GENERALIZING THE JACKET TRANSFORM BY SUB ORTHOGONALITY EXTENSION

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ABSTRACT

The Jacket transform is a generalization of the Hadamard (Walsh) transform and useful in signal and image processing. In this paper, we will further generalize the Jacket transform defined in previous papers. We use the sub orthogonality property of the columns of the Walsh transform to define a more general form of the Jacket transform. For an N-point Jacket transform, there are N parameters that can be freely chosen. Therefore, it is possible to make the generalized Jacket transform have a certain form (such as the sinusoid-like form) while preserving the advantages of the original Walsh transform (reversibility, no multiplication, and the fast algorithm). As with the original Walsh and Jacket transforms, the proposed generalized Jacket transform will be helpful for CDMA and signal analysis.

1. INTRODUCTION

The Jacket transform is a generalization of the Walsh (Hadamard) transform. In [1], Lee found that the 4-point Walsh transform can be generalized as:

$$\mathbf{J}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 2 & -1 \\ 1 & 2 & -2 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \tag{1}$$

Then, in [2], the 4-point Walsh transform was further generalized into the following form:

$$\mathbf{J}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -w & w & -1 \\ 1 & w & -w & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \tag{2}$$

or more general,

$$\mathbf{J}_4 = \begin{bmatrix} a & b & b & a \\ b & -c & c & -b \\ b & c & -c & -b \\ a & -b & -b & a \end{bmatrix}. \tag{3}$$

They are called the **Jacket transform**. The values of w in (2) and a, b, and c can in (3) can be

$$2^k \quad \text{or} \quad j2^k. \tag{4}$$

These coefficients can also be chosen as the $2n^{\text{th}}$ root of unity [3]. The inverse of the 4-point Jacket transform is:

$$\mathbf{H}_4 = \frac{1}{4}\begin{bmatrix} 1/a & 1/b & 1/b & 1/a \\ 1/b & -1/c & 1/c & -1/b \\ 1/b & 1/c & -1/c & -1/b \\ 1/a & -1/b & -1/b & 1/a \end{bmatrix}, \quad \mathbf{H}_4\mathbf{J}_4 = \mathbf{I}, \tag{5}$$

and the 2N-point Jacket transform can be obtained by the Kronecker product of the N-point Jacket transform and the 2-point Hadamard matrix:

$$\mathbf{J}_{2\mathbf{N}} = \mathbf{J}_\mathbf{N} \otimes \mathbf{W}_2, \quad \mathbf{W}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{6}$$

where $\otimes$ means the Kronecker product.

Since the Jacket transform is more flexible than the Walsh transform and can preserve the advantages (such as the fast algorithm) of the original Walsh transform, it is suitable for signal processing. The Jacket transform can be used for image coding, ECG signal analysis, error control coding, CDMA, spread spectrum communication, and MIMO system analysis [1]-[6].

In this paper, we will **further generalize** the Jacket transform. To preserve the properties of the original Walsh transform, we want the derived generalized Jacket transform satisfies the following three constraints:

(1) **Bi-orthogonality**: If $\mathbf{J}_\mathbf{N}$ and $\mathbf{H}_\mathbf{N}$ are the transform matrices of the forward and inverse N-point Jacket transforms,

$$\mathbf{H}_\mathbf{N}\mathbf{J}_\mathbf{N} = \mathbf{I}. \tag{7}$$

(2) The entries of the forward and inverse transform matrices have the **reciprocal relation** [6]:

$$\mathbf{H}_\mathbf{N}[m,n] = \frac{1}{N\,\mathbf{J}_\mathbf{N}[n,m]}. \tag{8}$$

With the second constraint, we can assure that if the entries of the forward transform are powers of two, then those of the inverse transform are also powers of two.

(3) The **fast algorithm** of the original Walsh transform is **preserved**. (9)

If these three properties are satisfied, the advantages of the original Walsh transform can be preserved.

We find that, using the rule of **sub orthogonality extension**, the derived Jacket transform can satisfy all the three constraints described above. Moreover, the derived Jacket transform is more general than proposed in previous papers and is more suitable for CDMA and other signal processing applications.

2. SUB ORTHOGONAL EXTENSION

We have known that the columns of the Walsh transform are orthogonal to each other. In fact, they are not only orthogonal but also form a **sub orthogonal set**.

[Definition] Sub Orthogonality

Suppose that the two sequences $x[n]$ and $y[n]$ are orthogonal.

$$\sum_{n=0}^{N-1} x[n]y^*[n] = 0 . \tag{10}$$

If the index {0, 1, 2, …., N–1} are classified into two sets:

$\mathbf{N_1} = \{v_1, v_2, \ldots, v_k\}$, $\mathbf{N_2} = \{w_1, w_2, \ldots, w_{N-k}\}$,

$$\mathbf{N_1} \cap \mathbf{N_2} = \varnothing, \qquad \mathbf{N_1} \cup \mathbf{N_2} = \{0, 1, 2, \ldots, N-1\}, \tag{11}$$

and **x** and **y** are orthogonal with respect to the two index subsets:

$$\sum_{n\in\mathbf{V}} x[n]y^*[n] = \sum_{n\in\mathbf{W}} x[n]y^*[n] = 0 , \tag{12}$$

then we say that **x**[n] and **y**[n] are **sub orthogonal** with respect to the index subsets $\mathbf{N_1}$ and $\mathbf{N_2}$.

For example, for 8-point Walsh transform:

$$\mathbf{W_8} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} . \tag{13}$$

The 1st and the 2nd columns of $\mathbf{W_8}$ are

$\mathbf{e_0} = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]^T$ and $\mathbf{e_1} = [1\ {-1}\ 1\ {-1}\ 1\ {-1}\ 1\ {-1}]^T$. (14)

It is obviously that they are orthogonal. Furthermore, if we divide the index into

$$\mathbf{N_1} = \{0, 1, 2, 3\} \quad \text{and} \quad \mathbf{N_2} = \{4, 5, 6, 7\}, \tag{15}$$

then we find that $\mathbf{e_0}$ and $\mathbf{e_1}$ are **sub orthogonal** with respect to the two index subsets $\mathbf{N_1}$ and $\mathbf{N_2}$:

$$e_0[0]e_1[0] + e_0[1]e_1[1] + e_0[2]e_1[2] + e_0[3]e_1[3] = 0 ,$$

$$e_0[4]e_1[4] + e_0[5]e_1[5] + e_0[6]e_1[6] + e_0[7]e_1[7] = 0 . \tag{16}$$

Moreover, if we denote the k^{th} column of $\mathbf{W_8}$ by $\mathbf{e_{k-1}}$, then we find that $\mathbf{e_0}$ is also sub orthogonal to $\mathbf{e_2}$, $\mathbf{e_3}$, $\mathbf{e_5}$, $\mathbf{e_6}$, and $\mathbf{e_7}$ with respect to $\mathbf{N_1}$ and $\mathbf{N_2}$. In fact, we can classify the columns of the 8-point Walsh transform into two sets:

$$\{\mathbf{e_0}, \mathbf{e_4}\} \quad \text{and} \quad \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_5}, \mathbf{e_6}, \mathbf{e_7}\}. \tag{17}$$

Then the columns in the first set are **sub orthogonal** to the columns in the second set with respect to $\mathbf{N_1}$ and $\mathbf{N_2}$.

The concept of sub orthogonality is helpful for finding a more general form of the Jacket transform.

[Theorem 1] Sub Orthogonality Extension (I)

For an N-point Walsh transform matrix $\mathbf{W_N}$, we denote its columns by $\mathbf{e_k}[n]$ (k = 0, 1, ….. , N–1). If

$$\{\mathbf{e_p}[n] \mid p \in \mathbf{K_1}\} \quad \text{and} \quad \{\mathbf{e_q}[n] \mid q \in \mathbf{K_2}\} \tag{18}$$

are two subsets of $\{\mathbf{e_k}[n] \mid k = 0, 1, \ldots, N-1\}$ and

$$\mathbf{K_1} \cap \mathbf{K_2} = \varnothing, \qquad \mathbf{K_1} \cup \mathbf{K_2} = \{0, 1, 2, \ldots, N-1\}, \tag{19}$$

when the column in the first sets $\{\mathbf{e_p}[n] \mid p \in \mathbf{K_1}\}$ are **sub orthogonal** to the columns in the second set $\{\mathbf{e_q}[n] \mid q \in \mathbf{K_2}\}$ with respect to the index subsets $\mathbf{N_1}$ and $\mathbf{N_2}$ (see (11)), then we can generalize the Walsh transform into $\mathbf{J_N}$ where

$$\mathbf{J_N}[m,n] = a\mathbf{W_N}[m,n] \quad \text{when } m \in \mathbf{K_1} \text{ and } n \in \mathbf{N_1},$$

$$\mathbf{J_N}[m,n] = b\mathbf{W_N}[m,n] \quad \text{when } m \in \mathbf{K_1} \text{ and } n \in \mathbf{N_2},$$

$$\mathbf{J_N}[m,n] = c\mathbf{W_N}[m,n] \quad \text{when } m \in \mathbf{K_2} \text{ and } n \in \mathbf{N_1},$$

$$\mathbf{J_N}[m,n] = d\mathbf{W_N}[m,n] \quad \text{when } m \in \mathbf{K_2} \text{ and } n \in \mathbf{N_2}. \tag{20}$$

The inverse transform of $\mathbf{J_N}$ can be defined as

$$\mathbf{H_N}[m,n] = \frac{1}{N}\mathbf{J}_{\mathbf{N}}^{-1}[n,m] . \tag{21}$$

Then $\mathbf{J_N}$ satisfy the bi-orthogonality constraint in (7) and can be viewed as a more general form of Jacket transforms.

(Proof): If we define

$$\mathbf{H_N}\mathbf{J_N} = \mathbf{G_N} , \tag{22}$$

then from (21),

$$\mathbf{G_N}[m,n] = \sum_{l=0}^{N-1} \mathbf{H_N}[m,l]\mathbf{J_N}[l,n] = \frac{1}{N}\sum_{l=0}^{N-1} \frac{1}{\mathbf{J_N}[l,m]}\mathbf{J_N}[l,n] . \tag{23}$$

We divide the discussion into several cases. First, suppose that $m \in \mathbf{K_1}$ and $n \in \mathbf{K_2}$. Then from (20),

$$\mathbf{G_N}[m,n] = \frac{1}{N}\frac{b}{a}\sum_{l\in\mathbf{N_1}} \frac{1}{\mathbf{W_N}[l,m]}\mathbf{W_N}[l,n] + \frac{1}{N}\frac{d}{c}\sum_{l\in\mathbf{N_2}} \frac{1}{\mathbf{W_N}[l,m]}\mathbf{W_N}[l,n] . \tag{24}$$

Since for the Walsh transform, $\mathbf{W_N}[m, n] = 1/\mathbf{W_N}[m, n]$,

$$\mathbf{G_N}[m,n] = \frac{1}{N}\frac{b}{a}\sum_{l\in\mathbf{N_1}} \mathbf{W_N}[l,m]\mathbf{W_N}[l,n] + \frac{1}{N}\frac{d}{c}\sum_{l\in\mathbf{N_2}} \mathbf{W_N}[l,m]\mathbf{W_N}[l,n]$$

$$= \frac{1}{N}\frac{b}{a}\sum_{l\in\mathbf{N_1}} \mathbf{e_m}[l]\mathbf{e_n}[l] + \frac{1}{N}\frac{d}{c}\sum_{l\in\mathbf{N_2}} \mathbf{e_m}[l]\mathbf{e_n}[l] . \tag{25}$$

Then, since $m \in \mathbf{K_1}$ and $n \in \mathbf{K_2}$, $\mathbf{e_m}$ and $\mathbf{e_n}$ are sub orthogonal with respect to the index sets $\mathbf{N_1}$ and $\mathbf{N_2}$, therefore

$$\sum_{l\in\mathbf{N_1}} \mathbf{e_m}[l]\mathbf{e_n}[l] = \sum_{l\in\mathbf{N_2}} \mathbf{e_m}[l]\mathbf{e_n}[l] = 0 , \tag{26}$$

$$\mathbf{G_N}[m,n] = 0 \quad \text{when } m \in \mathbf{K_1} \text{ and } n \in \mathbf{K_2}. \tag{27}$$

For the case where $m \in \mathbf{K_1}$ and $n \in \mathbf{K_1}$, (25) is written as

$$\mathbf{G_N}[m,n] = \frac{1}{N}\frac{a}{a}\sum_{l\in\mathbf{N_1}} \mathbf{W_N}[l,m]\mathbf{W_N}[l,n] + \frac{1}{N}\frac{c}{c}\sum_{l\in\mathbf{N_2}} \mathbf{W_N}[l,m]\mathbf{W_N}[l,n]$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} \mathbf{W_N}[l,m]\mathbf{W_N}[l,n] = \frac{1}{N}\sum_{l=0}^{N-1} \mathbf{e_m}[l]\mathbf{e_n}[l] . \tag{28}$$

Since the Walsh transform is an orthogonal transform: $\sum_{l=0}^{N-1} \mathbf{e_m}[l]\mathbf{e_n}[l] = N\delta_{m,n}$, therefore,

$$\mathbf{G_N}[m,n] = \delta_{m,n} \quad \text{when } m \in \mathbf{K_1} \text{ and } n \in \mathbf{K_1}. \tag{29}$$

From the similar processes, we can also prove that

$$\mathbf{G_N}[m,n] = 0 \quad \text{when } m \in \mathbf{K_2} \text{ and } n \in \mathbf{K_1}, \tag{30}$$

$$\mathbf{G_N}[m,n] = \delta_{m,n} \quad \text{when } m \in \mathbf{K_2} \text{ and } n \in \mathbf{K_2}. \tag{31}$$

Combining (27), (29), (30), and (31), we can conclude that

$$\mathbf{G_N} = \mathbf{I} , \qquad \mathbf{H_N}\mathbf{J_N} = \mathbf{I} , \tag{32}$$

and the bi-orthogonality property is hence proved. #

For example, for the 4-point Walsh transform:

$$\mathbf{W_4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} , \tag{33}$$

if we denote the k^{th} column by $\mathbf{e}_{k-1}$ and classify the columns into two sets

$$\{\mathbf{e}_0, \mathbf{e}_3\} \quad \text{and} \quad \{\mathbf{e}_1, \mathbf{e}_2\} \tag{34}$$

then the columns in the first set are **sub orthogonal** to the columns in the second set with respect to the index sets

$$\mathbf{N}_1 = \{0, 3\} \quad \text{and} \quad \mathbf{N}_2 = \{1, 2\}. \tag{35}$$

Therefore, from Theorem 1, we can generalize the 4-point Walsh transform into:

$$\mathbf{J}_4 = \begin{bmatrix} a & b & b & a \\ c & -d & d & -c \\ c & d & -d & -c \\ a & -b & -b & a \end{bmatrix} \tag{36}$$

and its inverse is

$$\mathbf{H}_4 = \frac{1}{4}\begin{bmatrix} 1/a & 1/c & 1/c & 1/a \\ 1/b & -1/d & 1/d & -1/b \\ 1/b & 1/d & -1/d & -1/b \\ 1/a & -1/c & -1/c & 1/a \end{bmatrix}, \quad \mathbf{H}_4\mathbf{J}_4 = \mathbf{I}. \tag{37}$$

Compared with (3), the Jacket transform in (37) has four parameters a, b, c, and d, which is more general.

Theorem 1 is also helpful for defining the 2^k-point Jacket transform where $k > 2$. Instead of performing the Kronecker product with the 2-point Walsh transform as in (6), we can first search the sub orthogonal sets of the 2^k-point Walsh transform then applying (20) and (21).

For example, for the 8-point Walsh transform, in (17), we have known that $\{\mathbf{e}_0, \mathbf{e}_4\}$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}$ form a **sub orthogonal** set pair with respect to the index subsets $\mathbf{N}_1 = \{0, 1, 2, 3\}$ and $\mathbf{N}_2 = \{4, 5, 6, 7\}$. Therefore, we can construct the 8-point Jacket transform as

$$\mathbf{J}_8 = \left[\begin{array}{c|ccc|c|ccc} a & b & b & b & a & b & b & b \\ a & -b & b & -b & a & -b & b & -b \\ a & b & -b & -b & a & b & -b & -b \\ a & -b & -b & b & a & -b & -b & b \\ \hline c & d & d & d & -c & -d & -d & -d \\ c & -d & d & -d & -c & d & -d & d \\ c & d & -d & -d & -c & -d & d & d \\ c & -d & -d & d & -c & d & d & -d \end{array}\right] \tag{38}$$

and its inverse is

$$\mathbf{H}_8 = \frac{1}{8}\left[\begin{array}{cccc|cccc} 1/a & 1/a & 1/a & 1/a & 1/c & 1/c & 1/c & 1/c \\ \hline 1/b & -1/b & 1/b & -1/b & 1/d & -1/d & 1/d & -1/d \\ 1/b & 1/b & -1/b & -1/b & 1/d & 1/d & -1/d & -1/d \\ 1/b & -1/b & -1/b & 1/b & 1/d & -1/d & -1/d & 1/d \\ \hline 1/a & 1/a & 1/a & 1/a & -1/c & -1/c & -1/c & -1/c \\ \hline 1/b & -1/b & 1/b & -1/b & -1/d & 1/d & -1/d & 1/d \\ 1/b & 1/b & -1/b & -1/b & -1/d & -1/d & 1/d & 1/d \\ 1/b & -1/b & -1/b & 1/b & -1/d & 1/d & 1/d & -1/d \end{array}\right],$$

$$\mathbf{H}_8\mathbf{J}_8 = \mathbf{I} \tag{39}$$

(38) is an 8-point Jacket transform. Note that it cannot be expressed as the form of the Kronecker product of the 4-point Jacket transform and the 2-point Walsh transform as in (6). Therefore, Theorem 1 is a new way to define the 2^k-point Jacket transform where $k \geq 2$.

3. Multiple Sub Bi-Orthogonal Extensions

In fact, Theorem 1 can be further generalized.

Note that, in (17), we divide the columns of $\mathbf{W}_8$ into two sets. However, in the second set, $\mathbf{e}_1$ and $\mathbf{e}_5$ are also sub orthogonal to $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_6$, and $\mathbf{e}_7$. It means that, instead of (18), we can further decompose the columns of the Walsh transform **more than two sub orthogonal sets**. Moreover, in Theorem 1, we stated that the columns of the Walsh transform are sub orthogonal with respect to two index subsets $\mathbf{N}_1$ and $\mathbf{N}_2$. In fact, the columns can be sub orthogonal with respect to **more than two index subsets**. For example, for the 8-point Walsh transform in (13) and (14), since

$$e_0[0]e_1[0] + e_0[1]e_1[1] = 0, \quad e_0[2]e_1[2] + e_0[3]e_1[3] = 0,$$
$$e_0[4]e_1[4] + e_0[5]e_1[5] = 0, \quad e_0[6]e_1[6] + e_0[7]e_1[7] = 0, \tag{40}$$

thus, $\mathbf{e}_0$ and $\mathbf{e}_1$ are also orthogonal with respect to the index subsets of {0, 1}, {2, 3}, {4, 5}, and {6, 7}. These are helpful for defining a more general form of the Jacket transform

[Theorem 2] Sub Orthogonality Extension (II)

For an N-point Walsh transform matrix $\mathbf{W}_N$, we denote its column by $\mathbf{e}_k[n]$ ($k = 0, 1, \ldots, N-1$, $n = 0, 1, \ldots, N-1$). Suppose that the columns are classified into S subsets:

$$\{\mathbf{e}_{p_1} \mid p_1 \in \mathbf{K}_1\}, \quad \{\mathbf{e}_{p_2} \mid p_2 \in \mathbf{K}_2\}, \quad \ldots\ldots, \quad \{\mathbf{e}_{p_S} \mid p_S \in \mathbf{K}_S\} \tag{41}$$

$\mathbf{K}_x \cap \mathbf{K}_y = \varnothing$ if $x \neq y$,

$$\mathbf{K}_1 \cup \mathbf{K}_2 \cup \ldots\ldots \cup \mathbf{K}_S = \{0, 1, 2, \ldots, N-1\}, \tag{42}$$

and the index n ($n = 0, 1, 2, \ldots, N-1$) are also classified into T subsets $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \ldots, \mathbf{N}_T$,

$\mathbf{N}_x \cap \mathbf{N}_y = \varnothing$ if $x \neq y$,

$$\mathbf{N}_1 \cup \mathbf{N}_2 \cup \ldots\ldots \cup \mathbf{N}_S = \{0, 1, 2, \ldots, N-1\}. \tag{43}$$

When the columns belonging to different column sets are **sub orthogonal** with respect to the index subsets $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$, ….., $\mathbf{N}_T$,

$$\sum_{n \in \mathbf{N}_1} e_{p_x}[n] e_{p_y}[n] = \sum_{n \in \mathbf{N}_2} e_{p_x}[n] e_{p_y}[n] = \cdots\cdots = \sum_{n \in \mathbf{N}_S} e_{p_x}[n] e_{p_y}[n] = 0$$

$$\text{for} \quad p_x \in \mathbf{K}_x, p_y \in \mathbf{K}_y, \quad \text{and } x \neq y, \tag{44}$$

then we can generalize the N-point Walsh transform matrix $\mathbf{W}_N$ into $\mathbf{J}_N$ where

$$\mathbf{J}_N[m,n] = a_{s,t}\mathbf{W}_N[m,n] \quad \text{when } m \in \mathbf{K}_s \text{ and } n \in \mathbf{N}_t. \tag{45}$$

The inverse of $\mathbf{J}_N$ is $\mathbf{H}_N$ where

$$\mathbf{H}_N[m,n] = \frac{1}{N}\mathbf{J}_N^{-1}[n,m]. \tag{46}$$

Then, using the process similar to (22)-(32), we can prove that $\mathbf{J}_N$ and $\mathbf{H}_N$ are bi-orthogonal:

$$\mathbf{H}_N\mathbf{J}_N = \mathbf{I}. \tag{47}$$

The Jacket transform generated from (45) will be even more general than that generated from Theorem 1.

For example, for the 8-point Walsh transform in (13), if we use $\mathbf{e}_{k-1}$ to denote the k^{th} column of $\mathbf{W}_8$, then we find that the columns can be divided into four sets:

$$\{\mathbf{e}_0, \mathbf{e}_4\}, \quad \{\mathbf{e}_1, \mathbf{e}_5\}, \quad \{\mathbf{e}_2, \mathbf{e}_6\}, \quad \{\mathbf{e}_3, \mathbf{e}_7\}. \tag{48}$$

The columns vector in one set is sub orthogonal to the vectors in other sets with respect to the index set $\mathbf{N}_1 = \{0, 1, 2, 3\}$ and $\mathbf{N}_2 = \{4, 5, 6, 7\}$, therefore, from (47), we can define the 8-point Jacket transform as:

$$\mathbf{J_8} = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} & a_{11} & a_{21} & a_{31} & a_{41} \\ a_{11} & -a_{21} & a_{31} & -a_{41} & a_{11} & -a_{21} & a_{31} & -a_{41} \\ a_{11} & a_{21} & -a_{31} & -a_{41} & a_{11} & a_{21} & -a_{31} & -a_{41} \\ a_{11} & -a_{21} & -a_{31} & a_{41} & a_{11} & -a_{21} & -a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} & -a_{12} & -a_{22} & -a_{32} & -a_{42} \\ a_{12} & -a_{22} & a_{32} & -a_{42} & -a_{12} & a_{22} & -a_{32} & a_{42} \\ a_{12} & a_{22} & -a_{32} & -a_{42} & -a_{12} & -a_{22} & a_{32} & a_{42} \\ a_{12} & -a_{22} & -a_{32} & a_{42} & -a_{12} & a_{22} & a_{32} & -a_{42} \end{bmatrix}, \quad (49)$$

and its inverse is:

$$\mathbf{H_8} = \frac{1}{8}\begin{bmatrix} 1/a_{11} & 1/a_{11} & 1/a_{11} & 1/a_{11} & 1/a_{12} & 1/a_{12} & 1/a_{12} & 1/a_{12} \\ 1/a_{21} & -1/a_{21} & 1/a_{21} & -1/a_{21} & 1/a_{22} & -1/a_{22} & 1/a_{22} & -1/a_{22} \\ 1/a_{31} & 1/a_{31} & -1/a_{31} & -1/a_{31} & 1/a_{32} & 1/a_{32} & -1/a_{32} & -1/a_{32} \\ 1/a_{41} & -1/a_{41} & -1/a_{41} & 1/a_{41} & 1/a_{42} & -1/a_{42} & -1/a_{42} & 1/a_{42} \\ 1/a_{11} & 1/a_{11} & 1/a_{11} & 1/a_{11} & -1/a_{12} & -1/a_{12} & -1/a_{12} & -1/a_{12} \\ 1/a_{21} & -1/a_{21} & 1/a_{21} & -1/a_{21} & -1/a_{22} & 1/a_{22} & -1/a_{22} & 1/a_{22} \\ 1/a_{31} & 1/a_{31} & -1/a_{31} & -1/a_{31} & -1/a_{32} & -1/a_{32} & 1/a_{32} & 1/a_{32} \\ 1/a_{41} & -1/a_{41} & -1/a_{41} & 1/a_{41} & -1/a_{42} & 1/a_{42} & 1/a_{42} & -1/a_{42} \end{bmatrix}$$

$$\mathbf{H_8 J_8} = \mathbf{I} \quad (50)$$

The 8-point Jacket transform defined in (49) has 8 parameters that are free to choose and is even more general than the 8-point Jacket transform in (38).

In addition to (49), there are many possible ways to define the 8-pt Jacket transform. For example, we can choose

$$\mathbf{N_1} = \{0, 3, 5, 6\} \text{ and } \mathbf{N_2} = \{1, 2, 4, 7\}. \quad (51)$$

Then the columns of the 8-pt Walsh transform can be classified into the following 4 sets and the columns belonging to different sets are sub orthogonal with respect to $\mathbf{N_1}$, $\mathbf{N_2}$:

$$\{\mathbf{e_0}, \mathbf{e_7}\}, \quad \{\mathbf{e_1}, \mathbf{e_6}\}, \quad \{\mathbf{e_2}, \mathbf{e_5}\}, \quad \{\mathbf{e_3}, \mathbf{e_4}\}. \quad (52)$$

From (45), we can construct the 8-pt Jacket transform as

$$\mathbf{J_8} = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} & a_{41} & a_{31} & a_{21} & a_{11} \\ a_{12} & -a_{22} & a_{32} & -a_{42} & a_{42} & -a_{32} & a_{22} & -a_{12} \\ a_{12} & a_{22} & -a_{32} & -a_{42} & a_{42} & a_{32} & -a_{22} & -a_{12} \\ a_{11} & -a_{21} & -a_{31} & a_{41} & a_{41} & -a_{31} & -a_{21} & a_{11} \\ a_{12} & a_{22} & a_{32} & a_{42} & -a_{42} & -a_{32} & -a_{22} & -a_{12} \\ a_{11} & -a_{21} & a_{31} & -a_{41} & -a_{41} & a_{31} & -a_{21} & a_{11} \\ a_{11} & a_{21} & -a_{31} & -a_{41} & -a_{41} & -a_{31} & a_{21} & a_{11} \\ a_{12} & -a_{22} & -a_{32} & a_{42} & -a_{42} & a_{32} & a_{22} & -a_{12} \end{bmatrix} \quad (53)$$

We can also choose the following four index subsets:

$$\mathbf{N_1} = \{0, 1\}, \quad \mathbf{N_2} = \{2, 3\}, \quad \mathbf{N_3} = \{4, 5\}, \quad \mathbf{N_4} = \{6, 7\}. \quad (54)$$

Then, if we classify the columns into the following two sets:

$$\{\mathbf{e_0}, \mathbf{e_2}, \mathbf{e_4}, \mathbf{e_6}\}, \quad \{\mathbf{e_1}, \mathbf{e_3}, \mathbf{e_5}, \mathbf{e_7}\}, \quad (55)$$

the columns belonging to different sets are sub orthogonal with respect to $\mathbf{N_1}$, $\mathbf{N_2}$, $\mathbf{N_3}$, and $\mathbf{N_4}$. Thus, from Theorem 2, we can define the 8-point Jacket transforms as:

$$\mathbf{J_8} = \begin{bmatrix} a_{11} & a_{21} & a_{11} & a_{21} & a_{11} & a_{21} & a_{11} & a_{21} \\ a_{11} & -a_{21} & a_{11} & -a_{21} & a_{11} & -a_{21} & a_{11} & -a_{21} \\ a_{12} & a_{22} & -a_{12} & -a_{22} & a_{12} & a_{22} & -a_{12} & -a_{22} \\ a_{12} & -a_{22} & -a_{12} & a_{22} & a_{12} & -a_{22} & -a_{12} & a_{22} \\ a_{13} & a_{23} & a_{13} & a_{23} & -a_{13} & -a_{23} & -a_{13} & -a_{23} \\ a_{13} & -a_{23} & a_{13} & -a_{23} & -a_{13} & a_{23} & -a_{13} & a_{23} \\ a_{14} & a_{24} & -a_{14} & -a_{24} & -a_{14} & -a_{24} & a_{14} & a_{24} \\ a_{14} & -a_{24} & -a_{14} & a_{24} & -a_{14} & a_{24} & a_{14} & -a_{24} \end{bmatrix}, \quad (56)$$

Therefore, from Theorem 2, there are many ways to define the N-point Jacket transform. In fact, when $N = 4$, there are 5 possible ways to choose the index subsets:

(1) $\mathbf{N_1} = \{0, 1, 2, 3\}$, (2) $\mathbf{N_1} = \{0, 1\}$, $\mathbf{N_2} = \{2, 3\}$,
(3) $\mathbf{N_1} = \{0, 2\}$, $\mathbf{N_2} = \{1, 3\}$, (4) $\mathbf{N_1} = \{0, 3\}$, $\mathbf{N_2} = \{1, 2\}$,
(5) $\mathbf{N_1} = \{0\}$, $\mathbf{N_2} = \{1\}$, $\mathbf{N_3} = \{2\}$, $\mathbf{N_4} = \{3\}$. (57)

Therefore, there are **5 possible ways** to define the 4-point Jacket transform from Theorem 2.

When $N = 8$, there are 16 ways to choose the index subsets. Thus, we have **16 possible ways** to define the 8-point Jacket transform from Theorem 2. In the following, we just list the index subset $\mathbf{N_1}$ for each case. Since if $\mathbf{N_1}$ is determined, other index subsets can also be determined.

(1) $\mathbf{N_1} = \{0, 1, 2, 3, 4, 5, 6, 7\}$,
(2) $\mathbf{N_1}=\{0, 1, 2, 3\}$, (3) $\mathbf{N_1}=\{0, 1, 6, 7\}$, (4) $\mathbf{N_1}=\{0, 1, 4, 5\}$,
(5) $\mathbf{N_1}=\{0, 3, 4, 7\}$, (6) $\mathbf{N_1}=\{0, 3, 5, 6\}$, (7) $\mathbf{N_1}=\{0, 2, 5, 7\}$,
(8) $\mathbf{N_1} = \{0, 2, 4, 6\}$, (9) $\mathbf{N_1} = \{0, 1\}$, (10) $\mathbf{N_1} = \{0, 2\}$,
(11) $\mathbf{N_1} = \{0, 3\}$, (12) $\mathbf{N_1} = \{0, 4\}$, (13) $\mathbf{N_1} = \{0, 5\}$,
(14) $\mathbf{N_1} = \{0, 6\}$, (15) $\mathbf{N_1} = \{0, 7\}$, (16) $\mathbf{N_1} = \{0\}$. (58)

Moreover, there are **67 possible ways** to define the 16-point Jacket transform and **374 possible ways** to define the 32-point Jacket transform.

[Theorem 3] The N-point Jacket transform generated from Theorem 2 has N parameters that are free to choose and each parameter affects the values of N entries.

For example, for the 8-point Jacket transforms in (49), (53), and (56), there are 8 parameters that are free to choose.

Since Theorem 2 makes the generation of the Jacket transform much more flexible, we can use it to define the desired form of the Jacket transform. For example, in (53), we can choose $a_{11} = a_{32} = a_{42} = 1$, $a_{21} = a_{22} = 2$, and $a_{31} = a_{41} = a_{12} = 4$. Then, (53) becomes

$$\mathbf{J_8} = \begin{bmatrix} 1 & 2 & 4 & 4 & 4 & 4 & 2 & 1 \\ 4 & -2 & 1 & -1 & 1 & -1 & 2 & -4 \\ 4 & 2 & -1 & -1 & 1 & 1 & -2 & -4 \\ 1 & -2 & -4 & 4 & 4 & -4 & -2 & 1 \\ 4 & 2 & 1 & 1 & -1 & -1 & -2 & -4 \\ 1 & -2 & 4 & -4 & -4 & 4 & -2 & 1 \\ 1 & 2 & -4 & -4 & -4 & -4 & 2 & 1 \\ 4 & -2 & -1 & 1 & -1 & 1 & 2 & -4 \end{bmatrix}. \quad (59)$$

We plot the former four rows of the 8-pt Jacket transform in (61) in Fig.1. Compared with the rows of the 8-point Walsh transform, the Jacket transform in (59) is more sinusoid-like.

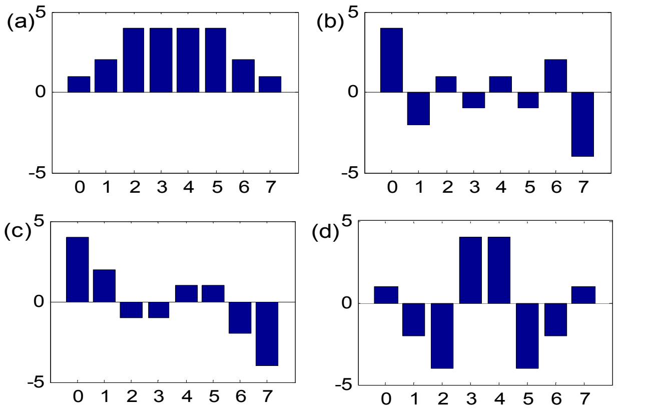

Fig. 1 The former four rows of the 8-point Jacket transform in (59).

4. Fast Algorithm

Then we discuss the third requirement described in Section 1, i.e., whether the derived Jacket transform has the fast algorithm. In fact, **the Jacket transform defined from Theorem 2 always has a fast algorithm for implementation**. The fast algorithm can be designed as follows.

(Step 1) The order of inputs is determined by the column subsets in (41). For example, for the 8-point Jacket transform in (49), the columns are divided into four sets: {$\mathbf{e}_0$, $\mathbf{e}_4$}, {$\mathbf{e}_1$, $\mathbf{e}_5$}, {$\mathbf{e}_2$, $\mathbf{e}_6$}, and {$\mathbf{e}_3$, $\mathbf{e}_7$}. Thus, in Fig. 2, {$x[0]$, $x[4]$}, {$x[1]$, $x[5]$}, {$x[2]$, $x[6]$}, and {$x[3]$, $x[7]$} are placed together.

(Step 2) If there are M columns for each column subset, then we should perform an M-point Walsh transform for the inputs that correspond to the same column subset. For the example in (49), we should perform 2-point Walsh transforms for {$x[0]$, $x[4]$}, {$x[1]$, $x[5]$}, {$x[2]$, $x[6]$}, and {$x[3]$, $x[7]$}, as the 1st stage in Fig. 2.

(Step 3) Then, we multiply proper coefficients for the output of Step 2, as the output of the 1st stage in Fig. 2.

(Step 4) Then for we perform the (N/M)-point Walsh transform for the

$$(k+gM)^{\text{th}} \text{ outputs} \qquad (g = 1, 2, \ldots, N/M) \qquad (60)$$

of Step 3 and for each of the (N/M)-point Walsh transform k is fixed. See the example of the 2nd and the 3rd stage of Fig. 2, which perform the 4-points Walsh transform for the even outputs and the odd outputs of Step 3.

(Step 5) The order of the outputs is determined by the index subsets in (43). If there are N/M indexes in a index subset, then the outputs correspond to the k^{th} index subset should be placed on the $(k+gM)^{\text{th}}$ locations of the outputs. For the example in (49), the index subsets are $\mathbf{N}_1 = \{0, 1, 2, 3\}$ and $\mathbf{N}_2 = \{4, 5, 6, 7\}$. Thus, in Fig. 2, {$X[0]$, $X[1]$, $X[2]$, $X[3]$} should be placed on the 1st, 3rd, 5th, and 7th locations of the outputs and {$X[4]$, $X[5]$, $X[6]$, $X[7]$} should be placed on the 2nd, 4th, 6th, and 8th locations of the outputs.

(Step 6) The inverse of the Jacket transform can be implemented by just reversing the direction of the forward Jacket transform, as Fig. 3.

From these six steps, the fast algorithms of the generalized Jacket transform derived in Theorem 2 can be constructed easily. In Fig, 4, we give another example to show the fast algorithm of the 8-point Jacket transform in (53).

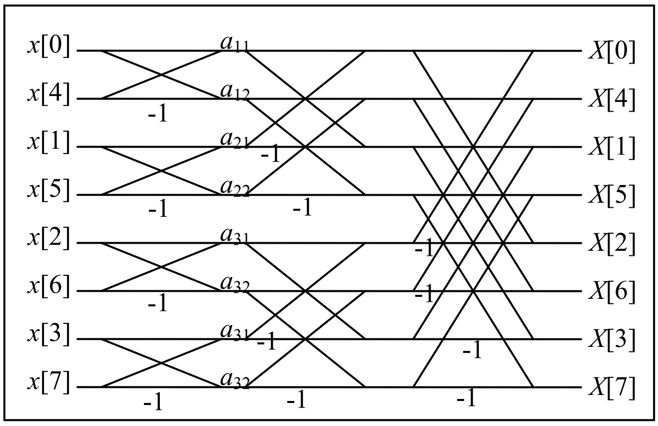

Fig. 2 The fast algorithm of the 8-point Jacket transform in (49).

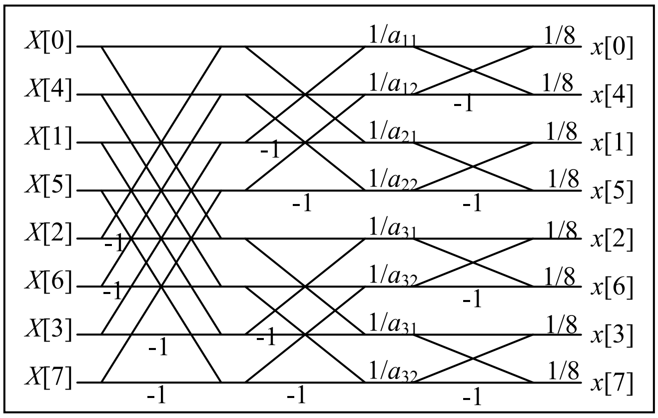

Fig.3 Fast algorithm of the inverse 8-point Jacket transform in (50).

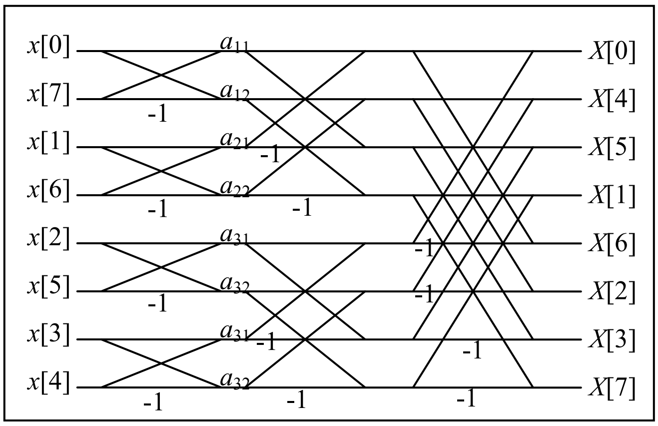

Fig. 4 Fast algorithm of the 8-point Jacket transform in (53).

5. Conclusions

In this paper, we use the sub orthogonality property of the original Walsh transform to define a more general form of the Jacket transform. The derived Jacket transform preserves the orthogonality property, no multiplication requirement, and the fast algorithm of the original Walsh transform. Since the proposed Jacket transform has a variety of prototypes and many parameters are free to choose, it will be more flexible and useful in signal processing applications.

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