ALIASING EFFECTS IN SAMPLING SPECTRALLY CORRELATED PROCESSES

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ABSTRACT

In this paper, the problem of sampling continuous-time spectrally correlated (SC) processes is addressed. SC processes have Loève bifrequency spectrum with spectral masses concentrated on a countable set of support curves. This class of nonstationary processes extends that of the almost-cyclostationary processes and occurs in wide-band mobile communications. The class of the discrete-time SC processes is introduced and characterized. It is shown that such processes can be obtained by uniformly sampling the continuous-time SC processes. Sampling theorems are presented and a sufficient condition to avoid aliasing in the whole bifrequency domain is provided.

1. INTRODUCTION

For wide-sense stationary processes no correlation exists between spectral components at distinct frequencies. Thus, they have Loève bifrequency spectrum [11] with support contained in the main diagonal of the bifrequency plane. The density of the Loève bifrequency spectrum on such diagonal is coincident with the power spectrum. The presence of spectral correlation between spectral component at distinct frequencies is an indicator of the nonstationarity of the process. In particular, when correlation exists only between spectral components separated by quantities belonging to a countable set of values (called cycle frequencies), the process is said to be almost-cyclostationary (ACS) or almost-periodically correlated. In such a case the Loève bifrequency spectrum has support contained in lines parallel to the main diagonal and the autocorrelation function is an almost-periodic function of time whose (generalized) Fourier series expansion has frequencies coincident with the cycle frequencies [6]. ACS processes occur in many fields of application. In particular, almost all modulated signals adopted in communications, radar, and telemetry can be modeled as ACS [6].

A new class of nonstationary stochastic processes, the spectrally correlated (SC) processes, has been introduced and characterized in [14]. SC processes exhibit Loève bifrequency spectrum with spectral masses concentrated on a countable set of support curves in the bifrequency plane. Thus, ACS processes are obtained as a special case of SC processes when the support curves are lines with unit slope. In communications, SC processes are obtained when ACS processes pass through Doppler channels that operate frequency warping on the input signal [4]. For example, let us consider the case of relative motion between transmitter and receiver in the presence of moving reflecting objects. If the involved relative radial speeds are constant and the product signal-bandwidth times data-record length is not much smaller than the ratios between the medium propagation speed and the radial speeds, the resulting multipath Doppler channel introduces a different complex gain, time-delay, frequency shift, and nonunit time-scale factor for each path [14]. Since the time-scale factors are non unit, when an ACS process passes through such a channel, the output signal is SC. Further situations where nonunit time-scale factors should be accounted for can be encountered in radar and sonar applications [22, pp. 339-340], communications with wide-band and ultra wide-band (UWB) signals [8], [20], and space communications [18]. In all these cases, SC processes are appropriate models for the involved signals [14]. Finally, in [19] it is shown that fractional Brownian motion (fBm) processes have Loève bifrequency spectrum with spectral masses concentrated on three lines of the bifrequency plane.

In [14], continuous-time SC processes are introduced and characterized and the problem of the spectral correlation density estimation for SC processes is addressed in the case of unknown support curves. In [15], the case of known support curves is treated. The special case of support lines is addressed in [10].

In this paper, the property of strict band-limitedness is analyzed for continuous-time SC processes. Then, discrete-time SC processes are introduced and characterized. It is shown that an SC discrete-time process can be obtained by uniformly sampling a continuous-time SC process and its Loève bifrequency spectrum is constituted by the superposition of replicas of the Loève bifrequency spectrum of the continuous-time SC process. It is shown that for strictly band-limited SC processes a sufficient condition to avoid non overlapping replicas in the Loève bifrequency spectrum is that the sampling frequency f_s is at least two times the process bandwidth which is the classical Shannon condition. However, unlike the case of the wide-sense stationary processes, this condition is not sufficient to assure that the mappings $\nu_i = f_i/f_s$, $i = 1,2$, between the frequencies $f_i \in [-f_s/2, f_s/2]$ of the Loève bifrequency spectrum of the continuous-time process and the frequencies ν_i of the Loève bifrequency spectrum of the discrete-time process hold for $\nu_i \in [-1/2, 1/2]$, for every support curve. A sufficient condition is derived in the paper and known results on the sampling frequency for ACS processes [6] are obtained as special cases.

2. CONTINUOUS-TIME SPECTRALLY CORRELATED PROCESSES

Definition 2.1 Let $x_a(t)$ be a continuous-time complex-valued second-order harmonizable stochastic process. Its *Loève bifrequency spectrum* [11], also called dual-frequency

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spectrum [19], is defined as

$$\mathcal{S}_{\boldsymbol{x}_a}(f_1,f_2) \triangleq \mathrm{E}\{X_a(f_1)\,X_a^*(f_2)\} \tag{1}$$

where

$$X_a(f) \triangleq \int_{\mathbb{R}} x_a(t)\,e^{-j2\pi ft}\,\mathrm{d}t \tag{2}$$

is the Fourier transform of $x_a(t)$ and is assumed to exist (at least) in the sense of distributions [24] for almost all sample paths of $x_a(t)$. See [14] for a link of this representation with the Cramer representation of stochastic processes in the spectral domain. In (1), superscript $*$ denotes complex conjugation and subscript $\boldsymbol{x}_a \triangleq [x_a x_a^*]$. □

For complex-valued processes, also the second-order spectral moment $\mathrm{E}\{X_a(f_1)\,X_a(f_2)\}$ must be considered for a complete second-order characterization [21]. Results for $\mathrm{E}\{X_a(f_1)\,X_a(f_2)\}$ will not be considered here.

Definition 2.2 Let $x_a(t)$ be a continuous-time complex-valued second-order harmonizable stochastic process. The process is said to be *spectrally correlated* [14] if its Loève bifrequency spectrum can be expressed as

$$\mathcal{S}_{\boldsymbol{x}_a}(f_1,f_2) = \sum_{k\in\mathbb{I}} S_{\boldsymbol{x}_a}^{(k)}(f_1)\,\delta\left(f_2 - \Psi_{\boldsymbol{x}_a}^{(k)}(f_1)\right) \tag{3}$$

where $\delta(\cdot)$ is Dirac delta, $\mathbb{I}$ is a countable set, the curves $f_2 = \Psi_{\boldsymbol{x}_a}^{(k)}(f_1)$ describe the support of $\mathcal{S}_{\boldsymbol{x}_a}(f_1,f_2)$, and the complex-valued functions $S_{\boldsymbol{x}_a}^{(k)}(f_1)$, called *spectral correlation density functions*, represent the density of the Loève spectrum on its support curves. The real-valued functions $\Psi_{\boldsymbol{x}_a}^{(k)}(\cdot)$ can always be chosen invertible. □

In the special case of linear support curves with unit slope, SC processes reduce to ACS processes. For ACS processes the separation between correlated spectral components assumes values belonging to a countable set, the set of cycle frequencies $\{\alpha_k\}$, which are also the frequencies of the (generalized) Fourier series expansion of the almost-periodically time-variant statistical autocorrelation function [6]. That is, for ACS processes, $\Psi_{\boldsymbol{x}_a}^{(k)}(f_1) = f_1 - \alpha_k$, the spectral correlation density functions $S_{\boldsymbol{x}_a}^{(k)}(f_1)$ are coincident with the cyclic spectra $S_{\boldsymbol{x}_a}^{\alpha_k}(f_1)$, and the Loève bifrequency spectrum is given by

$$\mathcal{S}_{\boldsymbol{x}_a}(f_1,f_2) = \sum_{k\in\mathbb{I}} S_{\boldsymbol{x}_a}^{\alpha_k}(f_1)\,\delta(f_2 - f_1 + \alpha_k)\,. \tag{4}$$

Note that the class of the ACS processes turns out to be the intersection between two wider classes of nonstationary processes that both generalize the class of the ACS processes: The class of the SC processes and that of the generalized almost-cyclostationary (GACS) processes [16], [17]. The GACS processes exhibit an almost-periodically time-variant statistical autocorrelation function whose (generalized) Fourier series expansion has both coefficients and frequencies (cycle frequencies) depending on the lag parameter. In the special case of ACS processes the cycle frequencies are independent of the lag parameter.

SC processes are encountered in several applications in communications. When an ACS signal is transmitted by a moving source with constant relative radial speed with respect to two sensors, then the received signals on the two sensors are jointly SC but not jointly ACS [1]. Moreover, reverberation mechanisms generate coherency relationships ensemblewise between spectral components [13]. In [14], it is shown that an ACS signal passing through a multipath Doppler channel gives rise to an output SC signal when, for each path, the relative radial speeds between transmitter, receiver, and reflecting moving objects can be considered constant within the observation interval. In such a case, for the input complex-envelope signal $x_a(t)$, the output complex-envelope $y_a(t)$ is given by

$$y_a(t) = \sum_{k=1}^{K} a_k\,x_a(s_k t - d_k)\,e^{j2\pi\nu_k t} \tag{5}$$

where, for each path of the channel, a_k is the complex gain, d_k the delay, s_k the time-scale factor, and ν_k the frequency shift. In [14], it is shown that the Loève bifrequency spectrum of $y_a(t)$ has support in the bifrequency plane constituted by lines with slopes s_{k_2}/s_{k_1}, $k_1,k_2 \in \{1,\ldots,K\}$. The time-scale factor s_k can be considered unitary if the condition $BT \ll c/\mathrm{v}_k$ is fulfilled, where B is input-signal bandwidth, T is the data-record length, c is the medium propagation speed, and v_k is the relative radial speed for the kth-path [22, pp. 240-242]. If $s_k \simeq 1\ \forall k$, then the multipath Doppler channel can be modeled as linear almost-periodically time variant and the output signal $y_a(t)$ is ACS. However, wide-band modulated signals and mobile environments of interest in modern communications systems give rise to time-variant channels with time-scale factors that cannot be considered unitary even for moderate speeds and/or data-record lengths [14]. Further cases of interest where $BT \not\ll c/\mathrm{v}_k$ and, hence, nonunit time-scale factors should be accounted for, can be encountered in wide-band MIMO communication systems, radar and sonar applications [22, pp. 339-340], time-delay and Doppler estimation of wide band signals [8], UWB channel modeling [20], and space communications [18]. SC processes with nonlinear support curves arise in the presence of linear time-variant transformations operating nonlinear frequency-warping of the input signals [3], [4]. Cross spectral analysis by frequency warping [2], [5], [12], [23] gives rise to jointly SC processes. Finally, fBm processes and their linear time-invariant filtered versions are SC processes with Loève bifrequency spectrum concentrated on the lines $f_2 = f_1$, $f_1 = 0$, and $f_2 = 0$ [19].

It is well known that for wide-sense stationary processes, the strictly band-limitedness condition allows to avoid aliasing after uniform sampling, provided that the sampling frequency exceeds twice the process bandwidth. Nonstationary processes should be carefully handled. In particular, not every nonstationary structure is compatible with the band limitedness property. ACS processes can be strictly bandlimited [7]. In contrast, in [16], [17] it is shown that the GACS processes cannot be strictly bandlimited . In the sequel it will be shown that SC processes can be strictly band-limited. For this purpose, the definition of strict band-limitedness is given for the general case of nonstationary processes. Then a characterization of the Loève bifrequency spectrum of strictly band-limited SC processes is provided.

Definition 2.3 A continuous-time nonstationary stochastic process $x_a(t)$ is said to be strictly band-limited with bandwidth B if B is the smallest value such that

$$x_a(t) \otimes h_{\mathrm{lp}}(t) = x_a(t) \tag{6}$$

for almost all sample paths, where $h_{\mathrm{lp}}(t)$ is the impulse response function of the ideal low-pass filter with harmonic response $H_{\mathrm{lp}}(f) = \mathrm{rect}(f/2B)$. □

Theorem 2.1 Let $x_a(t)$ be a strictly band-limited SC process with bandwidth B. It results that $\forall k \in \mathbb{I}$

$$S^{(k)}_{\boldsymbol{x}_a}(f_1) = 0 \quad \text{if} \quad \max\left\{|f_1|, \left|\Psi^{(k)}_{\boldsymbol{x}_a}(f_1)\right|\right\} > B \tag{7}$$

In addition, $\Psi^{(k)}_{\boldsymbol{x}_a}(f_1)$ can be assumed to be zero for $f_1 \notin [-B,B]$.

Conversely, if (7) holds, then $x_a(t)$ is strictly bandlimited with bandwidth B.

Proof:

The strictly band-limitedness condition (6), considered in the frequency domain, for a SC process implies that

$$\mathrm{E}\left\{H_{\mathrm{lp}}(f_1)X_a(f_1)H_{\mathrm{lp}}(f_2)X^*_a(f_2)\right\} = \mathrm{E}\left\{X_a(f_1)\,X^*_a(f_2)\right\} \tag{8}$$

from which, accounting for (3), it follows that

$$\begin{aligned} H_{\mathrm{lp}}(f_1)H_{\mathrm{lp}}(f_2) &\sum_{k\in\mathbb{I}} S^{(k)}_{\boldsymbol{x}_a}(f_1)\delta\left(f_2 - \Psi^{(k)}_{\boldsymbol{x}_a}(f_1)\right) \\ = &\sum_{k\in\mathbb{I}} S^{(k)}_{\boldsymbol{x}_a}(f_1)\delta\left(f_2 - \Psi^{(k)}_{\boldsymbol{x}_a}(f_1)\right) \end{aligned} \tag{9}$$

Thus, it necessarily results that $\forall k \in \mathbb{I}$

$$\begin{cases} f_1 \notin [-B,B] \;\Rightarrow\; S^{(k)}_{\boldsymbol{x}_a}(f_1) = 0 \\ f_1 \in [-B,B] \;\Rightarrow\; f_2 \neq \Psi^{(k)}_{\boldsymbol{x}_a}(f_1)\;\forall f_2 \notin [-B,B] \end{cases} \tag{10}$$

that is

$$f_1 \notin [-B,B] \;\text{ or }\; \Psi^{(k)}_{\boldsymbol{x}_a}(f_1) \notin [-B,B] \;\Rightarrow\; S^{(k)}_{\boldsymbol{x}_a}(f_1) = 0 \tag{11}$$

from which (7) immediately follow. Furthermore, accounting for (11), form (3) it follows that for strictly bandlimited SC processes the functions $\Psi^{(k)}_{\boldsymbol{x}_a}(f_1)$ are undetermined for $f_1 \notin [-B,B]$. Thus, $\Psi^{(k)}_{\boldsymbol{x}_a}(f_1) = 0$ for $f_1 \notin [-B,B]$ can be assumed in order to have these functions with compact support.

The proof of the converse is straightforward. □

3. DISCRETE-TIME SPECTRALLY CORRELATED PROCESSES

Definition 3.1 Let $x(n)$ be a discrete-time complex-valued second-order harmonizable stochastic process. Its *Loève bifrequency spectrum* is defined as

$$\widetilde{\mathbb{S}}_{\boldsymbol{x}}(\nu_1,\nu_2) \triangleq \mathrm{E}\left\{X(\nu_1)X^*(\nu_2)\right\} \tag{12}$$

where

$$X(\nu) \triangleq \sum_{n\in\mathbb{Z}} x(n)\,e^{-j2\pi\nu n} \tag{13}$$

is the Fourier transform of $x(n)$ and is assumed to exist (at least) in the sense of distributions [24] for almost all sample paths of $x(n)$. Subscript $\boldsymbol{x}$ denotes $[xx^*]$. □

Definition 3.2 The discrete-time process $x(n)$ is said to be *spectrally correlated* if its Loève bifrequency spectrum can be expressed as

$$\widetilde{\mathbb{S}}_{\boldsymbol{x}}(\nu_1,\nu_2) = \sum_{k\in\mathbb{I}} \widetilde{S}^{(k)}_{\boldsymbol{x}}(\nu_1)\,\widetilde{\delta}\left(\nu_2 - \widetilde{\Psi}^{(k)}_{\boldsymbol{x}}(\nu_1)\right) \tag{14}$$

where $\mathbb{I}$ is a countable set, $\widetilde{\delta}(\nu) \triangleq \sum_{p\in\mathbb{Z}} \delta(\nu - p)$, and $\widetilde{S}^{(k)}_{\boldsymbol{x}}(\nu)$ and $\widetilde{\Psi}^{(k)}_{\boldsymbol{x}}(\nu)$ are complex- and real-valued, respectively, periodic functions of ν with period 1. □

From (14) it follows that discrete-time SC processes have spectral masses concentrated on the countable set of support curves

$$\nu_2 \bmod 1 = \widetilde{\Psi}^{(k)}_{\boldsymbol{x}}(\nu_1) \qquad k \in \mathbb{I} \tag{15}$$

where mod 1 is the modulo 1 operation with values in $[-1/2, 1/2)$. Moreover, the spectral mass distribution is periodic with period 1 in both frequency variables.

Let us observe that the functions in the right-hand side of (14) in general are not univocally determined. By opportunely selecting the support of the functions $\widetilde{S}^{(k)}_{\boldsymbol{x}}(\nu_1)$, the corresponding functions $\nu_2 = \widetilde{\Psi}^{(k)}_{\boldsymbol{x}}(\nu_1)$ can always be chosen to be locally invertible in intervals $[p-1/2, p+1/2)$, with p integer. In addition, since $\widetilde{\Psi}^{(k)}_{\boldsymbol{x}}(\nu_1)$ are in the argument of a periodic delta, they can always be chosen with values in $[-1/2, 1/2)$.

Every periodic function $\widetilde{\Psi}^{(k)}_{\boldsymbol{x}}(\nu)$ can be expressed as the periodic replication, with period 1, of a $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$ generator function $\Psi^{(k)}_{\boldsymbol{x}}(\nu)$ [9]:

$$\widetilde{\Psi}^{(k)}_{\boldsymbol{x}}(\nu) = \sum_{p\in\mathbb{Z}} \Psi^{(k)}_{\boldsymbol{x}}(\nu - p)\,. \tag{16}$$

The generator, in general, is not univocally determined and can have support of width larger than 1. Let us consider the (unique) generator with compact support contained in $[-1/2, 1/2)$. That is, such that $\Psi^{(k)}_{\boldsymbol{x}}(\nu) = \widetilde{\Psi}^{(k)}_{\boldsymbol{x}}(\nu)$, $\nu \in [-1/2, 1/2)$. With this choice for the generator, the following useful expression holds for the periodic delta train in (14):

$$\begin{aligned} \widetilde{\delta}&\left(\nu_2 - \widetilde{\Psi}^{(k)}_{\boldsymbol{x}}(\nu_1)\right) \\ &= \sum_{p_2\in\mathbb{Z}} \delta\Big(\nu_2 - p_2 - \sum_{p_1\in\mathbb{Z}} \Psi^{(k)}_{\boldsymbol{x}}(\nu_1 - p_1)\Big) \\ &= \sum_{p_2\in\mathbb{Z}}\sum_{p_1\in\mathbb{Z}} \delta\Big(\nu_2 - p_2 - \Psi^{(k)}_{\boldsymbol{x}}(\nu_1 - p_1)\Big)\,. \end{aligned} \tag{17}$$

4. SAMPLING THEOREMS

In this section, a sampling theorem for SC processes is stated to link the Loève bifrequency spectrum of the sampled discrete-time process with that of the continuous-time process. Furthermore, in the case of strictly band-limited continuous-time SC processes, sufficient conditions on the sampling frequency to avoid aliasing effects are provided.

Theorem 4.1 Let $x_a(t)$ be a continuous-time SC process with Loève bifrequency spectrum (3) and let

$$x(n) \triangleq x_a(t)|_{t=nT_s} \quad n \in \mathbb{Z} \tag{18}$$

be the discrete-time process obtained by uniformly sampling $x_a(t)$ with sampling period $T_s \triangleq 1/f_s$. The process $x(n)$ is a discrete-time SC process with Loève bifrequency spectrum given by

$$\mathrm{E}\Big\{X(\nu_1)X^*(\nu_2)\Big\} = \sum_{k\in\mathbb{I}}\frac{1}{T_s}\sum_{p_1\in\mathbb{Z}} S_{\boldsymbol{x}_a}^{(k)}((\nu_1-p_1)f_s) \sum_{p_2\in\mathbb{Z}}\delta\Big(\nu_2-p_2-\Psi_{\boldsymbol{x}_a}^{(k)}((\nu_1-p_1)f_s)\,T_s\Big) \quad (19)$$

Proof:

The Fourier transform (13) of the sequence $x(n)$ is linked to the Fourier transform (2) of the continuous-time signal $x_a(t)$ by the relationship [9, Sec. 9.5]

$$X(\nu) = \frac{1}{T_s}\sum_{p\in\mathbb{Z}} X_a((\nu-p)f_s)\,. \quad (20)$$

Thus, accounting for (3) it results that

$$\begin{aligned}\mathrm{E}&\Big\{X(\nu_1)X^*(\nu_2)\Big\}\\ &= \frac{1}{T_s^2}\sum_{p_1\in\mathbb{Z}}\sum_{p_2\in\mathbb{Z}}\mathrm{E}\Big\{X_a((\nu_1-p_1)f_s)\,X_a^*((\nu_2-p_2)f_s)\Big\}\\ &= \frac{1}{T_s^2}\sum_{p_1\in\mathbb{Z}}\sum_{p_2\in\mathbb{Z}}\sum_{k\in\mathbb{I}} S_{\boldsymbol{x}_a}^{(k)}((\nu_1-p_1)f_s)\\ &\qquad \delta\Big((\nu_2-p_2)f_s-\Psi_{\boldsymbol{x}_a}^{(k)}((\nu_1-p_1)f_s)\Big) \end{aligned} \quad (21)$$

from which (19) follows by a scale change in the argument of the Dirac delta [24, Sec. 1.7]. □

From Theorem 4.1 it follows that the discrete-time process $x(n)$ is SC. In fact, its Loève bifrequency spectrum exhibits spectral masses concentrated on a countable set of support curves and is periodic with period 1 in booth variables ν_1 and ν_2. Furthermore, at every fixed double $(\nu_1,\nu_2)\in[-1/2,1/2)^2$, nonzero contribution to the Loève bifrequency spectrum of $x(n)$ is given by all terms in (19) such that $(\nu_2-p_2)f_s = \Psi_{\boldsymbol{x}_a}^{(k)}((\nu_1-p_1)f_s)$ and $S_{\boldsymbol{x}_a}^{(k)}((\nu_1-p_1)f_s)\neq 0$ for some $(p_1,p_2,k)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{I}$. That is, aliasing effects are present. Specifically, the support curves in the bifrequency principal domain $(\nu_1,\nu_2)\in[-1/2,1/2)^2$ are described by the set

$$\begin{aligned}\mathbb{S} \triangleq \bigcup_{k\in\mathbb{I}}\bigcup_{p_1\in\mathbb{Z}}\Big\{&(\nu_1,\nu_2)\in[-1/2,1/2)^2 :\\ &\nu_2 = \Big[\Psi_{\boldsymbol{x}_a}^{(k)}((\nu_1-p_1)f_s)T_s\Big] \bmod 1,\\ &S_{\boldsymbol{x}_a}^{(k)}((\nu_1-p_1)f_s)\neq 0\Big\}\,. \end{aligned} \quad (22)$$

The set $\mathbb{S}$ can contain clusters of curves or can be dense in the open square $(-1/2,1/2)^2$ if, for some $\nu_1\in[-1/2,1/2)$, f_s is incommensurate with a countable infinity of values of $\Psi_{\boldsymbol{x}_a}^{(k)}((\nu_1-p_1)\,f_s)$, $k\in\mathbb{I}$, $p_1\in\mathbb{Z}$, and in correspondence of these values the spectral densities $S_{\boldsymbol{x}_a}^{(k)}((\nu_1-p_1)f_s)$ are nonzero (see [10] for the special case of support lines).

Theorem 4.2 Let $x_a(t)$ be a strictly band-limited continuous-time SC process with Loève bifrequency spectrum (3) and bandwidth B. If $f_s \geqslant 2B$, then the process $x(n)$ defined in (18) is a discrete-time SC process with Loève bifrequency spectrum given by (14) with

$$\widetilde{S}_{\boldsymbol{x}}^{(k)}(\nu_1) \triangleq \frac{1}{T_s}\sum_{p\in\mathbb{Z}} S_{\boldsymbol{x}_a}^{(k)}((\nu_1-p)f_s) \quad (23)$$

$$\widetilde{\Psi}_{\boldsymbol{x}}^{(k)}(\nu_1) \triangleq T_s\sum_{p\in\mathbb{Z}} \Psi_{\boldsymbol{x}_a}^{(k)}((\nu_1-p)f_s) \quad (24)$$

and sums in (23) and (24) are with nonoverlapping replicas.

Proof:

Under the assumption $f_s\geqslant 2B$, it results that for each $k\in\mathbb{I}$ and for every $(p_1,p_2)\in\mathbb{Z}^2$ there exists only one double $(\nu_1,\nu_2)\in[p_1-1/2,p_1+1/2)\times[p_2-1/2,p_2+1/2)$ such that $(\nu_2-p_2)f_s = \widetilde{\Psi}_{\boldsymbol{x}_a}^{(k)}((\nu_1-p_1)f_s)$ in the argument of the Dirac delta in the right-hand side of (21). Consequently, (21) can be written as

$$\begin{aligned}\mathrm{E}&\Big\{X(\nu_1)X^*(\nu_2)\Big\}\\ &= \sum_{k\in\mathbb{I}}\Big[\frac{1}{T_s}\sum_{p_1\in\mathbb{Z}} S_{\boldsymbol{x}_a}^{(k)}((\nu_1-p_1)f_s)\Big]\\ &\qquad \frac{1}{T_s}\sum_{p_2\in\mathbb{Z}}\sum_{p_1\in\mathbb{Z}}\delta\Big((\nu_2-p_2)f_s-\Psi_{\boldsymbol{x}_a}^{(k)}((\nu_1-p_1)f_s)\Big)\\ &= \sum_{k\in\mathbb{I}}\Big[\frac{1}{T_s}\sum_{p_1\in\mathbb{Z}} S_{\boldsymbol{x}_a}^{(k)}((\nu_1-p_1)f_s)\Big]\\ &\qquad \sum_{p_2\in\mathbb{Z}}\delta\Big(\nu_2-p_2-\sum_{p_1\in\mathbb{Z}}\Psi_{\boldsymbol{x}_a}^{(k)}((\nu_1-p_1)f_s)T_s\Big) \end{aligned} \quad (25)$$

from which (14) with (23) and (24) immediately follow. In addition, since replicas in (25) are separated by 1 in both variables ν_1 and ν_2 and the functions (of ν_1) $S_{\boldsymbol{x}_a}^{(k)}(\nu_1 f_s)$ and $\Psi_{\boldsymbol{x}_a}^{(k)}(\nu_1 f_s)$ have compact support contained in $[-B/f_s, B/f_s]$, condition $f_s\geqslant 2B$ assures that replicas in (23) and (24) do not overlap. □

Under the assumption $f_s\geqslant 2B$ the relationships

$$\widetilde{S}_{\boldsymbol{x}}^{(k)}(\nu_1) = \frac{1}{T_s}S_{\boldsymbol{x}_a}^{(k)}(\nu_1 f_s)\,, \quad \widetilde{\Psi}_{\boldsymbol{x}}^{(k)}(\nu_1) = T_s\Psi_{\boldsymbol{x}_a}^{(k)}(\nu_1 f_s) \quad (26)$$

hold only for $\nu_1\in\operatorname{supp}\Big\{S_{\boldsymbol{x}_a}^{(k)}(\nu_1 f_s)\Big\}$ which, in general, is a proper subset of $[-1/2,1/2)$. Thus, for SC processes, the mappings $\nu_1 = f_1/f_s$ and $\nu_2 = f_2/f_s$ do not hold, for every support curve, for $\nu_1\in[-1/2,1/2]$ and $\nu_2\in[-1/2,1/2]$. In contrast, for wide-sense stationary processes, spectral masses are present only on the main diagonal of the bifrequency plane and condition $f_s\geqslant 2B$ assures that the mappings $\nu_1 = f_1/f_s$ and $\nu_2 = f_2/f_s$ hold in the whole principal domain. By considering a more stringent condition on the sampling frequency, the $\nu_i\leftrightarrow f_i$ mappings can be made valid in the whole principal frequency domain. We have the following result.

Theorem 4.3 Let $x_a(t)$ be a strictly band-limited continuous-time SC process with bandwidth B. If

$$f_s \geqslant B + \max\Big\{B, \sup_k |\Psi_{\boldsymbol{x}_a}^{(k)}(\pm f_s/2)|\Big\} \quad (27)$$

where "+" or "-" sign should be taken if $\Psi_{\boldsymbol{x}_a}^{(k)}(\cdot)$ is increasing or decreasing, respectively, then conditions (26) hold $\forall \nu_1 \in [-1/2, 1/2)$.

Proof: If $\Psi_{\boldsymbol{x}_a}^{(k)}(\cdot)$ is increasing, condition $|\Psi_{\boldsymbol{x}_a}^{(k)}(f_s/2)|T_s \leqslant 1 - B/f_s$, $\forall k \in \mathbb{I}$, assures in (25) that for $\nu_1 \in [-1/2, 1/2]$ support curves $\nu_2 = \Psi_{\boldsymbol{x}_a}^{(k)}(\nu_1 f_s)T_s$ of the replica with $p_1 = p_2 = 0$ can intersect support curves of other replicas only if on these other curves the spectral correlation density is zero. The case of $\Psi_{\boldsymbol{x}_a}^{(k)}(\cdot)$ decreasing is similar. □

In the special case of ACS processes, condition (27) leads to $f_s \geqslant 6B$. This condition, however, can be relaxed to $f_s \geqslant 4B$ since the support curves are lines with unit slope [6], [7].

5. CONCLUSION

The class of discrete-time SC processes is introduced and characterized and the problem of uniformly sampling continuous-time SC processes is addressed. It is shown that for a strictly band-limited SC process, sampling at twice the bandwidth leads to non overlapping replicas in the Loève bifrequency spectrum of the SC discrete-time process. However, a more stringent condition on the sampling frequency need to be satisfied in order to assure that, for the discrete-time process, the spectral correlation densities on the support curves are scaled version of those of the continuous-time process for all values of frequencies in $[-1/2, 1/2]$.

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