A LEAST SQUARES APPROACH TO THE DESIGN OF FREQUENCY INVARIANT BEAMFORMERS

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Abstract. A least squares formulation for the design of frequency invariant beamformers (FIBs) is proposed with two solutions provided. One is based on the Lagrange multipliers method and the other one is based on an orthogonal decomposition of the coefficient vector to transform the constrained design problem into an unconstrained one. Design examples including both broadside and off-broadside main beams are provided with satisfactory frequency invariant property and sidelobe attenuation.

Keywords. Least squares, Lagrange multipliers, orthogonal decomposition, Frequency invariant beamformer,

1. INTRODUCTION

Broadband beamforming has been studied extensively due to its wide applications to sonar, radar and communications [1]. Amongst them is a class of beamformers with frequency invariant responses [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], which can form beams pointing to the signal of interest with a constant beamwidth. In [5, 11, 12, 13, 14], the design was achieved based on a simple multi-dimensional inverse Fourier transform by exploiting the relationship between the array's spatial and temporal parameters and its beam pattern. More recently, a direct optimization approach was adopted using the convex optimization methods [15, 16].

The least squares approach is a conventional and well-known method for the design of both FIR filters and broadband beamformers [17, 18]. Compared with the convex optimization method, it can provide a closed-form solution to the problem and is more computationally efficient. Several broadband beamforming design methods based on the least squares formulation have been proposed in [18]. However those proposed methods are for the design of general broadband beamformers and not directly applicable to the design of FIBs. To solve the FIB design problem, we need to introduce a spatial variation formulation into the cost function as a frequency invariance controlling element. As a result, the unity response used as the desired response over the whole signal bandwidth at the look direction in [18] is replaced by constraining the response at a single reference frequency. Then a linearly constrained least squares formulation is derived and it can be solved in two different ways. One is to use the Lagrange multipliers method, which is a direct solution to the least squares problem with linear constraints. The other one aims to transform the constrained optimization problem into an unconstrained one by decomposing the coefficient vector into two orthogonal components.

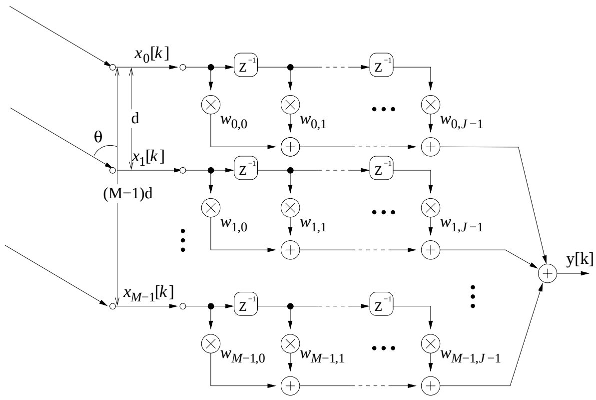

Figure 1: A signal impinges from an angle θ onto a uniformly spaced broadband linear array with M sensors, each followed by a J-tap filter.

This paper is organized as follows. A brief review of the general broadband beamforming structure with tapped delay-lines (TDLs) is provided in section 2 and the least squares formulation to the FIB design problem with two solutions is then proposed in Section 3. Design examples are provided in section 4 and conclusions drawn in section 5.

2. BROADBAND BEAMFORMING STRUCTURE WITH TDLS

A broadband beamforming structure based on a uniformly spaced linear array is shown in Fig. 1. Its response as a function of the signal frequency ω and arrival angle θ can be expressed as

$$\tilde{R}(\omega,\theta)=\sum_{m=0}^{M-1}\sum_{k=0}^{J-1} w_{m,k}e^{-jm\omega\Delta\tau}e^{-jk\omega T_s}\ , \quad (1)$$

where $\Delta\tau = \frac{d}{c}\cos\theta$, T_s is the delay between adjacent samples in the TDL, d is the array spacing, and c is the wave propagation speed.

With the normalized angular frequency $\Omega = \omega T_s$, we obtain the response as a function of Ω and θ

$$R(\Omega,\theta)=\sum_{m=0}^{M-1}\sum_{k=0}^{J-1} w_{m,k}e^{-jm\mu\Omega\cos\theta}e^{-jk\Omega} \text{ with } \mu=\frac{d}{cT_s} \quad (2)$$

We can rewrite the the response in a vector form

$$R(\Omega,\theta)=\mathbf{w}^T\mathbf{s}(\Omega,\theta) \quad (3)$$

where $\mathbf{w}$ is the coefficient vector defined as

$$\mathbf{w}=[w_{0,0},\cdots w_{M-1,0},\cdots,w_{0,J-1}\cdots w_{M-1,J-1}]^T, \quad (4)$$

and $\mathbf{s}(\Omega,\theta)$ is the $M\times J$ steering vector given by

$$\mathbf{s}(\Omega,\theta)=\mathbf{s}_{T_s}(\Omega)\otimes\mathbf{s}_{\Delta\tau}(\Omega,\theta) \quad (5)$$

where $\otimes$ represents the Kronecker product, and

$$\mathbf{s}_{T_S}(\Omega)=[1,e^{-j\Omega},\cdots,e^{-j(J-1)\Omega}]^T\ . \quad (6)$$

$$\mathbf{s}_{\Delta\tau}(\Omega,\theta)=[1,e^{-j\mu\Omega\cos\theta},\cdots,e^{-j(M-1)\mu\Omega\cos\theta}]^T \quad (7)$$

3. THE LEAST SQUARES APPROACH

The least squares cost function proposed in [18] is given as follows

$$J_{LS}=\int_{\Omega_i}\int_{\Theta}F(\Omega,\theta)\left|\mathbf{w}^T\mathbf{s}(\Omega,\theta)-D(\Omega,\theta)\right|^2 d\Omega d\theta \quad (8)$$

where Ω_i and Θ represent the frequency range of interest and the angle range, and $F(\Omega,\theta)$ is a positive real-valued weighting function and $D(\Omega,\theta)$ is a desired response function. In [18], it is focused on the specific design case with $F(\Omega,\theta)=1$ and $D(\Omega,\theta)=1$ in passband and $F(\Omega,\theta)=\alpha$ and $D(\Omega,\theta)=0$ over stopband.

Now let (Ω_n,θ_k) be the grid uniformly chosen from the continuous frequency and angle ranges. Then the formulation changes to

$$\begin{aligned} J_{LS_D} &= \sum_{\Omega_n\in\Omega_i}\sum_{\theta_k\in\Theta_m}\left|\mathbf{w}^T\mathbf{s}(\Omega_n,\theta_k)-1\right|^2 \\ &\quad+\alpha\sum_{\Omega_n\in\Omega_i}\sum_{\theta_k\in\Theta_s}\left|\mathbf{w}^T\mathbf{s}(\Omega_n,\theta_k)\right|^2 \end{aligned} \quad (9)$$

where Θ_s and Θ_m denote the sidelobe region and mainlobe region, respectively,

(9) can be rewritten as a quadratic function

$$J_{LS_D}=\mathbf{w}^T\mathbf{Q}_{LS}\mathbf{w}-2\mathbf{w}^T\mathbf{a}+d_{LS} \quad (10)$$

with

$$\mathbf{Q}_{LS}=\sum_{\Omega_n\in\Omega_i}\sum_{\theta_k\in\Theta_m}\mathbf{S}_R(\Omega_n,\theta_k)+\alpha\sum_{\Omega_n\in\Omega_i}\sum_{\theta_k\in\Theta_s}\mathbf{S}_R(\Omega_n,\theta_k), \quad (11)$$

$$\mathbf{a}=\sum_{\Omega_n\in\Omega_i}\sum_{\theta_k\in\Theta_m}\mathbf{s}_R(\Omega_n,\theta_k), \quad (12)$$

$$d_{LS}=\sum_{\Omega_n\in\Omega_i}\sum_{\theta_k\in\Theta_m}1, \quad (13)$$

where $\mathbf{S}_R(\Omega_n,\theta_k)$ is the real part of $\mathbf{S}(\Omega_n,\theta_k)=\mathbf{s}(\Omega_n,\theta_k)\mathbf{s}(\Omega_n,\theta_k)^H$ and $\mathbf{s}_R(\Omega_n,\theta_k)$ is the real part of $\mathbf{s}(\Omega_n,\theta_k)$.

The solution to the minimization of (10) is given by

$$\mathbf{w}_{LS}=\mathbf{Q}_{LS}^{-1}\mathbf{a} \quad (14)$$

In the above formulation, there is no constraint to guarantee a frequency invariance property. we need to introduce a frequency invariance controlling element into the design and it is denoted as SV (spatial variation), defined as

$$SV=\sum_{\Omega_n\in\Omega_i}\sum_{\theta_k\in\Theta_{FI}}\left|\mathbf{w}^T\mathbf{s}(\Omega_n,\theta_k)-\mathbf{w}^T\mathbf{s}(\Omega_r,\theta_k)\right|^2 \quad (15)$$

where Θ_{FI} represents the direction range in which frequency invariance is considered. It can be either the main beam direction area or the whole angle range, for example, from 0° to 180° for a linear array. Without loss of generality, here we will always consider the full angle range. Ω_r is a fixed reference frequency. The parameter SV is a measurement of the Euclidean distance between the response at the fixed reference frequency Ω_r and that at all the other operating frequencies over a range of directions in which frequency invariance is considered. When the beamformer has a frequency invariant response, the value of SV will be zero.

Since the frequency invariance property is expected to be held also in the sidelobe region, we only need to minimize the spectrum energy of the beamformer at the reference frequency Ω_r over the sidelobe region, which is given by

$$J_1=\sum_{\theta_k\in\Theta_s}\left|\mathbf{w}^T\mathbf{s}(\Omega_r,\theta_k)\right|^2 \quad (16)$$

Moreover, the unity response over the whole frequency band of interest in the look direction in the original formulation can be replaced by just constraining the response of the beamformer at the reference frequency in the look direction θ_r to be unity, which is given by

$$\mathbf{w}^T\mathbf{s}(\Omega_r, \theta_r) = 1 \tag{17}$$

Then, a constrained least squares formulation of the FIB design problem is obtained by combining (15), (16) and (17)

$$\begin{aligned} J_{CLS} = &\sum_{n=0}^{N-1}\sum_{k=0}^{K-1} \left|\mathbf{w}^T\mathbf{s}(\Omega_n, \theta_k) - \mathbf{w}^T\mathbf{s}(\Omega_r, \theta_k)\right|^2 \\ &+ \beta \sum_{\theta_k \in \Theta_s} \left|\mathbf{w}^T\mathbf{s}(\Omega_r, \theta_k)\right|^2 \\ &\text{subject to } \mathbf{w}^T\mathbf{s}(\Omega_r, \theta_r) = 1 \end{aligned} \tag{18}$$

where N and K are the number of samples chosen uniformly over the frequency and the angle ranges considered for frequency invariance, respectively, and β is a trade off parameter between the frequency invariance property and the sidelobe attenuation. We can rewrite (18) as

$$J_{CLS} = \mathbf{w}^T\mathbf{Q}_{CLS}\mathbf{w} \text{ subject to } \mathbf{s}(\Omega_r, \theta_r)^H\mathbf{w} = 1 \tag{19}$$

where

$$\begin{aligned} \mathbf{Q}_{CLS} = &\sum_{n=0}^{N-1}\sum_{k=0}^{K-1} \text{Re}\{(\mathbf{s}(\Omega_n, \theta_k) - \mathbf{s}(\Omega_r, \theta_k)) \\ &\qquad (\mathbf{s}(\Omega_n, \theta_k) - \mathbf{s}(\Omega_r, \theta_k))^H\} \\ &+ \beta \sum_{\theta_k \in \Theta_s} \mathbf{S}(\Omega_r, \theta_k)_R \end{aligned} \tag{20}$$

where Re$\{\bullet\}$ is the real part of its variable.

Note that $\mathbf{s}(\Omega_r, \theta_r)$ is complex-valued and we can change the single complex constraint into two real ones as follows

$$\mathbf{C}^T\mathbf{w} = \mathbf{f} \tag{21}$$

where $\mathbf{C} = [\mathbf{s}(\Omega_r, \theta_r)_R, \mathbf{s}(\Omega_r, \theta_r)_I]$ and $\mathbf{f} = [1, 0]^T$

Then we rewrite (19) as

$$J_{CLS} = \mathbf{w}^T\mathbf{Q}_{CLS}\mathbf{w} \text{ subject to } \mathbf{C}^T\mathbf{w} = \mathbf{f} \tag{22}$$

The problem in (22) can be solved by the Lagrange multipliers method directly and its solution is given by

$$\mathbf{w} = \mathbf{Q}_{CLS}^{-1}\mathbf{C}(\mathbf{C}^T\mathbf{Q}_{CLS}\mathbf{C})^{-1}\mathbf{f} \tag{23}$$

In addition, the constrained optimization problem in (22) can be transformed into an unconstrained one in a similar way as in the area of linear constrained minimum variance beamforming [19, 20]. The basic idea is to decompose the coefficient vector $\mathbf{w}$ into two orthogonal components $\mathbf{w}_q$ and $-\mathbf{v}$ as

$$\mathbf{w} = \mathbf{w}_q - \mathbf{v} \tag{24}$$

where $\mathbf{w}_q$ lies in the range of matrix $\mathbf{C}$ and $\mathbf{v}$ is in the null space of $\mathbf{C}$, i.e. the space of all $\mathbf{v}$ fulfilling $\mathbf{C}^T\mathbf{v} = 0$. Together the range and null space of a matrix span the entire space. So this decomposition can be used to represent any $\mathbf{w}$. To meet the constraint equation (21), we must have $\mathbf{C}^T\mathbf{w}_q = \mathbf{f}$, then we have

$$\mathbf{w}_q = (\mathbf{C}^T)^\dagger\mathbf{f} = \mathbf{C}(\mathbf{C}^T\mathbf{C})^{-1}\mathbf{f} \tag{25}$$

where $\{\cdot\}^\dagger$ is the pseudo-inverse operation.

The vector $\mathbf{v}$ can be expressed as a linear combination of the basis vectors of the null space of $\mathbf{C}$

$$\mathbf{v} = \mathbf{B}\mathbf{w}_a \tag{26}$$

where $\mathbf{B}$ satisfying $\mathbf{C}^T\mathbf{B} = 0$ can be obtained from $\mathbf{C}$ using orthogonalization methods, such as the singular value decomposition, and $\mathbf{w}_a$ is given by

$$\mathbf{w}_a = (\mathbf{B}^T\mathbf{Q}_{CLS}\mathbf{B})^{-1}\mathbf{B}^T\mathbf{Q}_{CLS}\mathbf{w}_q \,. \tag{27}$$

With $\mathbf{w}_a$, $\mathbf{w}_q$ and $\mathbf{B}$, we can obtain the final solution for $\mathbf{w}$.

4. DESIGN EXAMPLE

To show the effectiveness of the proposed methods, we give four design examples with either broadside or off-broadside main beams.

The dimension of the array is 14×16, i.e. 14 sensors and with each one followed by a 16-tap FIR filter. The array spacing is assumed to be half the wavelength corresponding to the maximum frequency π so that μ equals one. The frequency range of interest is set to be $[0.4\pi, 0.9\pi]$. The fixed reference frequency is $\Omega_r = 0.6\pi$ and $\beta = 0.05$.

Firstly we show examples with a broadside main beam, i.e. the look direction is $\theta = 90°$. The sidelobe region is $[0°, 75°] \cup [105°, 180°]$. The resultant beam pattern with the solution in (23) is shown in Fig. 2 and the one with the solution in (24) is shown in Fig. 3, both of which exhibit good frequency invariant properties over the frequency range $[0.4\pi, 0.9\pi]$ with a satisfactory sidelobe attenuation.

Now we change the look direction to $\theta = 60°$ and the sidelobe region to $[0°, 45°] \cup [75°, 180°]$. As seen from Fig. 4 and Fig. 5, the design results are satisfactory in terms of both frequency invariance and sidelobe attenuation.

5. CONCLUSION

A least squares formulation for the design of frequency invariant beamformers has been proposed with two solutions

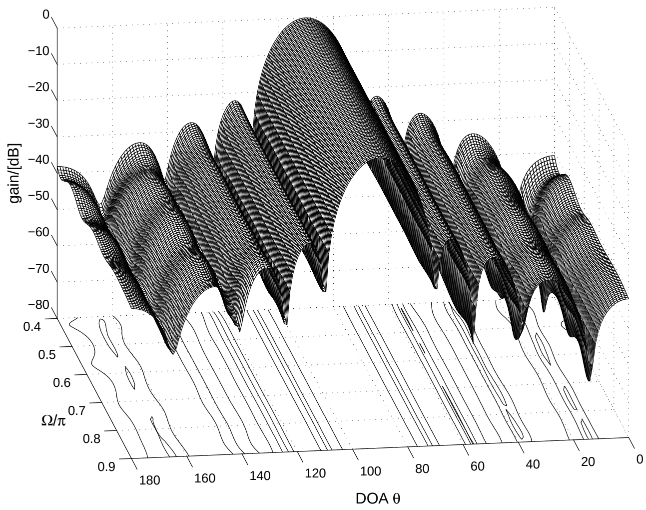

Figure 2: The designed beam pattern using the solution in (23) with its main beam at $\theta = 90^\circ$.

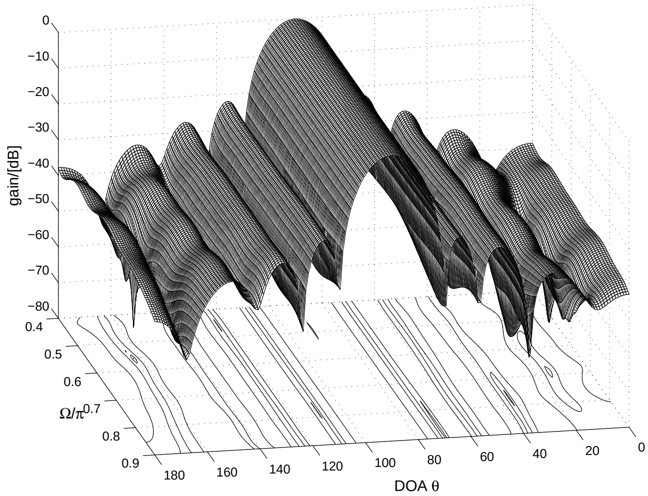

Figure 3: The designed beam pattern using the solution in (24) with its main beam at $\theta = 90^\circ$.

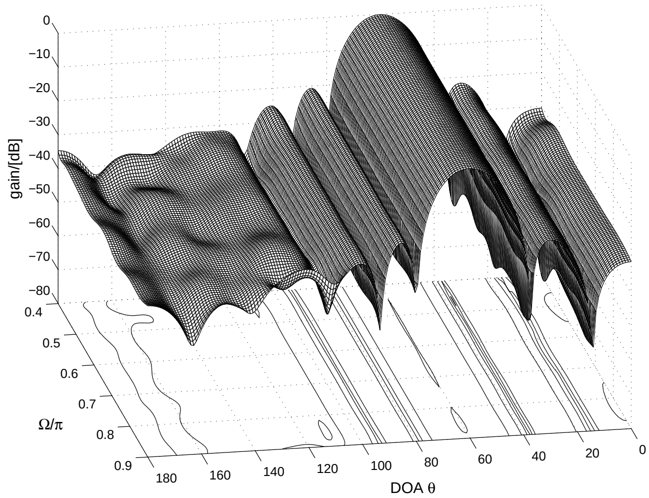

Figure 4: The designed beam pattern using the solution in (23) with its main beam at $\theta = 60^\circ$.

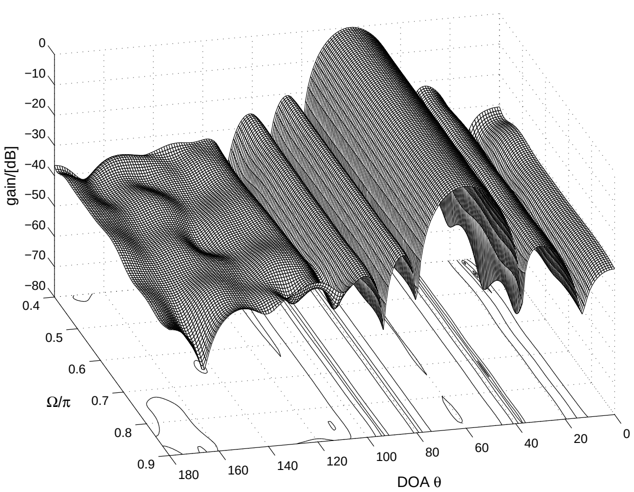

Figure 5: The designed beam pattern using the solution in (24) with its main beam at $\theta = 60^\circ$.

provided. One is based on the Lagrange multipliers method and the other one is based on an orthogonal decomposition of the coefficient vector to transform the constrained design problem into an unconstrained one. Examples including both broadside and off-broadside main beam designs were provided, clearly showing the effectiveness of the proposed approach.

6. REFERENCES

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