameter estimation and detection in array processing*," in *Radar Array Processing*, S. Haykin, J. Litva, and T. Shepherd, Eds. Berlin: Springer-Verlag, 1993, pp. 99–151, chapter 4, Springer Series in Information Sciences, vol 25.