ACCURATE GEOLOCATION IN THE PRESENCE OF OUTLIERS USING LINEAR PROGRAMMING

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ABSTRACT

Precise geolocation have attracted considerable interest in the engineering literature. Almost all previous publications consider small measurement errors. In this paper we discuss geolocation in the presence of outliers, where several measurements are severely corrupted while other measurements are reasonably accurate. It is known that Maximum Likelihood or Least Squares provide poor results under these conditions. We demonstrate how using the ℓ_1 norm and linear programming we can detect the outliers and use only the good measurements for providing the final location estimate. Moreover, we provide bounds on the number of outliers that can be detected and eliminated.

1. INTRODUCTION

This work has been motivated by a frequent problem in a large scale commercial geolocation system mainly used for tracking stolen luxury cars, controlling vehicle fleets and personal navigation [1]. The system consists of sensors distributed over a large geographical area, very much like rural cellular base-stations. The stations intercept signals transmitted by the mobile devices. The time synchronized receiving stations record the mobile device signal time of arrival (TOA). The TOA measurements are transferred to a central processing unit for obtaining an estimate of the mobile device location. Geolocation based on TOA has been known for long time and is a standard non-linear estimation problem with known solutions [2]. However, the commercial system mentioned above provides between 7 and 40 TOA measurements for each mobile device where a considerable number of these measurements (up to 20%) are unreliable (outliers). The large errors may be the result of multipath, interference, jamming, or even poor synchronization of some stations. In order to provide a reliable location estimate it is required to identify the outliers (or, equivalently, the set of consistent measurements) and either correct the large errors or eliminate the wrong measurements from the data selected for location estimation. The identification of the outliers is known to be NP hard and therefore the relatively simple problem of location estimation based on TOA requires excessive computer resources in the presence of outliers. For example, the identification of k (say 10) outliers out of m (say 40) TOA measurements requires the examination of $\binom{m}{m-k} = \binom{40}{30} = 847,660,528$ subsets of measurements when k is known. Since the number of outliers is not known in advance it is required to examine even more subsets.

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We propose to identify the outliers by using recent results in sparse representation of signals. Consider the linear set of equations $\mathbf{s} = \mathbf{B}\mathbf{r}$ where $\mathbf{s}$ is a vector that has a representation $\mathbf{r}$ in the span of $\mathbf{B}$, a.k.a. as dictionary. In some applications the sparsest $\mathbf{r}$ is desired. The representation is sparse if the dictionary is over complete, i.e., $\mathbf{B}$ has more columns than rows. In this case, $\mathbf{s} = \mathbf{B}\mathbf{r}$ is an underdetermined set of equations that cannot be solved uniquely by classical methods. In recent publications, it has been shown that if the sparsest representation is sufficiently sparse, then not only it is unique, but it can also be easily obtained by linear programming [3]-[5]. Conditions ensuring uniqueness of the sparsest solution were established in [6]. The problem has been extended to a wider variety of bases in [7],[8] including dictionaries built by concatenation of non-unitary matrices [9],[10]. Since these approaches are all based on ℓ_1-norm minimization, the validity of sparse representation by ℓ_1-norm relaxation was also studied in [11]. The theory of sparse signal representation has also been used for outliers identification in [12].

Recent works have exploited these advances by formulating geolocation as a sparse representation problem involving the signals collected at several sensors [13]. This method is known as Compressive Sensing [14], and sparsity is considered at the level of array signal processing.

Our approach is different since we address a problem occurring after the extraction of location parameters from the received signals. We assume that TOA, Angle of Arrival (AOA), Time Difference of Arrival (TDOA) or Received Signal Strength (RSS) measurements are available, although a subset is severely corrupted. In a linear formulation of the geolocation problem, an over-determined set of equations must be solved.

The solution that we are proposing here consists of several ingredients. First, we write the quadratic equations of the TOA measurements as linear equations. See [15]-[16] for earlier applications of transforming non-linear equations to linear equations in geolocation applications. Second, we define an optimization problem in terms of ℓ_1 norm minimization. Third, we solve the optimization problem using efficient linear programming methods. As a result we get a list of outliers, the size of each error and location estimate. Using the location estimate and the list of good measurements one can now apply a final, non-linear, localization step using digital terrain map (DTM) for the ultimate result.

The method described above can also be used for other geolocation systems such as geolocation based on AOA or geolocation based on RSS. The main goal of this paper is to find the limitations of the proposed outliers identification methods.

2. OVERVIEW

It is well known that geolocation based on AOA, TOA, TDOA and RSS can be based on a solution of a linear set of N equations (more details will be provided shortly),

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e} \tag{1}$$

The vector $\mathbf{x}$ is the vector of unknown parameters which obviously include the coordinates of the transmitter to be located. The matrix $\mathbf{A}$ and the $\mathrm{N} \times 1$ data vector $\mathbf{y}$ are functions of the available measurements and the known parameters such as the coordinates of the sensing devices. The entries of the error vector $\mathbf{e}$ are assumed to be zero (or "small") except for a few entries which are large and therefore referred to as outliers. The distinction between "small" and "large" errors is not obvious and therefore we assume, as a first attempt to solve the problem at hand, that most of the entries in $\mathbf{e}$ are zero and only few are different from zero. Thus, the error vector $\mathbf{e}$ is sparse. We search for the vector $\mathbf{x}$ that is associated with the minimal number of outliers. Define [12],

$$(P_0): \quad \min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_0 \tag{2}$$

$$(P_1): \quad \min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_1 \tag{3}$$

where $\|\cdot\|_0$ stands for the ℓ_0 -norm (the number of vector elements that are not zero) and $\|\cdot\|_1$ stands for the ℓ_1-norm (the sum of absolute values of the vector elements). Formally, we look for the solution of the minimization problem (P_0). Unfortunately, this problem is NP hard, and therefore one may consider solving (P_1). The problem (P_1) can be solved by linear programming and according to our numerical experience works amazingly well.

The challenge is to establish a condition ensuring that the solution of the tractable problem (P_1) is also the solution of the hard problem (P_0). In this paper, we derive this condition and show that it depends on the number of outliers. The idea is to identify a matrix $\mathbf{F} \neq \mathbf{0}$ so that $\mathbf{F}\mathbf{A} = \mathbf{0}$. Then equation (1) reduces to $\mathbf{F}\mathbf{y} = \mathbf{F}\mathbf{e}$. Define

$$(P_0'): \quad \min_{\mathbf{e}} \|\mathbf{e}\|_0 \quad s.t. \quad \mathbf{F}\mathbf{y} = \mathbf{F}\mathbf{e} \tag{4}$$

$$(P_1'): \quad \min_{\mathbf{e}} \|\mathbf{e}\|_1 \quad s.t. \quad \mathbf{F}\mathbf{y} = \mathbf{F}\mathbf{e} \tag{5}$$

In [12] it has been shown that the solution $\mathbf{e_0}$ of the alternative problem (P_0') yields the solution $\mathbf{x_0}$ of (P_0). Since the ℓ_0-norm problem (P_0') is hard, it is replaced by the alternative problem (P_1'). Define the matrix $\mathbf{U}$ whose columns are the normalized eigenvectors of $\mathbf{A}\mathbf{A}^T$ associated with the non-zero eigenvalues. We propose $\mathbf{F} = \mathbf{I}_N - \mathbf{U}\mathbf{U}^T$ where $\mathbf{I}_N$ is the $N \times N$ identity matrix. We look for a condition ensuring that the solution of (P_1') is identical to the solution of (P_0'). Denote by $\mathbf{e_0}$ the solution of (P_0') and by S the set of indices corresponding to non-zero entries of $\mathbf{e_0}$ i.e., $S = \{i \in [1,...,N] | e_0(i) \neq 0\}$. Denote by $\|.\|_2$ the ℓ_2-norm, by s the number of elements in S and by $\delta(S)$ the $s \times 1$ vector obtained from a vector δ by removing the entries δ_i, $i \notin S$.

Lemma 1 *If any vector $\delta \neq \mathbf{0}$ that satisfies $\mathbf{F}\delta = \mathbf{0}$ also satisfies $\|\delta(S)\|_1 < \frac{1}{2}\|\delta\|_1$ then the solution of (P_0') coincides with the solution of (P_1').*

Lemma 2 *Any $\delta \neq \mathbf{0}$ that satisfies $\mathbf{F}\delta = \mathbf{0}$ also satisfies* $\|\delta(S)\|_1 \leq s\|\delta\|_1 \max_{\ell}\left\{\|\mathbf{U}(\ell,:)\|_2^2\right\}$.

For the proofs of the lemmas, see [6], [8], [17] and [18]. By combining the two lemmas we can get a condition that guarantees equivalence between (P_1') and (P_0'). This condition is $s\|\delta\|_1 \max_{\ell}\{\|\mathbf{U}(\ell,:)\|_2^2\} < \frac{1}{2}\|\delta\|_1$which proves Theorem 1.

Theorem 1 *If the number of non-zero entries, s, in the solution of (P_0') satisfies*

$$s < \frac{1}{2\max_{\ell}\|\mathbf{U}(\ell,:)\|_2^2} \triangleq B \tag{6}$$

Then (P_1') and (P_0') have the same solution and the outliers can be identified exactly.

The bound above depends on $\mathbf{U}$ which is strongly related to $\mathbf{A}$. In the next section, we establish the structure of the matrix $\mathbf{A}$ for AOA, TOA and TDOA data. Then, in sections 4, 5 and 6 we find corresponding expressions for $\mathbf{U}$. The effects of small errors on outliers identification are discussed in section 8. Finally, numerical results are given in section 9.

3. REPRESENTATION OF GEOLOCATION BY LINEAR EQUATIONS

In this section we cast the usually non-linear geolocation equations as a set of linear equations.

3.1 Geolocation based on AOA

Assume that N stations collect AOA measurements of a single target. Let $\mathbf{p}_i = [x_i, y_i]^T$ represent the i-th station coordinates and $\mathbf{p}_t = [x_t, y_t]^T$ the target coordinates. Let ϕ_i be the AOA (measured anticlockwise w.r.t. the x-axis) at the i-th station. Then, in the absence of noise, the i-th AOA measurement satisfies $(y_t - y_i)\cos\phi_i = (x_t - x_i)\sin\phi_i$. Define

$$\begin{array}{llll} \mathbf{s} & \triangleq [\sin\phi_1, \cdots, \sin\phi_N]^T & \mathbf{c} & \triangleq [\cos\phi_1, \cdots, \cos\phi_N]^T \\ \mathbf{A} & \triangleq [\mathbf{s}, -\mathbf{c}] & \mathbf{x} & \triangleq [x_t, y_t]^T \\ \mathbf{y} & \multicolumn{3}{l}{\triangleq [(x_1\sin\phi_1 - y_1\cos\phi_1), \cdots, (x_N\sin\phi_N - y_N\cos\phi_N)]^T} \end{array}$$

Then, $\mathbf{y} = \mathbf{A}\mathbf{x}$, and in the presence of errors $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e}$.

3.2 Geolocation based on TOA

Let m_i be the measured TOA, c the propagation speed, and t_0 the *known* transmit time. In the absence of errors, the i-th TOA measurement is given by $m_i = c^{-1}\|\mathbf{p}_i - \mathbf{p}_t\|_2 + t_0$. Substituting the target and sensor coordinates we get, $c^2(m_i - t_0)^2 = \|\mathbf{p}_i\|^2 + \|\mathbf{p}_t\|^2 - 2\mathbf{p}_i^T\mathbf{p}_t$. Collecting the measurements from N sensors, we get $\mathbf{y} = \mathbf{A}\mathbf{x}$ with

$$\begin{array}{ll} \mathbf{y} & \triangleq [\|\mathbf{p}_1\|^2, \cdots, \|\mathbf{p}_N\|^2]^T - c^2\left(\mathbf{m}\odot\mathbf{m} + t_0^2\mathbf{1}_N - 2t_0\mathbf{m}\right) \\ \mathbf{A} & \triangleq \left[2[\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_N]^T, \mathbf{1}_N\right] \qquad \mathbf{x} \triangleq [\mathbf{p}_t^T, -\|\mathbf{p}_t\|^2]^T \end{array}$$

where $\mathbf{m}\odot\mathbf{m}$ is the element-wise product, and $\mathbf{1}_N$ is an $N \times 1$ vector of ones. In the presence of errors we get $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e}$.

3.3 Geolocation based on TDOA

If the transmit time, t_0 , is unknown the localization method is called TDOA. In this case we have,

$$\begin{array}{ll} \mathbf{y} & \triangleq [\|\mathbf{p}_1\|^2, \cdots, \|\mathbf{p}_N\|^2]^T - c^2(\mathbf{m}\odot\mathbf{m}) \\ \mathbf{A} & \triangleq \left[2[\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_N]^T, \mathbf{1}_N, c\mathbf{m}\right] \\ \mathbf{x} & \triangleq [\mathbf{p}_t^T, (c^2t_0^2 - \|\mathbf{p}_t\|^2), -2ct_0]^T \end{array}$$

Again, $\mathbf{y} = \mathbf{A}\mathbf{x}$, and in the presence of errors $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e}$.

4. NUMBER OF IDENTIFIABLE OUTLIERS IN AOA GEOLOCATION

We now establish the bound on the number of outliers s given in (6) for AOA geolocation. As shown in Section 3, we have $\mathbf{A} = [\mathbf{s}, -\mathbf{c}]$. The eigenvectors of $\mathbf{A}$ are $\mathbf{c}/\|\mathbf{c}\|$ and $\mathbf{s}/\|\mathbf{s}\|$ provided that $\mathbf{c}^T\mathbf{s} = 0$. This orthogonality condition can always be satisfied by rotating the coordinate system [18]. Thus, $\mathbf{U} = [\mathbf{s}/\|\mathbf{s}\|, \mathbf{c}/\|\mathbf{c}\|]$. According to theorem 1,

$$B = \frac{\|\mathbf{c}\|^2\|\mathbf{s}\|^2}{2\max_{\ell}\left(\|\mathbf{c}\|^2\sin^2\phi_\ell + \|\mathbf{s}\|^2\cos^2\phi_\ell\right)} \tag{7}$$

4.1 Application to Uniform Distribution of Sensors

Assume that the sensors are uniformly distributed within a circular area whose radius is R and its center is at $(0,0)$. Denote by (r_ℓ, θ_ℓ) the polar coordinates of the ℓ-th sensor. The angles θ_ℓ are uniformly distributed in $[0, 2\pi]$ and the ranges r_ℓ are distributed in $[0, R]$ according to the probability density function $f_r(r) = \frac{2r}{R^2}$. For large N, $\|\mathbf{c}\|^2$ and $\|\mathbf{s}\|^2$ tend to $\frac{N}{2}$. Then $B(N) = \frac{N}{4} + o(N)$.

4.2 Application to Circular Sensor Array

The sensors are uniformly distributed on a circle with radius R. The coordinates of the ℓ-th sensor are $x_\ell = R\cos\theta_\ell$, $y_\ell = R\sin\theta_\ell$ where $\theta_\ell = 2\pi\ell/N$. Assume that the transmitter is located on the x-axis at coordinates $(d, 0)$ where $d > 0$. We show in [18] that for large N the bound tends to $N/4$ for a transmitter within the circle of sensors, and to $N/6$ for a remote transmitter. Furthermore, for any N, including small N, if $d \in [0, R]$ then $B(N) \geq \frac{N}{4} - \frac{1}{2}$.

4.3 Application to Concentric Circular Sensor Arrays

Consider K concentric circles. Assume that N_k sensors are located on the k-th circle and that $N_k = R_k N_1$, where $N_1 \gg 1$ is the number of sensors on the smallest circle whose radius is $R_1 = 1$. Further, assume that $R_1 < R_2 < \cdots < R_K$. Then, for a transmitter located at coordinates $(d, 0)$,

$$B_a(N) = \begin{cases} \frac{N}{4} & d \leq R_1 \\ \frac{N_1}{4}\left(\frac{1}{d^2}\sum_{j=1}^{k-1} R_j^3 + \sum_{j=k}^{K} R_j\right) & d \in [R_{k-1}, R_k] \\ \frac{N}{2}\left(2R_K^2\frac{\sum_{j=1}^{K} R_j}{\sum_{j=1}^{K} R_j^3} + 1\right)^{-1} & d \gg R_K \end{cases}$$

converges asymptotically to a bound stricter than $B(N)$ [18].

5. NUMBER OF IDENTIFIABLE OUTLIERS IN TOA GEOLOCATION

As shown in section 3, $\mathbf{A} = [2\mathbf{x}_b, 2\mathbf{y}_b, \mathbf{1}_N]$ where $\mathbf{x}_b, \mathbf{y}_b$ are the x and y coordinates of all the sensors. The eigenvectors of $\mathbf{A}$ are $\mathbf{x}_b/\|\mathbf{x}_b\|$, $\mathbf{y}_b/\|\mathbf{y}_b\|$ and $\mathbf{1}_N/\sqrt{N}$ provided that $\mathbf{1}_N^T\mathbf{x}_b = \mathbf{1}_N^T\mathbf{y}_b = \mathbf{x}_b^T\mathbf{y}_b = 0$. These orthogonality assumptions can always be satisfied by a coordinate system rotation and an addition of an appropriate artificial outlier measurement [18], which slightly loosens the bound but in turn yields closed form expressions. Then $\mathbf{U} = \left[\frac{\mathbf{x}_b}{\|\mathbf{x}_b\|}, \frac{\mathbf{y}_b}{\|\mathbf{y}_b\|}, \frac{\mathbf{1}_N}{\sqrt{N}}\right]$. According to Theorem 1,

$$B = \frac{\|\mathbf{x}_b\|^2\|\mathbf{y}_b\|^2}{2\max_{\ell}\left\{\|\mathbf{y}_b\|^2 x_\ell^2 + \|\mathbf{x}_b\|^2 y_\ell^2\right\} + 2\|\mathbf{x}_b\|^2\|\mathbf{y}_b\|^2/N} \tag{8}$$

5.1 Application to Uniform distribution of sensors

Assume that the sensors are uniformly distributed as in 4.1. Then $\|\mathbf{x}_b\|^2 = \|\mathbf{y}_b\|^2 = \frac{NR^2}{4} + o(N)$. Thus, $B(N) = \frac{N}{10} + o(N)$ without any limitation on the location of the transmitter.

5.2 Application to Uniform Circular Sensor Array

Consider a uniform circular array of N sensors as in 4.2. Then $\|\mathbf{x}_b\|^2 = \|\mathbf{y}_b\|^2 = NR^2/2$. From (8) we obtain the bound on the number of outliers is $B = N/6$.

5.3 Application to Concentric Circular Sensor Arrays

Consider L uniform concentric circular sensor arrays centered on $(0,0)$. Assume that the ℓ-th circular array consists of $N_\ell \geq 3$ sensors. The coordinates of the k-th sensor of the ℓ-th array of radius r_ℓ are $\left[r_\ell\cos(\theta_{\ell,k} + \alpha_\ell), r_\ell\sin(\theta_{\ell,k} + \alpha_\ell)\right]$ where $\alpha_\ell \in [0, 2\pi]$ and $\theta_{\ell,k} = 2\pi k/N_\ell$ for $k = 1, \ldots, N_\ell$. Define $r^2 \triangleq \frac{1}{N}\sum_{\ell=1}^{L} N_\ell r_\ell^2$. Then we get from (8) the bound $B = N/(4\max\left\{r_\ell^2/r^2\right\} + 2)$ for any N.

5.4 Application to Uniform Linear Sensor Array

Consider a uniform linear array of sensors placed on the x axis between $-R/2$ and $R/2$. The sensor separation is $R/(N-1)$. Observe that $\|\mathbf{y}_b\|^2 = 0$ and $\|\mathbf{x}_b\|^2 = R^2\frac{N}{12}\frac{N+1}{N-1}$. Since $\mathbf{y}_b = \mathbf{0}$ the eigenvectors of the matrix $\mathbf{A}$ are $\mathbf{U} = [\frac{\mathbf{x}_b}{\|\mathbf{x}_b\|}, \frac{\mathbf{1}_N}{\sqrt{N}}]$. Then, $B = N/(6\frac{N-1}{N+1} + 2)$ and B tends to $\frac{N}{8}$.

6. NUMBER OF IDENTIFIABLE OUTLIERS IN TDOA GEOLOCATION

As shown in section 3, $\mathbf{A} = [2\mathbf{x}_b, 2\mathbf{y}_b, \mathbf{1}_N, c\mathbf{m}]$. The eigenvectors of $\mathbf{A}$ are $\frac{\mathbf{x}_b}{\|\mathbf{x}_b\|}$, $\frac{\mathbf{y}_b}{\|\mathbf{y}_b\|}$, $\frac{\mathbf{1}_N}{\sqrt{N}}$ and $\frac{\mathbf{m}}{\|\mathbf{m}\|}$ provided that they are orthogonal. These orthogonality conditions can always be satisfied by coordinate system rotation and translation and an addition of an appropriate artificial outlier measurement [18]. Under these orthogonality conditions the bound is

$$B = \left(2\max_{\ell}\left\{\frac{x_\ell^2}{\|\mathbf{x}_b\|^2} + \frac{y_\ell^2}{\|\mathbf{y}_b\|^2} + \frac{m_\ell^2}{\|\mathbf{m}\|^2}\right\} + \frac{2}{N}\right)^{-1} \tag{9}$$

6.1 Application to Uniform Distribution of Sensors

Consider a uniform distribution of sensors as in 5.1. Without loss of generality, select $t_0 = -\frac{E\{r_\ell\}}{c}$. We show in [18] that $B(N) = \frac{N}{18} + o(N)$.

6.2 Application to Uniform Circular Sensor Array

Consider a uniform circular array of sensors as in 4.2. Again, assume that $t_0 = -\frac{E\{r_\ell\}}{c}$ where r_ℓ is the range between the transmitter and the ℓ-th sensor. We show in [18] that for $N \geq 3$ the bound is $B(N) = \left(2\max_{\ell}\frac{m_\ell^2}{\|\mathbf{m}\|^2} + \frac{6}{N}\right)^{-1}$ and

$$B(N) = \begin{cases} N/6 & d = 0 \\ N/10 & d = 0^+ \text{ or } d \gg R \\ Np/(4p+4) + o(N) & d = R \quad \text{where} \quad p = 2 - \frac{16}{\pi^2} \end{cases}$$

Further, for finite N a good approximation for B is

$$B_a(N) = \begin{cases} (N/10)\left(1 - \frac{1}{3}\frac{d}{R}\right) & 0 < d \leq R \\ (N/10)\left(1 - \frac{R}{5d-2R}\right) & d > R \end{cases}$$

7. HIDDEN ERRORS

We show now that some measurement errors do not affect the consistency of the linear equations set. According to section 3, the AOA k-th linear equation is $\{\mathbf{y}\}_k = x_k \sin\phi_k - y_k \cos\phi_k$ where ϕ_k is the errorless AOA measurement. In the presence of perturbations, denote by ε_k the k-th measurement error and by $\hat{\phi}_k = \phi_k + \varepsilon_k$ the k-th actual measurement. We show in [18] that $e_k = 2r_k \sin(\varepsilon_k/2)\cos(\varepsilon_k/2 + \phi_k - \theta_k)$.The cosine term is zero for $\varepsilon_k = 2(\theta_k - \phi_k) \pm \pi$. Then $e_k = 0$ although $\varepsilon_k \neq 0$. Thus, there are cases of errors in the measurements that do not appear as an error in the linear equations and therefore do not affect the correct solution.

Similarly, according to section 3, and under the assumption that $t_0 = 0$, the exact TOA measurements are $m_k = \frac{1}{c}\|\mathbf{p}_k - \mathbf{p}_t\|$ and the corresponding linear equation is $\{\mathbf{y}\}_k = \|\mathbf{p}_k\|^2 - c^2 m_k^2$. If the k-th TOA measurement $\hat{m}_k$ is imprecise, we have $\hat{m}_k = m_k + \frac{\varepsilon_k}{c}$ where ε_k is the ranging error. Then $e_k = -\varepsilon_k(2\|\mathbf{p}_k - \mathbf{p}_t\| + \varepsilon_k)$, see [18] for details. If $\varepsilon_k = -2\|\mathbf{p}_k - \mathbf{p}_t\|$ then $e_k = 0$, meaning that this measurement error does not affect the linear equations and their solution. Note that this measurement error corresponds to $\hat{m}_k < t_0$ and therefore is likely to be removed from the TOA data set. If t_0 is unknown, as in TDOA, the error cannot be detected easily before emitter localization.

We have revealed the existence of hidden errors that affect the measurements and therefore the nonlinear estimation problem, but do not affect the linear set of equations. Of course, hidden errors are not likely to occur since they must satisfy the condition $e_k = 0$ described above. However, the probability of small values for e_k may not be negligible. Thus, we extend the concept of hidden outliers to any large ε_k for which e_k is small.

Proposition 2 Robustness to Hidden Outliers - *In the presence of small errors, solving nonlinear localization problems by means of a linear set of equations ensures robustness to hidden outliers.*

8. STABILITY IN THE PRESENCE OF SMALL ERRORS

Recall that our model in (1) assumed that the error vector $\mathbf{e}$ consists of a few large entries called outliers and all other entries are zero. This model assumes that small errors do not exist. The original problem can be written as $(P'_{1,0})$ where

$$(P'_{1,\varepsilon}): \quad \min_{\mathbf{e}} \|\mathbf{e}\|_1 \quad s.t. \quad \|\mathbf{F}\mathbf{y} - \mathbf{F}\mathbf{e}\|_2^2 \leq \varepsilon$$

In the presence of small errors we add a non-sparse vector $\mathbf{n}$, $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e} + \mathbf{n}$. Assuming bounded small errors, projection using $\mathbf{F}$ yields $\mathbf{F}\mathbf{y} = \mathbf{F}\mathbf{e} + \mathbf{F}\mathbf{n}$ and $\|\mathbf{F}\mathbf{n}\| \leq \varepsilon$. Thus we are interested in solving $(P'_{1,\varepsilon})$ with $\varepsilon > 0$. Denote by $\mathbf{e}$ and $\mathbf{e}_n$ the solutions of $(P'_{1,\varepsilon})$ in the absence and in the presence of small errors. By continuity of the penalty and constraint functions, we can justify the existence of a function $f(\varepsilon)$ so that $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$ and $|e(k) - e_n(k)| < f(\varepsilon)$ for any k. For a non-outlier entry we have $e(k) = 0$ and $|e_n(k)| < f(\varepsilon)$. Thus, the solutions $\mathbf{e}$ and $\mathbf{e}_n$ have identical support provided that all the entries of $\mathbf{e}_n$ satisfying $|e_n(k)| < f(\varepsilon)$ are set to 0. Therefore, outliers identification in the presence of sufficiently small errors can be achieved by linear programming.

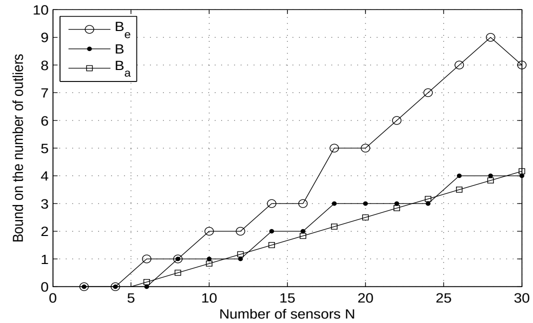

Figure 1: The experimental bound, (B_e), the theoretical bound on the number of outliers, (B), and its approximation, (B_a), versus the number N of TOA sensors in the circular array

9. NUMERICAL EXAMPLES

In previous sections, we discussed a sufficient condition ensuring the identification of outliers within a set of AOA, TOA or TDOA measurements by means of linear programming. The condition limits the number of identifiable outliers by an upper bound. In this section we examine the tightness of the upper bound by numerical simulations.

9.1 Outliers identification in TOA models

Recall that a range measurement error ε_k generates a linear equation error $e_k = -\varepsilon_k(2r_k + \varepsilon_k)$ where r_k is the distance of the k-th sensor from the transmitter. We define an error as an outlier if $|\varepsilon_k| \geq \alpha r_k$ with $\alpha = 0.1$ Furthermore, for a given topology one can generally establish an upper bound on the largest distance between the transmitter and the sensors. Range measurements with excessively large errors can be removed before processing. Thus, one can define a topology-dependent scalar D so that $0 \leq r_k + \varepsilon_k \leq D$ for any k. Thus, an outlier error e_k resulting from ε_k, satisfies

$$e_k \in \left[-D^2 + r_k^2,\ -\alpha(2+\alpha)r_k^2\right] \cup \left[\alpha r_k^2,\ (2-\alpha)r_k^2\right] \qquad (10)$$

In the simulations the outlier error have been uniformly distributed within the interval defined in (10).

In this example, small errors have not been simulated. Consider a uniform circular array of TOA sensors. Assume the array has a radius of 1 distance unit [DU] and that the emitter is located within this array. The maximum distance between the transmitter and the sensors is 2 [DU]. Thus, $D = 2$ [DU]. The bound $B_a = N/6$ is independent of the transmitter location. For each of the $N_e = 200$ experiments the transmitter location is selected at random within the circular area of the sensor array, and we identify the smallest number of outliers for which identification fails. The result of the i-th experiment is denoted by k_i. The experimental bound is $B_e = \min_{1 \leq i \leq N_e} k_i$. Denote by $\hat{\mathbf{e}}$ the estimate of $\mathbf{e}$ by solving (P'_1) using linear programming. We define failure when some outliers are not identified or some good measurements are classified as outliers. Since the experiments are based on numerical solutions, we define perfect outliers identification when $\max_{1 \leq i \leq N} |\hat{e}_i - e_i| < 10^{-6}$. In Fig. 1, we plotted the bounds B_e, B and B_a versus N. The bound B appears to be rather tight.

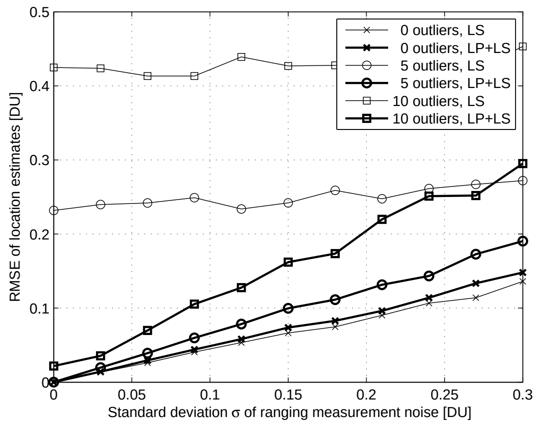

Figure 2: RMSE of LS location estimates versus standard deviation of ranging small errors. Thin lines show results of straightforward LS while thick lines show results with outliers removal step.

9.2 Localization in the Presence of Small Errors and Outliers

In this section we examine the precision of least squares (LS) location estimates with and without outliers identification. The measurements in the presence of noise (small errors) and outliers are given by $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e} + \mathbf{n}$. The LS estimate of $\mathbf{x}$ is given by $\hat{\mathbf{x}}_0 = \mathbf{A}^\dagger \mathbf{y}$ where $\mathbf{A}^\dagger$ is the pseudo-inverse of $\mathbf{A}$. A better way to estimate $\mathbf{x}$ may be to first apply linear programming in order to estimate the error vector $\widehat{\mathbf{e}+\mathbf{n}}$. Then apply LS to the modified data $\hat{\mathbf{x}}_e = \mathbf{A}^\dagger(\mathbf{y} - \widehat{\mathbf{e}+\mathbf{n}})$.

We performed 200 experiments with a uniform circular array of radius 1 [DU] consisting of $N = 30$ TOA sensors. The number of outliers s was successively set to 0, 5 and 10, corresponding to 0%, 16% and 33% of the equations. The outliers size correspond to (10). The small ranging errors are realizations of zero-mean gaussian random variables with standard deviation σ. Fig. 2 shows the RMSE of location estimates versus σ. Note that $\sigma = 0.3$ [DU] is not negligible compared with the error-free measurements and therefore the number of outliers is occasionally increased. According to Fig. 2, in the presence of 5 or 10 outliers, it is always significantly advantageous to remove the outliers by linear programming before LS estimation, regardless of the noise variance. In the absence of outliers, a slight performance degradation may be observed if preprocessing is used. Note however that if no outlier is found the preprocessing can be ignored.

10. CONCLUSION

We have invoked the theory of sparse signal representation in the context of positioning. We have shown that in the absence of noise, AOA, TOA and TDOA data sets corrupted by unknown outliers can be corrected using linear programming, provided that the number of outliers does not exceed a limit given in Theorem 1. In Sections 4, 5 and 6 the limit given in (6) has been evaluated for several geometries. For each case, the exact expression of the bound and simple approximations were presented. In followup work we focus on outliers in the presence of small errors.

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