FIRST-ORDER ANALYSIS OF THE MODE AND AMPLITUDE ESTIMATES OF A DAMPED SINUSOID USING MATRIX PENCIL

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ABSTRACT

This article presents a statistical analysis of the Matrix Pencil method for estimating the mode and the amplitude of a single damped complex exponential. This study is based on a perturbation analysis of the mode and the amplitude, assuming a high signal-to-noise ratio. Closed-form expressions of the mean and variance of these perturbations are derived. It is shown that the estimates are unbiased and that the estimator can be tuned in order to obtain a minimal variance. The theoretical results are verified using Monte Carlo simulations.

1. INTRODUCTION

Parameter estimation of exponential signals is a fundamental problem in signal processing. Several methods have been proposed to solve this problem, such as maximum likelihood approaches [1, 19] and subspace-based methods (MUSIC [15], ESPRIT [14], backward linear prediction (BLP) [8], direct data approximation (DDA) [9], Matrix Pencil [6], etc). In the case of undamped sinusoids, statistical performances of these methods have been extensively studied, either for estimating the frequencies [5, 11–13] or the amplitudes [4, 20]. A few studies have analyzed the case of damped signals [7, 10, 16]. Nevertheless, the expressions obtained in this case are complicated and usually written in a matrix form. So, it is difficult to use them in order to, for instance, examine precisely the dependency of the method's design parameters on the variance.

In this paper, the estimation method considered is the well-known "Matrix Pencil" (MP) method developed by Hua and Sarkar [6]. It is based on a matrix prediction equation in which the data matrices have a Hankel structure. As for all subspace-based methods, the MP method uses a singular value decomposition (SVD) followed by the truncation of the lowest singular values to reduce noise effect. The perturbation analysis developed consists in studying error propagation, starting from the singular values of the data matrix towards the signal mode and amplitude. Thus, we use the same approach as in [2, 7, 10] to derive the expression of the damped mode (frequency and damping factor) variance. As compared to these works, the first contribution of the present paper is the derivation of a compact form for the final expression of the mode variance which is moreover more suitable for practical use. As a second contribution, the noise sensitivity of the amplitude estimate is also presented and compared to previous results. From the expressions of the mode and amplitude variances, we give the optimal design parameters of MP allowing one to achieve a minimum variance. Of course, this work can be easily extended to other subspace-based estimation methods for damped sinusoids implying an SVD truncation, such as BLP and DDA.

The remaining of the paper is organized as follows. In section 2, a brief outline of the MP method is given. Sections 3 and 4 are devoted to the first-order perturbation analysis of the method, for the mode and amplitude estimates. In section 5, some Monte Carlo simulations are presented in order to verify the analytical expressions obtained.

2. MATRIX PENCIL METHOD

Consider the following complex signal composed of M damped exponentials:

$$\tilde{x}(n) = x(n) + e(n) = \sum_{i=1}^{M} a_i p_i^n + e(n) \tag{1}$$

for $n = 0, \dots, N-1$, where $\{p_i = \exp(\alpha_i + j\omega_i)\}_{i=1}^{M}$ are the damped modes ($\alpha_i < 0$) with complex amplitudes $a_i = A_i \exp(j\phi_i)$. The signal $e(n)$ is a zero-mean complex white Gaussian noise with variance σ_e^2. Let $r_i = \exp(\alpha_i)$. Model (1) is used in this section in order to describe the MP method. Afterwards, we will use the following model ($M = 1$):

$$\tilde{x}(n) = x(n) + e(n) = ap^n + e(n). \tag{2}$$

Throughout this paper, the tilde symbol $(\tilde{\cdot})$ indicates the noisy version of the variable $(\cdot)$ which may be a scalar or a matrix.

All subspace-based methods consist first in estimating the modes $\{p_i\}$ by using SVD. Then, the amplitudes are obtained by solving a system of linear equations. The MP method is based on the following steps [6]:

1. With the available data, form two Hankel matrices $\tilde{\mathbf{X}}_0$ and $\tilde{\mathbf{X}}_1$ where $\tilde{\mathbf{X}}_i = [\tilde{\mathbf{x}}_i, \tilde{\mathbf{x}}_{i+1}, \dots, \tilde{\mathbf{x}}_{i+L-1}]$ and $\tilde{\mathbf{x}}_k = [\tilde{x}(k), \tilde{x}(k+1), \dots, \tilde{x}(N-L-1+k)]^T$ for $k = 0, \dots, L$. The parameter $L \in \mathbb{N}$ is similar to the prediction order in the well-known Prony method with $M \leqslant L \leqslant N-M$.
2. Perform the SVD of matrix $\tilde{\mathbf{X}}_1$ and set to 0 all but the first M largest singular values. The resulting matrix is the best rank M approximation of $\tilde{\mathbf{X}}_1$.

3. Compute the reduced rank pseudoinverse of $\tilde{\mathbf{X}}_1$ (denoted by $\tilde{\mathbf{X}}_1^\dagger$) to obtain matrix $\tilde{\mathbf{Z}}$:

$$\tilde{\mathbf{Z}} = \tilde{\mathbf{X}}_1^\dagger \tilde{\mathbf{X}}_0. \tag{3}$$

4. The estimated modes $\{\tilde{p}_i\}_{i=1}^M$ correspond to the inverse of the M eigenvalues of matrix $\tilde{\mathbf{Z}}$ lying outside the unit circle in the complex plan.
5. Once the modes are estimated, the amplitudes can be computed by solving the following linear system in the least-squares sense:

$$\begin{bmatrix} 1 & \cdots & 1 \\ \tilde{p}_1 & \cdots & \tilde{p}_M \\ \vdots & & \vdots \\ \tilde{p}_1^{K-1} & \cdots & \tilde{p}_M^{K-1} \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_M \end{bmatrix} \approx \begin{bmatrix} \tilde{x}(0) \\ \vdots \\ \tilde{x}(K-1) \end{bmatrix}$$

$$\tilde{\mathbf{P}}_K \tilde{\mathbf{a}} \approx \tilde{\mathbf{x}}_K.$$

It leads to $\tilde{\mathbf{a}} = (\tilde{\mathbf{P}}_K^H \tilde{\mathbf{P}}_K)^{-1} \tilde{\mathbf{P}}_K^H \tilde{\mathbf{x}}_K$. Here, $K \leqslant N$ represents the number of equations used to estimate the amplitudes. The role of this parameter will be discussed later.

3. PERTURBATION OF THE MODE

In the single-mode case (model (2)), the matrix $\mathbf{X}_1$ is rank 1 and it can be shown [10] that its SVD is given by:

$$\mathbf{X}_1 = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^H$$

where $\sigma_1 = Ar(k_v k_u)^{1/2}$ is the unique nonzero singular value of $\mathbf{X}_1$, and

$$\mathbf{u}_1 = \frac{\exp(j\phi)}{\sqrt{k_u}}[1, p, \cdots, p^{N-L-1}]^T$$
$$\mathbf{v}_1 = \frac{\exp(-j\omega)}{\sqrt{k_v}}[1, p^*, \cdots, p^{*L-1}]^T$$

are the corresponding principal singular vectors. The terms k_u and k_v are the normalization of vectors $\mathbf{u}_1$ and $\mathbf{v}_1$ ($k_u = \sum_{i=0}^{N-L-1} r^{2i}$, $k_v = \sum_{i=0}^{L-1} r^{2i}$), respectively. The superscripts "T" and "H" denote matrix transpose and conjugate transpose, respectively. According to perturbation theory, a small error in $\mathbf{X}_1$ produces a perturbation in the eigenvalue σ_1^2 of $\mathbf{X}_1^H \mathbf{X}_1$. At high SNR, the first order approximation of this perturbation is given by [17]:

$$\begin{aligned} \Delta\sigma_1^2 &= \mathbf{v}_1^H (\tilde{\mathbf{X}}_1^H \tilde{\mathbf{X}}_1 - \mathbf{X}_1^H \mathbf{X}_1) \mathbf{v}_1 \\ &= \mathbf{v}_1^H (\mathbf{E}_1^H \mathbf{X}_1 + \mathbf{X}_1^H \mathbf{E}_1) \mathbf{v}_1. \end{aligned} \tag{4}$$

where $\mathbf{E}_1$ is a Hankel matrix of noise entries. It is built as $\mathbf{X}_1$, that means that $\tilde{\mathbf{X}}_1 = \mathbf{X}_1 + \mathbf{E}_1$. Now, at high SNR, the matrix $\tilde{\mathbf{X}}_1$ is approximately rank 1, thus:

$$\tilde{\mathbf{X}}_1^\dagger \approx \frac{1}{\tilde{\sigma}_1^2} \tilde{\mathbf{X}}_1^H. \tag{5}$$

Replacing this expression in (3) and using the fact that $\tilde{\mathbf{X}}_0 = \mathbf{X}_0 + \mathbf{E}_0$ (so $\mathbf{E}_0$ is also a Hankel matrix of noise elements), it yields:

$$\Delta\mathbf{Z} = \tilde{\mathbf{Z}} - \mathbf{Z} \approx \frac{1}{\sigma_1^2}(-\mathbf{Z}\Delta\sigma_1^2 + \mathbf{X}_1^H \mathbf{E}_0 + \mathbf{E}_1^H \mathbf{X}_0).$$

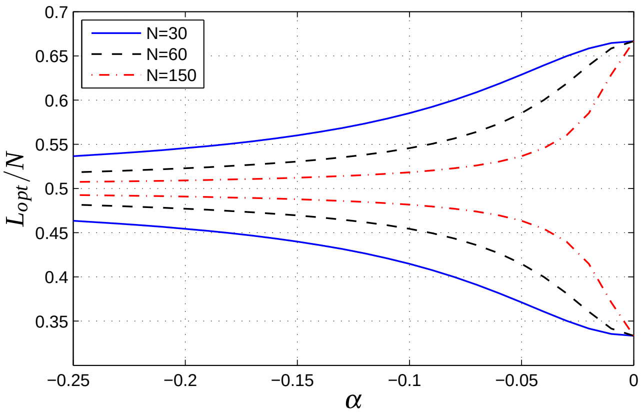

Figure 1: Plot of L_{opt}/N as a function of α for some values of N.

The first order perturbation of the nonzero eigenvalue z_1 of the matrix $\mathbf{Z}$ is then:

$$\Delta z_1 = \frac{1}{\sigma_1^2} \mathbf{v}_1^H (-\mathbf{Z}\Delta\sigma_1^2 + \mathbf{X}_1^H \mathbf{E}_0 + \mathbf{E}_1^H \mathbf{X}_0) \mathbf{v}_1.$$

Since $\tilde{p} = 1/\tilde{z}_1$, it can be shown that $\Delta p \approx -p^2 \Delta z_1$. Moreover, as $\mathbf{Z} = \frac{1}{p}\mathbf{v}_1 \mathbf{v}_1^H$ and by using the expression of $\Delta\sigma_1^2$ in (4), we finally obtain

$$\Delta p = \frac{p}{\sigma_1} \mathbf{u}_1^H (\mathbf{E}_1 - p\mathbf{E}_0) \mathbf{v}_1. \tag{6}$$

This expression of the mode error shows that the estimate is unbiased, that is $\mathbb{E}\{\Delta p\} = 0$. Hence, it is easy to show that $\mathbb{E}\{\Delta\omega\} = \mathbb{E}\{\Delta\alpha\} = 0$. Moreover, under the assumption of white Gaussian noise, after some calculations we found that $\mathbb{E}\left\{(\Delta p)^2\right\} = 0$, and [3]

$$\mathbb{E}\left\{|\Delta p|^2\right\} = \frac{\sigma_e^2}{A^2} \cdot \begin{cases} \dfrac{(1-r^2)^3(1+r^{2N-2L})}{(1-r^{2N-2L})^2(1-r^{2L})}, & L \leqslant \frac{N}{2} \\ \dfrac{(1-r^2)^3(1+r^{2L})}{(1-r^{2N-2L})(1-r^{2L})^2}, & L \geqslant \frac{N}{2} \end{cases} \tag{7}$$

From (7), it is possible to obtain the variances of the frequency and the damping factor since $\operatorname{var}(\Delta\omega) = \operatorname{var}(\Delta\alpha) = \frac{1}{2r^2}\mathbb{E}\left\{|\Delta p|^2\right\}$. In addition, the closed-form expression (7) can now be exploited to study the role of parameter L in the estimation variance. Indeed, $\mathbb{E}\left\{|\Delta p|^2\right\}$ is minimized with respect to L when:

$$\begin{aligned} L_{opt} &= \arg \min_{1 \leqslant L < N} \mathbb{E}\left\{|\Delta p|^2\right\} \\ &= \frac{N}{2} \pm \frac{1}{2\log(r)} \log\left(\tan \frac{\pi - \tan^{-1} r^{-N}}{3}\right). \end{aligned} \tag{8}$$

In the case of a pure sinusoid ($r = 1$), it gives the well-known optimal values $L_{opt} \in \{N/3, 2N/3\}$ [6]. For $r < 1$ and N sufficiently large so that $r^N \ll 1$, L_{opt} tends towards $N/2$ (see figure 1). This means that the value of L_{opt} lies in the interval $[N/3, 2N/3]$ according to the value of the damping factor. In fact, since $\mathbb{E}\left\{|\Delta p|^2\right\}$ is symmetric about $N/2$, it is sufficient to choose the value of L between $N/3$ and $N/2$.

4. PERTURBATION OF THE AMPLITUDE

In the single-mode case given by model (2), the amplitude estimation calls for the resolution of the following linear system:

$$\tilde{\mathbf{p}}_K.\tilde{a} \approx \tilde{\mathbf{x}}_K$$

where $\tilde{\mathbf{p}}_K = [1, \tilde{p}, \ldots, \tilde{p}^{K-1}]^T$, $\tilde{\mathbf{x}}_K = [\tilde{x}(0), \ldots, \tilde{x}(K-1)]^T$ and $K \leqslant N$. The least-squares solution is then given by:

$$\tilde{a} = (\tilde{\mathbf{p}}_K^H \tilde{\mathbf{p}}_K)^{-1} \tilde{\mathbf{p}}_K^H \tilde{\mathbf{x}}_K.$$

By considering the noise in the vector $\tilde{\mathbf{x}}_K$ and the estimation errors in $\tilde{\mathbf{p}}_K = \mathbf{p}_K + \Delta\mathbf{p}_K$, the first-order perturbation in the amplitude can be derived:

$$\Delta a = \frac{1}{k_p} \mathbf{p}_K^H (\mathbf{e}_K - a\mathbf{p}'_K \Delta p) \tag{9}$$

where $\Delta\mathbf{p}_K = \mathbf{p}'_K \Delta p$, $\mathbf{p}'_K = [0, 1, 2p, \cdots, (K-1)p^{K-2}]^T$, $\mathbf{e}_K = [e(0), \cdots, e(K-1)]^T$, and $k_p = \sum_{i=0}^{K-1} r^{2i}$. From this expression it is easy to conclude that, if the estimate of p is unbiased (i.e. $\mathbb{E}\{\Delta p\} = 0$), then the least-squares estimate of a is also unbiased since $\mathbb{E}\{\Delta a\} = -(a/k_p)\mathbf{p}_K^H \mathbf{p}'_K \mathbb{E}\{\Delta p\} = 0$. Thus, $\mathbb{E}\{\Delta A\} = \mathbb{E}\{\Delta\phi\} = 0$. Moreover, it can be easily shown that $\mathbb{E}\left\{(\Delta a)^2\right\} = 0$. The calculation of the variance of the complex amplitude ($\mathbb{E}\left\{|\Delta a|^2\right\}$) is more tedious. The main steps are now given. First, from (9) we have

$$\begin{aligned} |\Delta a|^2 =& (\mathbf{p}_K^H \mathbf{e}_K \mathbf{e}_K^H \mathbf{p}_K + A^2 |\Delta p|^2 \mathbf{p}_K^H \mathbf{p}'_K \mathbf{p}'^H_K \mathbf{p}_K \\ & - 2\Re[a\mathbf{p}_K^H \mathbf{p}'_K \Delta p \mathbf{e}_K^H \mathbf{p}_K])/k_p^2. \end{aligned}$$

where $\Re[\boldsymbol{.}]$ denotes the real part of a complex number. Applying mathematical expectation and using the fact that $e(n)$ is zero-mean uncorrelated complex noise, we obtain

$$\begin{aligned} \mathbb{E}\left\{|\Delta a|^2\right\} =& (\mathbf{p}_K^H \mathbb{E}\left\{\mathbf{e}_K \mathbf{e}_K^H\right\} \mathbf{p}_K + A^2 |\mathbf{p}_K^H \mathbf{p}'_K|^2 \mathbb{E}\left\{|\Delta p|^2\right\} \\ & - 2\Re[a\mathbf{p}_K^H \mathbf{p}'_K \mathbb{E}\left\{\Delta p \mathbf{e}_K^H\right\} \mathbf{p}_K])/k_p^2 \\ =& (\sigma_e^2 k_p^2 + A^2 |\mathbf{p}_K^H \mathbf{p}'_K|^2 \mathbb{E}\left\{|\Delta p|^2\right\} \\ & - 2\Re[a\mathbf{p}_K^H \mathbf{p}'_K \mathbb{E}\left\{\Delta p \mathbf{e}_K^H\right\} \mathbf{p}_K])/k_p^2. \end{aligned}$$

Finally, replacing Δp in (6) leads to the following expression:

$$\mathbb{E}\left\{|\Delta a|^2\right\} = \frac{\sigma_e^2}{k_p} + \frac{A^2 s_K^2}{r^2 k_p^2} \mathbb{E}\left\{|\Delta p|^2\right\} + \frac{2\sigma_e^2 s_K m_K r^{2K}}{r^2 k_v k_u k_p^2} \tag{10}$$

where

$$\begin{aligned} s_K &= \sum_{i=0}^{K-1} i r^{2i} \\ m_K &= \min(K, N-K, L, N-L). \end{aligned}$$

Now, replacing all the variables depending on K in (10) leads to the final expression (11) given at the bottom of the page. Here again, it is possible to express the variances of A and ϕ since $\mathrm{var}(\Delta A) = A^2 \mathrm{var}(\Delta\phi) = \frac{1}{2}\mathbb{E}\left\{|\Delta a|^2\right\}$. Note that this derivation may be extended to other subspace-based methods for which the expression of Δp in (9) in known. For instance, the expression of Δp for BLP and DDA may be found in [3].

4.1 The case of a pure sinusoid

In order to compare the variance (11) with those derived in the literature, let us consider the particular case of an undamped sinusoid. For $r = 1$, we have:

$$\mathbb{E}\left\{|\Delta a|^2\right\} = \frac{\sigma_e^2}{K} + \frac{1}{4} A^2 \mathbb{E}\left\{|\Delta p|^2\right\} (K-1)^2 + \frac{\sigma_e^2 m_K (K-1)}{L(N-L)K}.$$

The two first terms in the right-hand side (RHS) of this equation correspond to the results already presented in [4, 20] for the undamped case. The last term, corresponding to the correlation between $\mathbf{e}_K$ and Δp in (9), is neglected in [4, 20]. Now, this term is not always negligible; it depends on the values of L and K. Moreover, we can show that the global minimum of the variance of the amplitude can be achieved for $K_{opt} \approx 0.86N$ if the prediction order L is fixed to minimize the variance of the mode (i.e. $L = N/3$). Note that this result is more accurate than the one presented in [20] (i.e. $K_{opt} \approx \frac{2}{3}N$). The two minima of $\mathbb{E}\left\{|\Delta a|^2\right\}$ for $r = 1$ and $L = N/3$ are:

$$K_{opt} = \begin{cases} 0.53N, & \text{for } N/3 \leqslant K \leqslant 2N/3 \\ 0.86N, & \text{for } 2N/3 \leqslant K \leqslant N. \end{cases} \tag{12}$$

Here we assumed that $L \leqslant N/2$ thus $m_K = \min(L, K, N-K)$.

4.2 Approximation of K_{opt} for a damped sinusoid

The values of K_{opt} derived in the previous subsection are valid for an undamped sinusoid and stand approximately for a moderately damped one. For a general damped sinusoid, the optimal value of K should be derived from the expression in (11). But this is not a simple task due to its non-linear dependence upon K. Hence, for practical issues, we give here approximate values of K_{opt} under the following assumptions:

i) $L \in [N/3, 2N/3]$ and $K \geqslant L$,

ii) $r^{2L} \ll 1$.

Note that assuming $L \in [N/3, 2N/3]$ is not very restrictive since we have seen that the optimal value of L lies in this interval. Assumption (ii) implies that the number of samples is large enough so that the damped sinusoid vanishes sufficiently ($r^{2L} \ll 1 \Rightarrow r^{2N} \ll 1$). The combination of the two assumptions will lead us to a large sample approximation of K_{opt}. First, $\mathbb{E}\left\{|\Delta p|^2\right\}$ may be approximated by:

$$\mathbb{E}\left\{|\Delta p|^2\right\} \approx \frac{\sigma_e^2}{A^2} \cdot \begin{cases} 1/k_u^2 k_v, & L \leqslant \frac{N}{2} \\ 1/k_u k_v^2, & L \geqslant \frac{N}{2} \end{cases}$$

$$\mathbb{E}\left\{|\Delta a|^2\right\} = \frac{1-r^2}{1-r^{2K}}\sigma_e^2 + \frac{A^2\mathbb{E}\left\{|\Delta p|^2\right\}}{r^2}\left(\frac{r^2}{1-r^2} - \frac{Kr^{2K}}{1-r^{2K}}\right)^2 + \frac{2\sigma_e^2}{r^2 k_v k_u}\left(\frac{r^2}{1-r^2} - \frac{Kr^{2K}}{1-r^{2K}}\right)\frac{(1-r^2)m_K r^{2K}}{1-r^{2K}} \tag{11}$$

where $k_v \approx k_u \approx 1/(1-r^2)$, using assumption (ii). Then, deriving (11) with respect to K and using the aforementioned two assumptions leads to the following approximate optimal values of K:

$$K_{opt} \approx \begin{cases} \min(L, N-L) + \dfrac{0.5+r^2}{1-r^2}, & \text{if } m_K \in \{L, N-L\} \\ 0.5N + \dfrac{1+r^2}{4(1-r^2)}, & \text{if } m_K = N-K. \end{cases} \tag{13}$$

5. NUMERICAL SIMULATIONS

Consider a complex sinusoidal signal of amplitude $a = 1$ and angular frequency $\omega = \pi/2$, with $N = 30$ samples. The damping factor is set to two values: $\alpha = 0$ (pure sinusoid) and $\alpha = -0.1$. The noise variance is chosen so that the peak SNR is equal to 40 dB. The theoretical and experimental (for 1000 Monte Carlo runs) variances of the mode estimates versus the prediction order L are given in figure 2. We observe that the theoretical and simulated variances are very close. Moreover, the theoretical expression of L_{opt} in (8) is confirmed by simulations. Indeed, we can observe that $L_{opt} \in \{\frac{1}{3}N, \frac{2}{3}N\}$ for $\alpha = 0$ ($r = 1$) and $L_{opt} \in \{12, 18\}$ for $\alpha = -0.1$, which correspond to the values found with relation (8). At these points, the variance is close to the Cramér-Rao bound (CRB) [18].

Now, let us set L to its optimal value ($L = 10$ for $\alpha = 0$ and $L = 12$ for $\alpha = -0.1$). A plot of the amplitude variance versus K is given in figure 3. Here again, one can check the precision of the theoretical expression. In addition, we observe that the minimum variance is achieved in the undamped case for $K \approx 16$ and $K \approx 26$, which correspond to the values found using equation (12). For a damped signal, this minimum is reached for $K \approx 20$ which is near the value given by (13): $K_{opt} \approx 19.3$. The position of the second (local) minimum ($K_{opt} \approx 10$) does not satisfy assumption (i) in subsection 4.2, therefore it cannot be expressed by (13). Generally speaking, the plots of both simulation and theoretical variances show that a damped mode requires less equations than an undamped one to obtain an accurate estimate of the amplitude.

The last simulation is intended to assess the value of the SNR from which the derived expression of the amplitude variance is valid. This is an important point which depends not only on the actual value of the SNR but also on the damping factor α. Thus, the simulation presented here is performed with $\alpha = -0.1$. The parameters L and K are set to their optimal values: $L = 10$ and $K = 20$. The results achieved for different values of the SNR are given in figure 4. We observe that the theoretical expression of the mean-square error is valid beyond a threshold SNR, which is about 10 dB in our case. Of course, this is not a general result because the threshold SNR also depends on the damping factor. Hence, the threshold will be smaller for a pure sinusoid and larger for a strongly damped one.

6. CONCLUSION

We have presented a first-order perturbation analysis of a well-known subspace-based method in the estimation of a single damped sinusoidal signal. The analytical expressions obtained allow one to tune accurately the estimator in order to obtain a minimal variance on both the mode (frequency and damping factor) and the amplitude. These expressions have been confirmed by numerical simulations. The main conclusion stemming out from this work is the importance of the damping factor: the optimal design parameters (L and K), allowing to reach the minimum variance, strongly depend on it. In practice, a heuristic approach is to choose $L \in [N/3, N/2]$ and $K \in [0.53N, 0.86N]$, because the variances do not vary much in these intervals, both for a damped and an undamped sinusoid. This conclusion remains approximately valid for well separated multiple modes if the damping factors are almost equal.

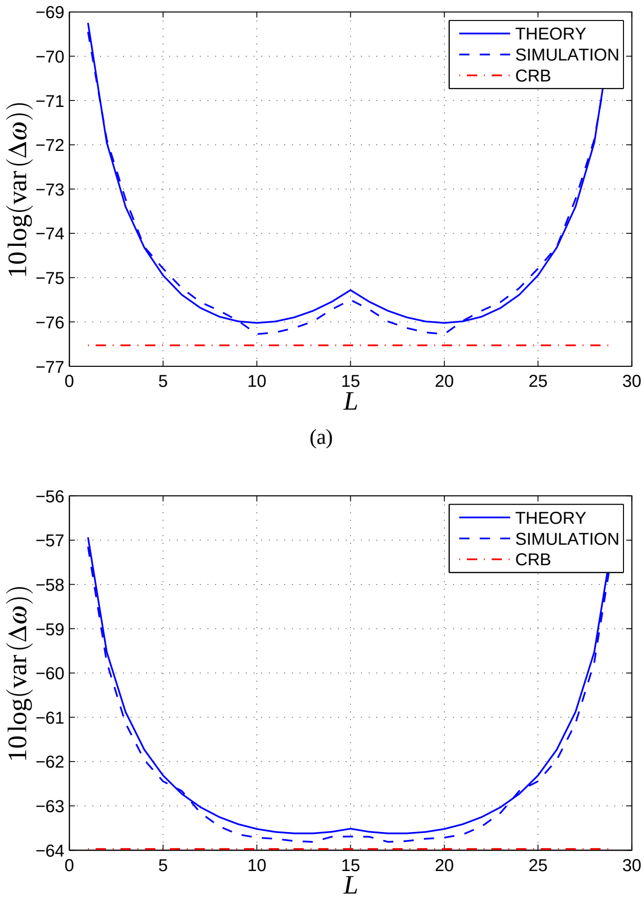

Figure 2: Comparison between theoretical and estimated variances of the pulsation for various values of L. (a) $\alpha = 0$; (b) $\alpha = -0.1$.

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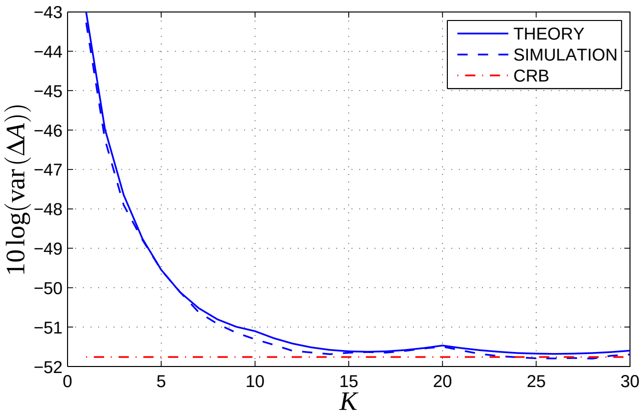
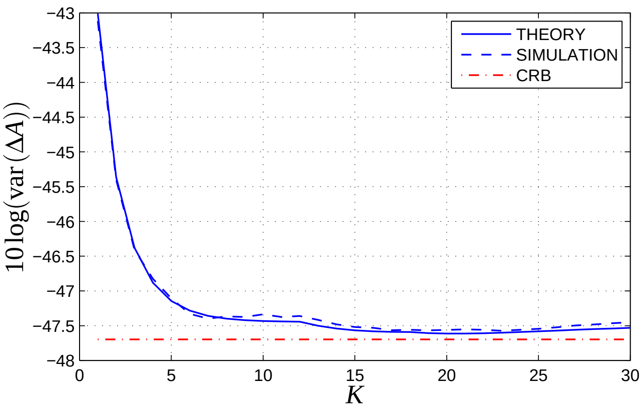


(a)



(b)

Figure 3: Comparison between theoretical and estimated variances of the amplitude for various values of K. (a) $\alpha = 0$; (b) $\alpha = -0.1$.

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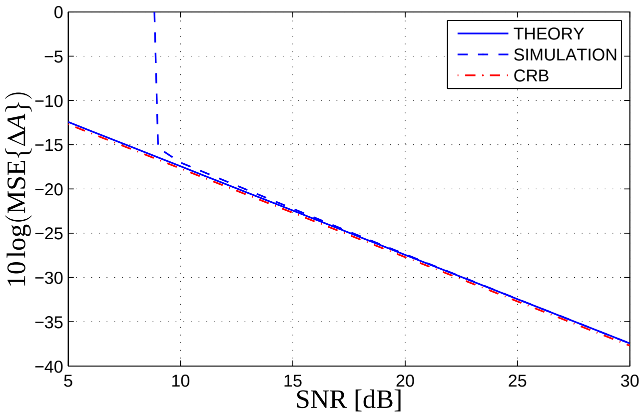

Figure 4: Theoretical and empirical amplitude mean-square errors (MSEs) versus the SNR for $\alpha = -0.1$

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