PARAMETER ESTIMATION FOR POLYNOMIAL PHASE SIGNALS WITH A FAST AND ROBUST ALGORITHM

Olivier Fourt, Messaoud Benidir

Laboratoire des Signaux et Systèmes, Supélec, Université Paris-Sud
3 rue Joliot-Curie 91190 Gif-sur-Yvette (France)
email: Olivier.FOURT@lss.supelec.fr

ABSTRACT

Polynomial phase signals belong to a wide class of non-stationary signals used for modeling and engineering applications. In this paper, we take benefits of some advances in robust estimation in order to propose a new algorithm for estimating the parameters of a polynomial phase signal. This algorithm has the advantages to be fast and its structure is robust to the shape of the noise.

1. INTRODUCTION

Polynomial phase signals (PPS) are a wide class of signals commonly used to model real signals. In radar and sonar applications, Doppler signals can be approximated by a polynomial signal. Even if PPS are of a classical use, the estimation of their parameters is not an easy task. Several methods have been proposed [1], often related with frequency tracking [2]. MUSIC algorithms [3] [4] [5] could be easily adapted for this kind of estimation [6]. Another one is the PHAF algorithm, which is an upgrade of HAF proposed for multi-component signals, with a better threshold towards noise [7] [8] [9]. These methods often have a good accuracy but also have some limits: they are often restricted to gaussian environments, some are sub-optimal [1] and have a high computing cost, especially MUSIC.

Let us now recall the following definition of a polynomial phase signal of order p:

$$s(t) \;=\; \alpha(t)\exp\left(\; i\sum_{k=0}^{p}\beta_k t^k \;\right),\; \beta_k\in\mathbb{R}\; k=0,\ldots,p \quad (1)$$

where $\alpha(t)$ is the amplitude of the signal. Due to the estimators used in this work, the parameters are restricted so that the signals $s(t)$ is real and we have chosen a case with constant amplitude, i.e. $s(t)=\alpha\cos\left(\sum_{k=0}^{p}\beta_k t^k\right)$, with $\alpha\in\mathbb{R}^+$ and $\beta_k\in\mathbb{R}\; k=0,\ldots,p$.

The method used in this paper relies in the combination of several existing algorithms and the use of the instantaneous frequency. The estimation of the instantaneous frequency is a common and old task which could sometime suffer from gross errors and so, many fast algorithms generally have a very low accuracy – 10^{-1} of relative errors – only for an estimation of the range of the parameters. The key of our method is to take benefits from the robust estimation algorithms in order to perform a polynomial robust regression on the instantaneous frequency with a M-estimator. In the experiments, our method was proved to be fast and robust to the kind of additive noise (gaussian or impulse noise) with no changes needed and the accuracy was quite better, even if limited compares to the best but slow algorithms. It's drawbacks are a high signal to noise ratio (SNR) threshold and those limits on accuracy. For this paper, we give only empirical results and so there are many tables. As a model of impulse noise, we chose random variable with α-stable laws [10] $S_\alpha(\beta,\gamma,\mu)$ where γ is the dispersion coefficient, and restricted to the symmetric and centered cases ($\beta=\mu=0$). The SNR's definition is not valid with α-stable noise (infinite power), we use instead the GSNR (Generalize Signal to Noise Ratio) defined in [10] for α-stable noise by:

$$\text{GSNR} \;=\; 10\log\left(\frac{1}{2N\gamma}\sum_{k=0}^{N}|x(k)|^2\right) \quad (2)$$

In the results, only the SNR is indicated but depending on the situation, it's in fact the GSNR with α-stable noise.

This paper is made of 5 sections. Section 2 is used for some recalls on instantaneous parameters. Section 3 deals with the amplitude estimation algorithm. Section 4 deals with the phasis parameters estimation algorithm. Then the section 5 gives the simulations results for both algorithms.

2. ESTIMATION OF PPS PARAMETERS: DEFINITIONS AND ASSUMPTIONS

We are working on the signals:

$x(t) \;=\; s(t)+b(t)$ noisy signal

$s(t)=A\cos\left(\sum_{k=0}^{p}\beta_k(t)^k\right)\; A\in\mathbb{R}^+,\beta_k\in\mathbb{R},k=0,\ldots,p$

$b(t)$ noise independent, equally distributed

We consider the sampled signal: $x(n/f_s)$, $n=1,\ldots,N$ where f_s denotes the sampling frequency.

The definition of the instantaneous frequency can take several formulations [11], we have chosen one of the most commonly used which relies on the use of the analytic signal $z(t)$ associated to $s(t)$:

$$z(t) \;=\; a(t)\exp\left(\; i\phi(t)\;\right) \;=\; s(t) \;+\; i\mathrm{H}[s(t)] \quad (3)$$

where H[.] is the Hilbert transform

$a(t)$ and $\phi(t)$ are respectively the instantaneous amplitude and phasis of $z(t)$. The instantaneous frequency of $s(t)$ is then given by:

$$f_i(t) \;=\frac{1}{2\pi}\frac{\mathrm{d}\phi}{\mathrm{d}t} \quad (4)$$

Usually, we can't evaluate $f_i(t)$ in an explicit form but for a PPS $s(t)$ of order p, we can define its instantaneous frequency by:

$$f_i(t) \;=\; \frac{1}{2\pi}\sum_{k=1}^{p} k\beta_k t^{k-1} \qquad (5)$$

so, as long as we can have a good estimation of the instantaneous frequency it is possible to estimate the parameters of the phasis with a polynomial linear regression.

Since we have chosen to make experiments with impulse noise, robust statistics were needed. Instead of using the variance, we have used the interquartile range. For a random variable V with probability law $P(V)$, one defines:

$$Q_1 = v/P(V \le v) \ge 0.25 \text{ and } P(V \ge v) \ge 0.75 \; 1^{st} \text{ quartile}$$
$$Q_3 = v/P(V \le v) \ge 0.75 \text{ and } P(V \ge v) \ge 0.25 \; 3^{rd} \text{ quartile}$$
$$iqr \;=\; |Q_3 - Q_1| \text{ interquartile range} \qquad (6)$$

3. AMPLITUDE ESTIMATION

Since the instantaneous amplitude of $z(t)$ is the amplitude of the PPS to be estimated up to perturbations and modeled by $a(t) = A + e(t)$, we chose a two step estimation:

- Estimation of the instantaneous amplitude: $a(n) = \mid z(n/f_s) \mid, \qquad n = 1, \dots, N$
- Robust selection of the amplitude: $\hat{A} \;=\; median(a(n))$

4. ESTIMATION OF THE PHASIS PARAMETERS

Due to the considerations on the instantaneous frequency of a PPS, we propose a two step algorithm for the phasis parameters estimation: firstly an estimation of the instantaneous frequency for several samples and then secondly, the estimation of the parameters β_k with a polynomial linear regression. Since the estimation of the instantaneous frequency often shows gross errors and the regression is an ill-posed problem, our contribution was to use a robust estimation algorithm instead of the ordinary least squares.

4.1 Estimation of the instantaneous frequency

We have used the "Parabolic Smoothed Central Finite Difference" (PSCFD) algorithm [12]. It relies on a analysis of the phasis of $z(t)$. The estimator of the instantaneous frequency is then:

$$\hat{f}_i(n) = \frac{f_s}{4\pi}\Bigg(\Bigg(\arg\Bigg[\sum_{p=-Q}^{Q} h(p)\exp\{\; i\arg[z(n-p+1)z^*(n-p-1)]\}\Bigg]\Bigg)\Bigg)_{2\pi} \qquad (7)$$

arguments are belonging to $[0;2\pi[$ and the smoothing window $h(p)$ is defined by:

$$h(p) = \begin{cases} \frac{3N_i}{2(N_i^2-1)}(1-(\frac{p}{N_i})^2) & \text{for p even and } -Q \le p \le Q \\ 0 & \text{otherwise} \end{cases} \qquad (8)$$

the window's length is $M = 2Q+1$ with $N_i = (M+3)/2$. In the simulations in this paper, we have taken $Q = 128$. It's a tradeoff between the accuracy and the variation rate allowed for the frequency.

We have then a set $\hat{f}_i(n), \quad n = 2, \dots, N-1$, of instantaneous frequency estimates. For implementation, the convolution operator implies some losses at the borders and the set used is $\hat{f}_i(n), \quad n = Q+1, \dots, N-Q-2$.

We recall below the conditions of use for PSCFD, given in [12]:

1. Signal is corrupted with additive noise, gaussian, centered, i.i.d. and SNR ≥ 5 thus 7dB.
2. Instantaneous frequency must remain in the Nyquist range $0 \le f_i \le f_s/2$.
3. Variations in frequency must be limited, if f_0 is the mean frequency and $\Delta\nu$ the variations in frequency, then $\frac{\Delta\nu}{f_0} \ll 1$.

Under those assumptions, the estimator is efficient, the Cramer-Rao bound is reached, there is a negligible bias and the variance is given by [12]:

$$\sigma^2 \;=\; \frac{f_s^2}{4\pi}\frac{6}{SNR \times N_i(N_i^2-1)} \qquad (9)$$

Those conditions are important because in our simulations, we are quite often out of the guaranteed domain and we will show that for PPS parameters estimations, some results remain valid.

4.2 Polynomial robust regression

Let us introduce the folowing notations:

$$\mathbf{fi} = \Big(\hat{f}_i(2), \dots, \hat{f}_i(N-1)\Big)^T \qquad \text{with} \qquad \hat{f}_i(k) = \hat{f}_i(kt_s)$$

$$\mathbf{b} = \Big(\beta_p, \dots, \beta_1\Big)^T$$

$$\mathbf{R} = \frac{1}{2\pi}\begin{pmatrix} p\Big(2t_s\Big)^{p-1} & \cdots & 2\Big(2t_s\Big) & 1 \\ p\Big(3t_s\Big)^{p-1} & \cdots & 2\Big(3t_s\Big) & 1 \\ \vdots & & \vdots & \vdots \\ p\Big((N-1)t_s\Big)^{p-1} & \cdots & 2\Big((N-1)t_s\Big) & 1 \end{pmatrix} \qquad (10)$$

The vector $\mathbf{b}$ is then the solution of the linear system:

$$\mathbf{fi} \;=\; \mathbf{Rb} \;+\; \mathbf{e} \qquad (11)$$

where $\mathbf{e}$ is the vector of errors.

Estimations errors of f_i don't have a gaussian distribution and could even show some aberrations. So the classical least squares can't be used and it is required to use a robust estimator. The choice was to use a M-estimator [13] with the algorithm "Iteratively Reweighted Least Squares" (IRLS) and the choice of the Welsch cost function [14]. The Welsch function has been chosen since it offered the best results in our tests. The parameters estimated are obtained via the following optimisation problem:

$$\min_{\beta_k}\sum_n \rho\Big(e(n)\Big) \text{ with } \rho(x) = \frac{c^2}{2}\left[1-\exp\left(-(x/c)^2\right)\right] \qquad (12)$$

The constant $c = 4.2$ in the simulations. For the special case of a PPS of order 1 (harmonic signal), only a constant is estimated and there's no need for the linear regression in that situation: we take the median of instantaneous frequency estimates as an estimator of $\hat{\beta}_1$.

5. SIMULATIONS AND RESULTS

A large class of tests have been made in order to check the performances of parameters'estimation for PPS of different orders and several noise conditions.

In all the experiments we have used signals with the following characteristics: amplitude $A=1$, sampling frequency $f_s=10^6$Hz, $N=8192$. A Butterworth anti-aliasing filter of order 8 with cutoff frequency of 4×10^5Hz has been used. For the phasis smoothing window, we took $Q=128$.

Parameters range: for all signals, $\beta_0\in[\pi\ ;\ 3\pi]$

- PPS1 (order 1) $\beta_1\in[0\ ;\ 3.5\times10^5]$
- PPS2 (order 2) $\beta_1\in[0\ ;\ 3.5\times10^5]$
  $\beta_2\in[-2\times10^7\ ;\ +2\times10^7]$
- PPS3 (order 3) $\beta_1\in[0\ ;\ 3.5\times10^5]$
  $\beta_2\in[-8\times10^7\ ;\ +8\times10^7]$
  $\beta_3\in[-6\times10^9\ ;\ +6\times10^9]$

Parameters have been chosen so that the instantaneous frequency of PPS remains between 0 and 3.5×10^5 Hz.

For each phasis order, we have randomly created a set of PPS for several SNR, both for gaussian and impulse noise. Then we estimate their parameters, chosing five gaussian noise level and five α-stable noise equivalent in power terms. In order to check the best performances achievable, we have also conduct an experiment with no additive noise. Each experiment was repeated 15000 times. The gaussian noise levels were respectively 30dB, 20dB, 10dB, 6dB et 3dB α-stable noise were defined by the following parameters: $\alpha=1.2$, $\beta=0$ (symmetric distribution), $\mu=0$ (centered distribution), $\gamma=0.0005$, 0.005, 0.05, 0.1 and 0.2.

The obtained results are only empirical. We give some statistics of the relatives errors of the parameters. The use of impulse noise implies the use of robust statistic: median, interquartile range (dispersion estimation) and a robust mean square error – a trimmed mean calculated by excluding the 5% highest values of the squared errors. In the results tables, the symbol α indicates the result of an experiment with α-stable noise whereas $\mathcal{N}$ indicates the result of an experiment with gaussian noise.

5.1 Amplitude estimation results

The results of the amplitude estimations are given in the tables 1 and 2. The table 1 gives the medians of the relatives error estimations and the table 2 gives the interquartiles ranges of the errors. A thing will be immediately noticed: for a given SNR and statistic, the results are almost the same for the different PPS phasis order p. The accuracy of the amplitude estimation seems independent from the phasis'parameters estimation, a result known for several years with gaussian noise [15] [16] that could have been expected and which remains true with α-stable noise. For a given SNR, the results with α-stable noise and with gaussian noise are almost the same. To be true the results are slightly better with α-stable noise: the estimator is robust to the kind of the noise.

The accuracy decrease with the SNR as we could see it on the Fig. 1 which is the plot of the robust MSE for the differents signals and kind of noise and we should choose a limit for the use at 10 dB, a limit we already had in our previous work [18] and that we should consider as a limit of use of analytical signal methods. For a fixed SNR, the median is always negative and its absolute value is of the same range of the interquartile range: $\hat{A}$ estimator seemed have a negative bias of the range of the median, bias due to the anti-aliasing filter whose effect is not corrected: in a noise-free experiment without this filter, the median and interquartile range of $\hat{A}$ decrease to values around 10^{-9} and MSE a value of 10^{-14}.

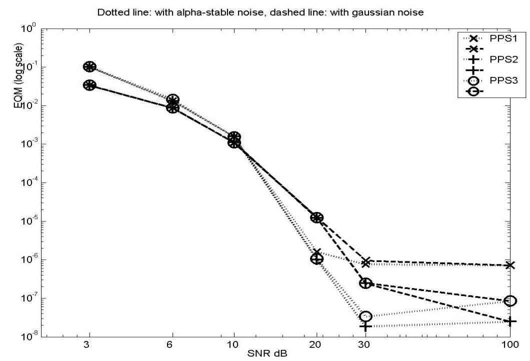

Figure 1: Robust MSE of $\hat{A}$ for PPS of order 1, 2 and 3 with different noises

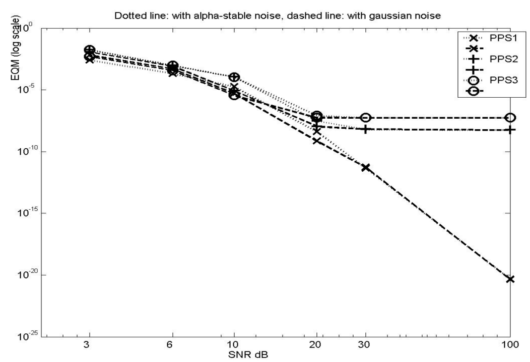

Figure 2: Robust MSE of $\hat{\beta_1}$ for PPS of order 1, 2 and 3 with different noises

5.2 Phasis parameters estimation results

The results of the phasis parameters β_1 to β_p estimations are given in the tables 3 and 4. The table 3 gives the medians of the relatives error estimations and the table 4 gives the interquartiles ranges of the errors.

Parameter $\hat{\beta_1}$ estimation is slightly biased, bias that increase with the noise's power which is coherent with the properties of PSCFD algorithm [12] as this parameter is a frequency translation in the time-frequency plane. For the other parameters, the alternate positive and negative values don't allow us to conclude to a bias of the estimates.

The Fig. 2 and Fig. 3 are the plots of MSE for the parameters β_1 and β_2 with different noises. Dotted lines are the plots of tests with gaussian noise, the others are tests with α-stable noise. The 100dB value is the noise free experiments. For space considerations, we don't give the plots for parameters β_3 and β_4 but they are quite the same.

With the whole results of the experiments, we could see that some limitations of the PSCFD use given in [12] are really needed whereas some others can be ignored for the PPS's parameters estimation task.

Firstly, we see that for all the tests, at a given power of the noise, the results with gaussian noise and with α-stable noise

SNR / PPS order	∞ dB	30 dB	20 dB	10 dB	6 dB	3 dB
PPS1	9.43×10^{-6}	α -5.82×10^{-6} $\mathcal{N}$ -2.13×10^{-4}	α -9.67×10^{-4} $\mathcal{N}$ -3.48×10^{-3}	α -3.87×10^{-2} $\mathcal{N}$ -3.39×10^{-2}	α -0.112 $\mathcal{N}$ -9.49×10^{-2}	α -0.312 $\mathcal{N}$ -0.186
PPS2	6.56×10^{-6}	α -4.78×10^{-6} $\mathcal{N}$ -2.15×10^{-4}	α -9.26×10^{-4} $\mathcal{N}$ -3.48×10^{-3}	α -3.86×10^{-2} $\mathcal{N}$ -3.39×10^{-2}	α -0.112 $\mathcal{N}$ -9.51×10^{-2}	α -0.311 $\mathcal{N}$ -0.186
PPS3	7.47×10^{-6}	α -2.18×10^{-6} $\mathcal{N}$ -2.18×10^{-4}	α -9.27×10^{-4} $\mathcal{N}$ -3.47×10^{-3}	α -3.86×10^{-2} $\mathcal{N}$ -3.39×10^{-2}	α -0.112 $\mathcal{N}$ -9.50×10^{-2}	α -0.311 $\mathcal{N}$ -0.186

Table 1: Median of the relative error of $\hat{A}$

SNR / PPS order	∞ dB	30 dB	20 dB	10 dB	6 dB	3 dB
PPS1	1.79×10^{-3}	α 1.78×10^{-3} $\mathcal{N}$ 1.68×10^{-3}	α 1.74×10^{-3} $\mathcal{N}$ 2.67×10^{-3}	α 1.30×10^{-2} $\mathcal{N}$ 7.08×10^{-3}	α 3.39×10^{-2} $\mathcal{N}$ 1.15×10^{-2}	α 8.36×10^{-2} $\mathcal{N}$ 1.47×10^{-2}
PPS2	1.35×10^{-4}	α 1.87×10^{-4} $\mathcal{N}$ 7.87×10^{-4}	α 8.79×10^{-4} $\mathcal{N}$ 2.44×10^{-3}	α 1.33×10^{-2} $\mathcal{N}$ 7.04×10^{-3}	α 3.33×10^{-2} $\mathcal{N}$ 1.13×10^{-2}	α 8.26×10^{-2} $\mathcal{N}$ 1.53×10^{-2}
PPS3	4.91×10^{-4}	α 3.17×10^{-4} $\mathcal{N}$ 7.74×10^{-4}	α 9.13×10^{-4} $\mathcal{N}$ 2.46×10^{-3}	α 1.31×10^{-2} $\mathcal{N}$ 7.02×10^{-3}	α 3.41×10^{-2} $\mathcal{N}$ 1.14×10^{-2}	α 8.35×10^{-2} $\mathcal{N}$ 1.51×10^{-2}

Table 2: Interquartile range of the relative error of $\hat{A}$

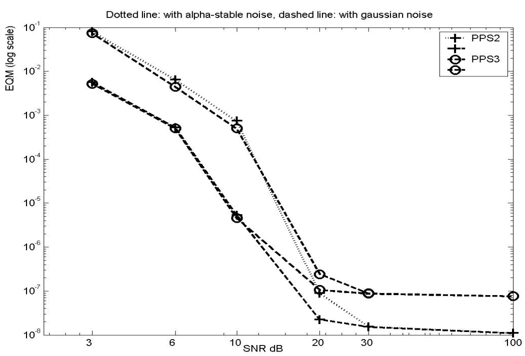

Figure 3: Robust MSE of $\hat{\beta}_2$ for PPS of order 2 and 3 with different noises

are of the same range: our method is robust to the kind of the noise's distribution and the restriction to gaussian noise is not needed.

Secondly, as the power of the noise increases, the accuracy of the estimations quickly decreases. For SNR below 6dB, the algorithm only gives an estimation of the range of the parameters. The condition SNR $\geq$ 7dB is needed although SNR$\geq$ 10dB seems to be a better guarantee.

Thirdly, for all kind of noises, we have a gap between the results for PPS of order 4 and the others. The results for phasis of order 4 are not given for space considerations and for their low accuracy: those signals are more complex and subject to greater frequency's variations. Our conclusion to this is that the limitation in frequency's variation given in [12] holds true but for the PPS's parameters estimation task, we could allow a larger range. For a signal of duration T sampled at f_s and a smoothing window of 257 points, we propose the following limit:

$$\left|\frac{\mathrm{d}f_i}{\mathrm{d}t}\right| < \frac{2f_e}{T}$$

This limit is the widest frequency's variation possible for a PPS of order 3 obtained for $f_i(0)=0$, $f_i(T/2)=f_s/2$ and $f_i(T)=0$ or $f_i(0)=f_s/2$, $f_i(T/2)=0$ and $f_i(T)=f_s/2$.

Even if the estimations'accuracies is good at high SNR – values of 10^{-7} or better – it is still fewer than the best algorithms such as MUSIC or PHAF methods [3] [4] [17] [7] [19] [9] which have accuracies of 10^{-12} or better. On the other hand, our algorithm is quite faster: for a N point long signal, MUSIC give a solution in $O(N^3)$ operations and PHAF in $O(N^2 log(N))$ whereas our method makes it in $O(N)$ for PSCFD [1] plus the computation of the regression which is of $O(Np^3)$ for a phasis of order p.

6. CONCLUSION

As a final point to this paper, we have build an algorithm for PPS parameters estimation made with several existing methods which is fast, with a linear complexity and with a limit of use toward the power of the noise which should be fixed at SNR= 10dB. The amplitude estimation is quite robust to the shape of the noise, it is slightly better with impulse noise and seems independent of the phasis order. The phasis's parameters estimation depend few of the order of the phasis as long as the variations of the frequency don't go out of range and it is robust to the kind of the noise's distribution.

A futur work will be to find a better instantaneous frequency estimation algorithm in order to work with heavy noise power.

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Parameters \ SNR	∞ dB	30 dB	20 dB	10 dB	6 dB	3 dB
$\hat{\beta}_1$ PPS1	-3.1×10^{-13}	α -6.4×10^{-8} $\mathcal{N}$ -2.7×10^{-8}	α -5.4×10^{-6} $\mathcal{N}$ -2.8×10^{-6}	α -1.2×10^{-3} $\mathcal{N}$ -2.8×10^{-4}	α -3.7×10^{-3} $\mathcal{N}$ -3.4×10^{-3}	α -1.2×10^{-2} $\mathcal{N}$ -1.1×10^{-2}
$\hat{\beta}_1$ PPS2	-2.2×10^{-5}	α -2.2×10^{-5} $\mathcal{N}$ -2.3×10^{-5}	α -3.9×10^{-5} $\mathcal{N}$ -3.2×10^{-5}	α -2.3×10^{-3} $\mathcal{N}$ -3.2×10^{-4}	α -4.3×10^{-3} $\mathcal{N}$ -3.8×10^{-3}	α -9.6×10^{-3} $\mathcal{N}$ -1.4×10^{-2}
$\hat{\beta}_1$ PPS3	-4.2×10^{-5}	α -2.6×10^{-5} $\mathcal{N}$ -2.7×10^{-5}	α -4.3×10^{-5} $\mathcal{N}$ -3.4×10^{-5}	α -2.2×10^{-3} $\mathcal{N}$ -2.7×10^{-4}	α -3.9×10^{-3} $\mathcal{N}$ -3.5×10^{-3}	α -8.8×10^{-3} $\mathcal{N}$ -1.3×10^{-2}
$\hat{\beta}_2$ PPS2	-9.4×10^{-8}	α -5.6×10^{-7} $\mathcal{N}$ -6.2×10^{-7}	α 3.1×10^{-5} $\mathcal{N}$ 1.2×10^{-5}	α 9.9×10^{-4} $\mathcal{N}$ 1.7×10^{-3}	α -5.4×10^{-3} $\mathcal{N}$ 1.8×10^{-2}	α -4.5×10^{-2} $\mathcal{N}$ 6.3×10^{-2}
$\hat{\beta}_2$ PPS3	-2.62×10^{-5}	α -1.2×10^{-5} $\mathcal{N}$ -2.9×10^{-5}	α 8.1×10^{-6} $\mathcal{N}$ -3.3×10^{-5}	α 8.6×10^{-4} $\mathcal{N}$ 1.6×10^{-3}	α -4.9×10^{-3} $\mathcal{N}$ 1.8×10^{-2}	α -4.3×10^{-2} $\mathcal{N}$ 6.3×10^{-2}
$\hat{\beta}_3$ PPS3	1.9×10^{-7}	α -1.7×10^{-8} $\mathcal{N}$ 2.4×10^{-6}	α 7.3×10^{-5} $\mathcal{N}$ 1.6×10^{-5}	α 9.9×10^{-4} $\mathcal{N}$ 1.7×10^{-3}	α -4.9×10^{-3} $\mathcal{N}$ 1.8×10^{-2}	α -4.4×10^{-2} $\mathcal{N}$ 6.4×10^{-2}

Table 3: Median of the relative error of the parameters $\hat{\beta}_k$

Parameters \ SNR	∞ dB	30 dB	20 dB	10 dB	6 dB	3 dB
$\hat{\beta}_1$ PPS1	1.1×10^{-11}	α 1.6×10^{-6} $\mathcal{N}$ 1.8×10^{-6}	α 5.8×10^{-5} $\mathcal{N}$ 2.8×10^{-5}	α 3.7×10^{-3} $\mathcal{N}$ 2.7×10^{-3}	α 8.2×10^{-3} $\mathcal{N}$ 2.8×10^{-2}	α 3.5×10^{-2} $\mathcal{N}$ 9.9×10^{-2}
$\hat{\beta}_1$ PPS2	7.8×10^{-5}	α 7.9×10^{-5} $\mathcal{N}$ 8.1×10^{-5}	α 1.2×10^{-4} $\mathcal{N}$ 8.8×10^{-5}	α 7.3×10^{-3} $\mathcal{N}$ 2.9×10^{-3}	α 1.5×10^{-2} $\mathcal{N}$ 3.1×10^{-2}	α 7.8×10^{-2} $\mathcal{N}$ 0.11
$\hat{\beta}_1$ PPS3	3.4×10^{-4}	α 3.6×10^{-4} $\mathcal{N}$ 3.4×10^{-4}	α 3.9×10^{-4} $\mathcal{N}$ 3.6×10^{-4}	α 9.8×10^{-3} $\mathcal{N}$ 2.5×10^{-3}	α 2.2×10^{-2} $\mathcal{N}$ 2.8×10^{-2}	α 8.1×10^{-2} $\mathcal{N}$ 0.11
$\hat{\beta}_2$ PPS2	1.4×10^{-4}	α 1.6×10^{-4} $\mathcal{N}$ 1.6×10^{-4}	α 3.3×10^{-4} $\mathcal{N}$ 1.9×10^{-4}	α 1.6×10^{-2} $\mathcal{N}$ 1.4×10^{-3}	α 4.3×10^{-2} $\mathcal{N}$ 1.5×10^{-2}	α 0.13 $\mathcal{N}$ 4.7×10^{-2}
$\hat{\beta}_2$ PPS3	2.7×10^{-4}	α 3.6×10^{-4} $\mathcal{N}$ 3.7×10^{-4}	α 5.8×10^{-4} $\mathcal{N}$ 4.1×10^{-4}	α 1.9×10^{-2} $\mathcal{N}$ 1.6×10^{-3}	α 4.9×10^{-2} $\mathcal{N}$ 1.6×10^{-2}	α 0.15 $\mathcal{N}$ 4.9×10^{-2}
$\hat{\beta}_3$ PPS3	2.6×10^{-4}	α 3.7×10^{-4} $\mathcal{N}$ 3.7×10^{-4}	α 6.1×10^{-4} $\mathcal{N}$ 4.2×10^{-4}	α 1.9×10^{-2} $\mathcal{N}$ 1.7×10^{-3}	α 5.1×10^{-2} $\mathcal{N}$ 1.6×10^{-2}	α 0.16 $\mathcal{N}$ 4.9×10^{-2}

Table 4: Interquartile range of the relative error of the parameters $\hat{\beta}_k$

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