BOUNDED REAL LEMMA FOR MULTIVARIATE TRIGONOMETRIC MATRIX POLYNOMIALS AND FIR FILTER DESIGN APPLICATIONS

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ABSTRACT

We propose a linear matrix inequality formulation of the Bounded Real Lemma (BRL) for multivariate trigonometric polynomials with matrix coefficients. This is a generalization of previous results regarding positive trigonometric polynomials. The proposed BRL allows the formulation of several FIR filter design problems as semidefinite programming (SDP) problems. We employ the new BRL in three applications: matrix filter design, 2-D deconvolution and design of 2-D filters with matrix coefficients. All applications are illustrated with examples that improve on previous work.

1. INTRODUCTION

The recent developments in the field of positive trigonometric polynomials [4] concern mainly polynomials with scalar coefficients. Although some basic results have been proven to hold in (almost) the same form for polynomials with matrix coefficients [7, 5, 1, 2, 8], there are still issues not yet investigated. Moreover, while the generalization to matrix coefficients may be relatively easy from a mathematical viewpoint, the applicative importance of the new results should be relevant enough to deserve the investigation.

Let us consider a causal matrix polynomial (filter) in d variables,

$$\mathbf{H}(\mathbf{z}) = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \mathbf{H}_{\mathbf{k}} \mathbf{z}^{-\mathbf{k}}. \tag{1}$$

We denote $\mathbf{z} = (z_1, \ldots, z_d)$ the complex variable and $\mathbf{z}^{\mathbf{k}}$ the monomial $z_1^{k_1} \ldots z_d^{k_d}$, with $\mathbf{k} \in \mathbb{Z}^d$. The matrix coefficients $\mathbf{H}_{\mathbf{k}}$ have size $\kappa_1 \times \kappa_2$; we can see $\mathbf{H}(\mathbf{z})$ as a MIMO system with κ_2 inputs and κ_1 outputs. The degree of the filter (1) is $\mathbf{n} \in \mathbb{Z}_+^d$ and the sum runs for all $\mathbf{0} \le \mathbf{k} \le \mathbf{n}$, where the inequalities are valid elementwise. A Bounded Real Lemma (BRL) is a characterization of the inequality

$$\|\mathbf{H}(\mathbf{z})\| \le \gamma, \tag{2}$$

where γ is a positive number and $\|\cdot\|$ is a system norm. We consider here the H_∞ norm, which makes (2) equivalent to

$$\sigma_{\max}(\mathbf{H}(e^{j\omega})) \le \gamma, \ \ \forall \omega \in \mathbb{T}^d, \tag{3}$$

where $\sigma_{\max}(\cdot)$ is the maximum singular value of its matrix argument and $\mathbb{T}$ is the unit circle. We actually treat the more general case where the inequality (3) is valid on a subset $\mathscr{D} \subset \mathbb{T}^d$ described by the positivity of some trigonometric polynomials (and so $\|\cdot\|$ from (2) is no longer a norm).

We provide in this paper a linear matrix inequality (LMI) characterization of (2), which allows solving optimization problems involving (2) via semidefinite programming (SDP). The result can be seen as a generalization of the BRL from [2] to the multivariate case (including frequency domains) or of the BRL from [3] to matrix polynomials. Due to space restrictions, we omit the proofs. We describe and give design examples for three applications: design of matrix filters [9, 13], 2-D deconvolution [11] and design of filters with matrix coefficients [12]. Although we present our results for polynomials with real coefficients, they can be extended easily to the complex case.

2. BOUNDED REAL LEMMA

The two main results we present in this section are intimately related to the theory of sum-of-squares polynomials. A symmetric trigonometric matrix polynomial has the form

$$\mathbf{R}(\mathbf{z}) = \sum_{\mathbf{k}=-\mathbf{n}}^{\mathbf{n}} \mathbf{R}_{\mathbf{k}} \mathbf{z}^{-\mathbf{k}}, \ \ \mathbf{R}_{-\mathbf{k}} = \mathbf{R}_{\mathbf{k}}^T. \tag{4}$$

The coefficients $\mathbf{R}_{\mathbf{k}}$ have size $\kappa \times \kappa$. For $\mathbf{z} \in \mathbb{T}^d$, $\mathbf{R}(\mathbf{z})$ is a Hermitian matrix and so it has real eigenvalues. The polynomial (4) is sum-of-squares if it can be expressed as

$$\mathbf{R}(\mathbf{z}) = \sum_{\ell=1}^{\nu} \mathbf{F}_\ell(\mathbf{z}) \mathbf{F}_\ell(\mathbf{z}^{-1})^T, \tag{5}$$

where $\mathbf{F}_\ell(\mathbf{z})$ are causal polynomials as in (1). It is clear that for $\mathbf{z} \in \mathbb{T}^d$, the sum-of-squares $\mathbf{R}(\mathbf{z})$ is a positive semidefinite matrix. Conversely, all polynomials (4) with $\mathbf{R}(\mathbf{z}) \succ 0$, $\forall \mathbf{z} \in \mathbb{T}^d$, are sum-of-squares, see e.g. [1]; however, the degrees of the polynomials $\mathbf{F}_\ell(\mathbf{z})$ from (5) may be arbitrarily high.

The connection between sum-of-squares and SDP is made by expressing causal polynomials (1) using the standard d-dimensional basis

$$\psi(\mathbf{z}) = \psi(z_d) \otimes \ldots \otimes \psi(z_1) \otimes \mathbf{I}_\kappa, \tag{6}$$

where $\otimes$ is the Kronecker product and

$$\psi(z_i) = [1 \ z_i \ \ldots \ z_i^{n_i}]^T \tag{7}$$

is the univariate basis. By stacking the matrix coefficients of (1) in the order of the monomials from the basis (6), we

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obtain a matrix $\overline{\mathbf{H}}$ of size $N\kappa_1 \times \kappa_2$, with $N = \prod_{i=1}^{d}(n_i+1)$ being the number of matrix coefficients in (1). For example, for a 2-D polynomial with $n_1 = 2$, $n_2 = 1$, the basis (6) is

$$\psi(\mathbf{z}) = [\mathbf{I} \;\; z_1\mathbf{I} \;\; z_1^2\mathbf{I} \;\; z_2\mathbf{I} \;\; z_1z_2\mathbf{I} \;\; z_1^2z_2\mathbf{I}]^T \tag{8}$$

and the stacked coefficients matrix is

$$\overline{\mathbf{H}} = [\mathbf{H}_{0,0}^T \;\; \mathbf{H}_{1,0}^T \;\; \mathbf{H}_{2,0}^T \;\; \mathbf{H}_{0,1}^T \;\; \mathbf{H}_{1,1}^T \;\; \mathbf{H}_{2,1}^T\,]^T. \tag{9}$$

Using the above ingredients, the causal filter can be expressed as

$$\mathbf{H}(z) = \psi(\mathbf{z}^{-1})^T \cdot \overline{\mathbf{H}}. \tag{10}$$

The parameterization of sum-of-squares trigonometric polynomials is the following [7]. A polynomial $\mathbf{S}(\mathbf{z})$ defined as in (4) is sum-of-squares (of order $\mathbf{n}$) if and only if there exists a positive semidefinite matrix $\mathbf{Q}$ of size $N\kappa \times N\kappa$ (named Gram matrix) such that

$$\mathbf{S}(\mathbf{z}) = \psi(\mathbf{z}^{-1})^T \cdot \mathbf{Q} \cdot \psi(\mathbf{z}). \tag{11}$$

This relation connects *linearly* the coefficients of a sum-of-squares polynomial to the elements of a positive semidefinite matrix.

We consider frequency domains

$$\mathscr{D} = \{\mathbf{z} \in \mathbb{T}^d \mid D_\ell(\mathbf{z}) \geq 0, \; \ell = 1 : L\} \tag{12}$$

defined by the positivity of trigonometric polynomials (with scalar coefficients). The next theorem describes trigonometric matrix polynomials that are positive definite on the domain $\mathscr{D}$.

Theorem 1 *A matrix polynomial (4) is positive definite on the set (12), i.e.* $\mathbf{R}(\mathbf{z}) \succ 0$, $\forall \mathbf{z} \in \mathscr{D}$, *if and only if there exist sum-of-squares* $\mathbf{S}_\ell(\mathbf{z})$, $\ell = 0 : L$, *such that*

$$\mathbf{R}(\mathbf{z}) = \mathbf{S}_0(\mathbf{z}) + \sum_{\ell=1}^{L} D_\ell(\mathbf{z})\mathbf{S}_\ell(\mathbf{z}). \tag{13}$$

Proof. The theorem can be proved similarly to the scalar coefficients result from [3]. The starting point is a result from [8] on multivariate *real* matrix polynomials that are positive definite on a domain described by the positivity of (real) polynomials. ■

As in other results of this type, the degree of the sum-of-squares $\mathbf{S}_\ell(\mathbf{z})$ from (13) can be arbitrarily high. Practically, we have to bound the degrees, usually to the degree of $\mathbf{R}(\mathbf{z})$, which makes (13) only a sufficient stability condition. However, in the 1-D case, when $\mathscr{D}$ is an interval described by the positivity of a single polynomial, Theorem 1 holds true for sum-of-squares $\mathbf{S}_0(z)$, $\mathbf{S}_1(z)$ whose degrees are minimial (i.e. n and $n-2$, respectively). (The proof has been provided by C.W. Scherer in a personal communication.) Using the representation (11) for the sum-of-squares appearing in (13), polynomial positivity is expressed as an LMI.

We can now present the BRL for trigonometric matrix polynomials.

Theorem 2 *Let* $\mathbf{H}(\mathbf{z})$ *be a causal matrix polynomial (1) and* γ *a positive real. The inequality*

$$\sigma_{\max}(\mathbf{H}(\mathbf{z})) < \gamma, \;\; \forall \mathbf{z} \in \mathscr{D}, \tag{14}$$

holds true if and only if there exist sum-of-squares $\mathbf{S}_\ell(\mathbf{z})$, $\ell = 0 : L$, *such that*

$$\gamma^2\mathbf{I}_{\kappa_1} = \mathbf{S}_0(\mathbf{z}) + \sum_{\ell=1}^{L} \mathbf{D}_\ell(\mathbf{z})\mathbf{S}_\ell(\mathbf{z}) \tag{15}$$

and

$$\begin{bmatrix} \mathbf{Q}_0 & \overline{\mathbf{H}} \\ \overline{\mathbf{H}}^T & \mathbf{I}_{\kappa_2} \end{bmatrix} \succeq 0, \tag{16}$$

where $\mathbf{Q}_0$ *is the Gram matrix associated with* $\mathbf{S}_0(\mathbf{z})$ *through (11) and the matrix* $\overline{\mathbf{H}}$ *contains the stacked coefficients of* $\mathbf{H}(z)$ *as in (10).*

Proof. The proof is similar to the scalar case treated in [3] and uses Theorem 1, a majorization result and the Schur complement. ■

Some comments on Theorem 2 are necessary. Relation (14) is equivalent to

$$\mathbf{H}(\mathbf{z})\mathbf{H}(\mathbf{z}^{-1})^T < \gamma^2\mathbf{I}_{\kappa_1}.$$

Accordingly, the matrix coefficients of the polynomials from (15) have size $\kappa_1 \times \kappa_1$. As the size of Gram matrices is proportional with the size of matrix coefficients and since the nonzero singular values of $\mathbf{H}(\mathbf{z})$ and $\mathbf{H}(\mathbf{z})^T$ are the same, this form of the BRL is convenient when $\kappa_1 \leq \kappa_2$. Otherwise, it is more efficient to rewrite Theorem 2 for $\mathbf{H}(\mathbf{z})^T$.

Similarly to Theorem 1, the degrees of the sum-of-squares can be arbitrarily high. In our use of Theorem 2, we will always consider the minimum degree, which e.g. implies that the degree of $\mathbf{S}_0(\mathbf{z})$ is $\mathbf{n}$. So (excepting the 1-D case), we implement only a sufficient boundedness condition. In the case where the degree of $\mathbf{S}_0(\mathbf{z})$ is larger, the stacked coefficients matrix $\overline{\mathbf{H}}$ that appears in (16) must contain zero coefficients (in the appropriate positions) for the monomials with degree not smaller than $\mathbf{n}$. Working with higher degrees of the sum-of-squares may improve the quality of the results, but only marginally in most cases; however, the complexity always increases; so, practical considerations and our experience with the scalar coefficient case suggest to use the minimum degree.

Using the Gram matrix representation (11) for the sum-of-squares appearing in (15), the relations (15)–(16) are an LMI in which the coefficients of $\mathbf{H}(z)$ appear linearly. So, a score of optimization problems can be solved via SDP. The simplest example is the computation of the H_∞ norm of a system (1). It consists of the minimization of γ^2, subject to (15) and (16). Since in this case $\mathscr{D} = \mathbb{T}^d$, the equality (15) is reduced to $\gamma^2\mathbf{I}_{\kappa_1} = \mathbf{S}_0(\mathbf{z}) = \psi(\mathbf{z}^{-1})^T \cdot \mathbf{Q}_0 \cdot \psi(\mathbf{z})$. The variables of the problem are γ^2, the coefficients of $\mathbf{H}(\mathbf{z})$ and the Gram matrix $\mathbf{Q}_0 \succeq 0$. This is an SDP problem, since all the variables appear linearly in (15) and (16). The optimal γ is (an upper approximation) of the desired H_∞ norm.

3. DESIGN PROBLEMS AND RESULTS

We discuss here three design applications of Theorem 2, pertaining to matrix filters, 2-D FIR deconvolution and 2-D FIR filters with matrix coefficients. Other possible applications, not touched here, are in filters for MIMO sampling and reconstruction [10] or the design (for nearly perfect reconstruction) of a (multidimensional) synthesis filter bank given the analysis bank.

3.1 Matrix filter design

Matrix filters [9] process blocks of data $\mathbf{x} \in \mathbb{C}^N$ through the linear transformation

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \tag{17}$$

where $\mathbf{A}$ is a real (or complex) matrix of size $N \times N$ (we consider square matrices only for the ease of presentation). Such processing is useful for example in antenna arrays for underwater acoustics.

We treat here the simplest design setup, in which we want to design a minimax lowpass matrix filter (with real coefficients), which satisfies the conditions

$$\|\mathbf{A}\psi(e^{-j\omega}) - \psi(e^{-j\omega})\| \leq \gamma_p, \ \ \forall\omega \in [0, \omega_p], \tag{18}$$

$$\|\mathbf{A}\psi(e^{-j\omega})\| \leq \gamma_s, \ \ \forall\omega \in [\omega_s, \pi], \tag{19}$$

where ω_p and ω_s are the edges of the passband and stopband, respectively, and γ_p and γ_s are error bounds with respect to the desired response. The passband desired response is a vector of delays, see (7). The norms in (18) and (19) are 2-norms.

The matrix filter has the form

$$\mathbf{H}(z) = \mathbf{A}\psi(z^{-1}) = \sum_{k=0}^{N-1} \mathbf{a}_k z^{-k}, \tag{20}$$

where $\mathbf{a}_k \in \mathbb{R}^N$ are the columns of $\mathbf{A}$. The polynomial (20) has $d = 1$ variable and the size of the matrix coefficients is $\kappa_1 = N$, $\kappa_2 = 1$. The inequality (19) is equivalent to

$$\sigma_{\max}(\mathbf{H}(e^{j\omega})) \leq \gamma_s, \ \ \forall\omega \in [\omega_s, \pi]. \tag{21}$$

This makes Theorem 2 applicable. The trigonometric polynomial whose positivity defines $\mathscr{D} = [\omega_s, \pi]$ is

$$D_s(z) = 2\cos\omega_s - z - z^{-1}. \tag{22}$$

Since $\kappa_1 > \kappa_2$, we apply Theorem 2 for the transposed filter. It results that (19) holds if and only if there exist sum-of-squares

$$S_0(z) = \psi(z^{-1})^T \mathbf{Q}_0 \psi(z), \tag{23}$$

$$S_1(z) = \psi(z^{-1})^T \mathbf{Q}_1 \psi(z) \tag{24}$$

(note that these are polynomials with scalar coefficients) such that

$$\gamma_s^2 = S_0(z) + D_s(z)S_1(z), \tag{25}$$

$$\begin{bmatrix} \mathbf{Q}_0 & \mathbf{A}^T \\ \mathbf{A} & \mathbf{I}_N \end{bmatrix} \succeq 0. \tag{26}$$

In the $d = 1$ case, which applies to (23,24), the Gram matrix parameterization (11) has the form

$$s_k = \mathrm{tr}[\boldsymbol{\Theta}_k \mathbf{Q}], \tag{27}$$

where s_k are the (scalar) coefficients of the sum-of-squares and $\boldsymbol{\Theta}_k$ is the Toeplitz matrix with ones on diagonal k and zeros elsewhere. Using this parameterization and the particular form of the polynomial $D_s(z)$, relation (25) becomes

$$\gamma_s^2 \delta_k = \mathrm{tr}[\boldsymbol{\Theta}_k \mathbf{Q}_0] + \mathrm{tr}[(2\cos\omega_s \cdot \boldsymbol{\Theta}_k - \boldsymbol{\Theta}_{k-1} - \boldsymbol{\Theta}_{k+1})\mathbf{Q}_1], \tag{28}$$

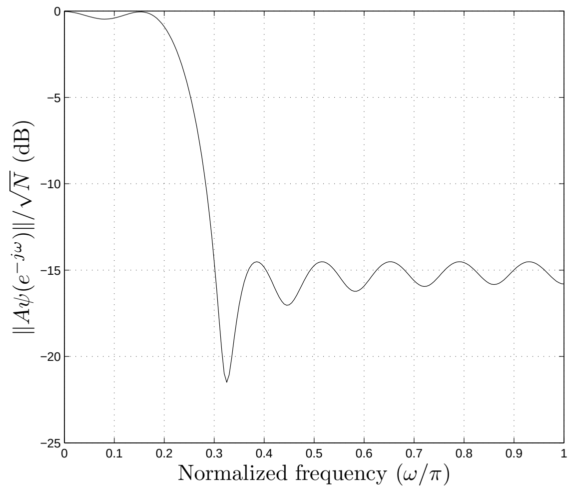

Figure 1: Power response of the matrix filter designed in Example 1.

for $k = 0 : N-1$ (δ_k is the Kronecker symbol). For the passband, a similar reasoning transforms (18) into

$$\gamma_p^2 \delta_k = \mathrm{tr}[\boldsymbol{\Theta}_k \tilde{\mathbf{Q}}_0] + \mathrm{tr}[(\boldsymbol{\Theta}_{k-1} + \boldsymbol{\Theta}_{k+1} - 2\cos\omega_p \cdot \boldsymbol{\Theta}_k)\tilde{\mathbf{Q}}_1], \tag{29}$$

$$\begin{bmatrix} \tilde{\mathbf{Q}}_0 & \mathbf{A}^T - \mathbf{I}_N \\ \mathbf{A} - \mathbf{I}_N & \mathbf{I}_N \end{bmatrix} \succeq 0. \tag{30}$$

The minimax optimization problem can be formulated as follows. Given the order N, stopband edge ω_p, passband edge ω_s, minimize the maximum passband and stopband error by solving the SDP problem

$$\begin{array}{rl} \min & \gamma_s^2 \\ \text{subject to} & (28), (26), (29), (30), \gamma_p^2 = \gamma_s^2 \\ & \mathbf{Q}_0 \succeq 0, \mathbf{Q}_1 \succeq 0, \tilde{\mathbf{Q}}_0 \succeq 0, \tilde{\mathbf{Q}}_1 \succeq 0 \end{array} \tag{31}$$

We note that the size of the matrices $\mathbf{Q}_0$ and $\tilde{\mathbf{Q}}_0$ is $N \times N$, while the size of $\mathbf{Q}_1$ and $\tilde{\mathbf{Q}}_1$ is $(N-1) \times (N-1)$. In (31), the passband and stopband errors are forced to be equal. In general, we can force a given ratio, or set one or both errors to preset values (in the latter case, the SDP problem requires only feasibility).

Example 1. We consider the specifications of the last example from [13], namely $N = 15$, $\omega_p = 0.2\pi$, $\omega_s = 0.3\pi$. The power response of the filter designed by solving (31) is shown in Figure 1. The optimal error is $\gamma_s/\sqrt{N} = -14.5$ dB. In [13], the stopband error was set to -12 dB, using semi-infinite optimization techniques. However, in [13] the passband error energy was optimized. Since this energy is a positive quadratic function of the elements of the matrix $\mathbf{A}$, it is easy to insert its optimization in the SDP problem (31). The time required for solving (31) was of about 4 seconds on a dual core PC at 1.86 GHz, with 4Gb memory.

3.2 2-D FIR deconvolution

In the general deconvolution scheme shown in Figure 2, the signal $\mathbf{s}$ passes through the channel $\mathbf{G}(\mathbf{z})$, whose model is known, and is contaminated by the noise η. We want to design a filter $\mathbf{X}(\mathbf{z})$ whose output $\hat{\mathbf{s}}$ approximates the ideal output $\mathbf{D}\mathbf{s}$. We assume that all filters are FIR. The output error

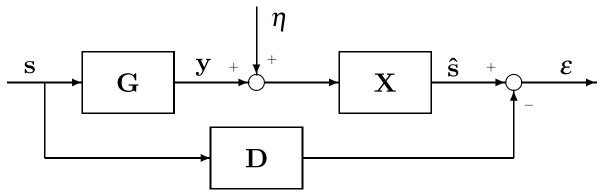

Figure 2: General deconvolution scheme.

is

$$\varepsilon = \hat{\mathbf{s}} - \mathbf{D}\mathbf{s} = (\mathbf{X}[\mathbf{G}\ \mathbf{I}] - [\mathbf{D}\ \mathbf{0}]) \begin{bmatrix} \mathbf{s} \\ \eta \end{bmatrix} \triangleq \mathbf{H} \begin{bmatrix} \mathbf{s} \\ \eta \end{bmatrix}. \quad (32)$$

The error function $\mathbf{H}(z)$ has the general form

$$\mathbf{H}(\mathbf{z}) = \mathbf{X}(\mathbf{z})\mathbf{A}(\mathbf{z}) - \mathbf{B}(\mathbf{z}), \quad (33)$$

where $\mathbf{A}(z)$, $\mathbf{B}(z)$ are given; in (32) we have $\mathbf{A}(\mathbf{z}) = [\mathbf{G}(\mathbf{z})\ \mathbf{I}]$, $\mathbf{B}(\mathbf{z}) = [\mathbf{D}(\mathbf{z})\ \mathbf{0}]$. Lacking knowledge on input and noise signals, the best way to control the output error is to minimize the norm of $\mathbf{H}(z)$, using inequalities like (2). Since the coefficients of $\mathbf{H}(z)$ depend linearly on those of $\mathbf{X}(z)$, the use of Theorem 2 transforms (2) into an LMI. Equality (15) does not depend on $\mathbf{X}(\mathbf{z})$, while in (16) $\overline{\mathbf{H}}$ is replaced by $\mathscr{L}(\mathbf{X})$ where $\mathscr{L}$ is the linear transformation that maps the coefficients of $\mathbf{X}(\mathbf{z})$ into those of $\mathbf{H}(\mathbf{z})$.

The optimization scheme outlined above can be used for several problems. Let us illustrate it for the case of 2-D H_∞ deconvolution of SISO systems. In this case, we have $d = 2$, $\kappa_1 = 1$, $\kappa_2 = 2$. We assume that $G(\mathbf{z})$ is FIR of order $\mathbf{n}_g$. We want to design the FIR filter $X(\mathbf{z})$ of order $\mathbf{n}_x$ such that the error norm inequality (2) hold for the smallest possible value γ. Taking into account the generalization of the parameterization (27) to the 2-D case (see [7, 3]), the use of Theorem 2 leads to the following optimization problem

$$\begin{array}{ll} \min & \gamma^2 \\ \text{s.t.} & \gamma^2 \delta_{k_1 k_2} = \operatorname{tr}[(\mathbf{\Theta}_{k_2} \otimes \mathbf{\Theta}_{k_1})\mathbf{Q}], \ \ -\mathbf{n} \le (k_1, k_2) \le \mathbf{n} \\ & \begin{bmatrix} \mathbf{Q} & \mathscr{L}(\mathbf{X}) \\ \mathscr{L}(\mathbf{X})^T & \mathbf{I}_2 \end{bmatrix} \succeq 0 \end{array} \quad (34)$$

where $\mathbf{n} = \mathbf{n}_g + \mathbf{n}_x$ is the degree of the error filter $H(\mathbf{z})$.

Example 2. We consider the example from [11], with

$$\begin{aligned} G(z_1, z_2) &= 0.1(z_1^{-1} + z_2^{-1})^3 + 0.1z_2^{-2} + 0.1z_2^{-1} + 8 \\ &= [1\ z_1^{-1}\ z_1^{-2}\ z_1^{-3}] \begin{bmatrix} 8 & 0.1 & 0.1 & 0.1 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ z_2^{-1} \\ z_2^{-2} \\ z_2^{-3} \end{bmatrix} \end{aligned}$$

and $D(\mathbf{z}) = 1$. Solving (34) with $\mathbf{n}_x = (2, 2)$, we obtain an optimal value of the H_∞ error norm of $\gamma = 0.1379 = -17.2$ dB. The error frequency response $\sigma_{\max}(e^{j\omega})$ is shown in Figure 3. Increasing the degree of $X(\mathbf{z})$ does not improve the result. For comparison, the state-space approach from [11] gives an error of 0.15 for a system of degree $(3, 3)$.

In this example we have used a global error bound, i.e. $\mathscr{D} = \mathbb{T}^d$ in Theorem 2, as so the error surface from Figure 3 is equiripple. By enforcing equalities (2) with different values of γ on different domains, it is possible to shape the error.

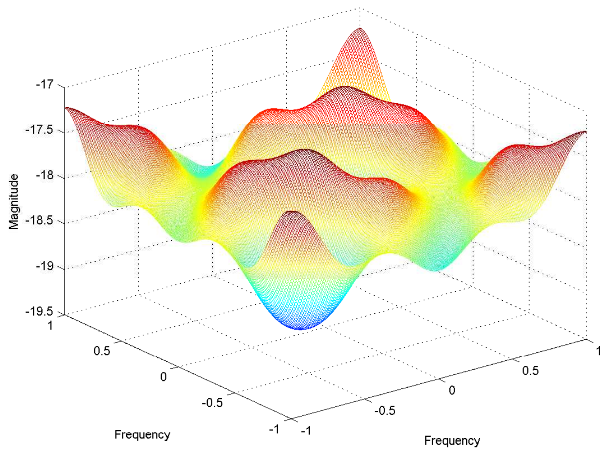

Figure 3: Frequency response of input-output error in the deconvolution scheme optimized in Example 2.

3.3 Design of 2-D MIMO filters

The design of lowpass MIMO multidimensional filters was discussed in [12] in the following setup. Given a desired response $\mathbf{D}(\mathbf{z})$ and a passband error bound γ_p and assuming that the passband and stopband have rectangular shapes defined by only two frequencies, ω_p and ω_s, find the optimal minimax filter (1) which is the solution of the problem

$$\begin{array}{ll} \min & \gamma_s \\ \text{s.t.} & \sigma_{\max}(\mathbf{H}(e^{j\omega}) - \mathbf{D}(e^{j\omega})) \le \gamma_p, \ \ \forall |\omega_i| \le \omega_p, \ i = 1:d \\ & \sigma_{\max}(\mathbf{H}(e^{j\omega})) \le \gamma_s, \ \ \exists i \in 1:d, |\omega_i| \ge \omega_s \end{array} \quad (35)$$

If $\mathbf{D}(\mathbf{z})$ is a FIR system (typically a constant matrix or a delay matrix), this problem can be expressed in SDP form using Theorem 2. The passband $\mathscr{D}_s$ can be described by a set (12) with $D_\ell(\mathbf{z}) = z_\ell + z_\ell^{-1} - 2\cos\omega_p$, $\ell = 1:d$. The stopband is a union

$$\mathscr{D}_s = \bigcup_{i=1}^{d} \mathscr{D}_{s,i},$$

with

$$\mathscr{D}_{s,i} = \{\mathbf{z} \in \mathbb{T}^d \mid D_s(z_i) \ge 0\}, \ i = 1:d,$$

where $D_s(\cdot)$ is the polynomial (22). The problem (35) is equivalent to

$$\begin{array}{ll} \min & \gamma_s \\ \text{s.t.} & \sigma_{\max}(\mathbf{H}(\mathbf{z}) - \mathbf{D}(\mathbf{z})) \le \gamma_p, \ \ \forall \mathbf{z} \in \mathscr{D}_p \\ & \sigma_{\max}(\mathbf{H}(\mathbf{z})) \le \gamma_s, \ \ \forall \mathbf{z} \in \mathscr{D}_{s,1} \\ & \vdots \\ & \sigma_{\max}(\mathbf{H}(\mathbf{z})) \le \gamma_s, \ \ \forall \mathbf{z} \in \mathscr{D}_{s,d} \end{array} \quad (36)$$

Each of the constraints of (36) can be transformed into an LMI via Theorem 2. We note that similar problems can be obtained for passband and stopbands that are not rectangular (see [3] for examples of other shapes), while the results from [12] cannot be apparently generalized.

Example 3. The particular case treated in [12] is 2-D ($d = 2$), with $\mathbf{D}(\mathbf{z}) = \mathbf{I}_2$. So, the MIMO systems has $\kappa_1 = 2$ inputs and $\kappa_2 = 2$ outputs. The design specifications are $\gamma_p = 0.1$, $\omega_p = 0.4\pi$, $\omega_s = 0.9\pi$. Due to the form of the desired response, the intuitive solution of (35) should be a

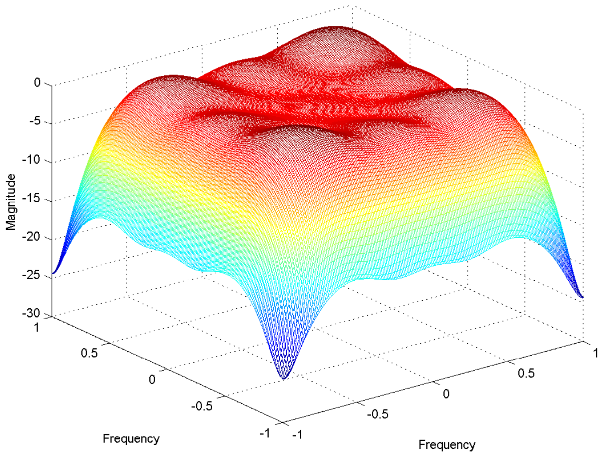

Figure 4: Frequency response of 2-D scalar filter lying on the diagonal of the optimal MIMO filter designed in Example 3.

diagonal filter, with identical scalar filters on the diagonal. This is indeed what we have obtained by solving (36). The frequency response of such a scalar filter of order $\mathbf{n} = (4,4)$ is shown in Figure 4. The optimal stopband error is $\gamma_s = 0.44$. The responses of the "cross-channels" filters (input 1 to output 2 and input 2 to output 1) are negligibly small. In contrast, the FIR filters designed in [12] are not diagonal and the optimal error is larger than 0.7. (The results in [12] are based on a state space implementation and so are difficult to compare to ours other but by examples.) Of course, it is more sensible to choose $\mathbf{D}(\mathbf{z}) = \mathrm{diag}(z_1^{-\tau_1} z_2^{-\tau_2})$; by taking $\tau_1 = \tau_2 = 2$, the optimal solution (again diagonal) has an error $\gamma_s = 0.028$.

3.4 Implementation details

We have implemented the SDP problems discussed in this section using the convex optimization library CVX [6]. We have taken advantage of the possibility to describe convex sets in CVX, and built functions for sum-of-squares polynomials, polynomials that are positive on domains (as in Theorem 1) and for the BRL described by Theorem 2. In the latter case, the variables are γ^2 and the (vectorized) coefficients of the filter (1). Although it might add a small computational overhead, this hierarchical construction leads to the very easy programming of end user applications. For example, the CVX program designing matrix filters based on the constraints (18) and (19) has only 12 easily readable lines. The direct implementation of the SDP problem (31) would have more than hundred lines.

4. CONCLUSIONS

We have presented a Bounded Real Lemma for trigonometric polynomials with matrix coefficients, as described by Theorem 2. Its LMI form allows the transformation of several optimization problems involving FIR filters into SDP form. We have given three examples of applications (matrix filter design, 2-D deconvolution and 2-D MIMO filter design), which cover several features of our BRL. In the first example, the filters depends on a single variable, while in the others there are two variables. In the second example, the BRL is global (holds on $\mathbb{T}^d$), while in the others the BRL holds on frequency domains (intervals in the first example). Finally, the implementation is modular and further applications can be programmed without intimate knowledge of the theory described here.

Further work will be devoted to a Positivstellensatz for polynomials with matrix coefficients, i.e. towards an LMI form of the condition $\det \mathbf{H}(\mathbf{z}) \neq 0$, $\forall \mathbf{z} \in \mathscr{D}$, without computation of the determinant.

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