

RANK TRANSFORMATION AND MANIFOLD LEARNING FOR MULTIVARIATE MATHEMATICAL MORPHOLOGY

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ABSTRACT

The extension of lattice based operators to multivariate images is still a challenging theme in mathematical morphology. In this paper, we propose to explicitly construct complete lattices and replace each element of a multivariate image by its rank, creating a rank image suitable for classical morphological processing. Manifold learning is considered as the basis for the construction of a complete lattice after reducing a multivariate image to its main data by Vector Quantization. A quantitative comparison between usual ordering criteria is performed and experimental results illustrate the abilities of our proposal.

1. INTRODUCTION

Mathematical Morphology (MM) is a nonlinear approach to image processing which relies on a fundamental structure, the complete lattice \mathcal{L} [1]. A complete lattice \mathcal{L} is a nonempty set equipped with an ordering relation, such that every non-empty subset \mathcal{H} of \mathcal{L} has a lower bound $\wedge \mathcal{H}$ and an upper bound $\vee \mathcal{H}$. In this context, images are modeled by functions mapping their domain space Ω , into a complete lattice \mathcal{L} . With the acceptance of complete lattice theory, it is possible to define morphological operators for any type of image data once a proper ordering is established [2]. Within this model, morphological operators are represented as mappings between complete lattices in combination with matching patterns called structuring elements that are subsets of Ω . In particular, the two fundamental operators in Mathematical Morphology, dilation and erosion, form the basis of many other morphological processes [3] such as opening ($\gamma = \delta \varepsilon$), closing ($\varphi = \varepsilon \delta$), etc. Erosion ε and dilation δ of a function $f \in \mathcal{L}$ for an element $x \in \Omega$ are defined by

$$\varepsilon(f, x|B) = \{f(y) : f(y) = \wedge f(z), z \in B(x)\} \quad (1)$$

$$\delta(f, x|B) = \{f(y) : f(y) = \vee f(z), z \in B(x)\} \quad (2)$$

where B denotes a structuring element that contains x and its neighbours in Ω . If Mathematical Morphology is well defined for binary and gray scale images, there exist no general extension which permits to perform basic operations on multivariate data since there is no natural ordering on vectors. Several ordering have been reported in literature to consider that problem but they are reduced to considering one specific type of images (e.g. color images [4]). In this paper, we propose to use a rank transformation with Manifold Learning for complete lattice creation. Our approach is mainly illustrated on color images but the principle holds for any arbitrary higher dimensions.

The remainder of this paper is organized as follows. Section 2 presents the concept of rank transformation for complete lattice construction. Section 3 recalls Vector Quantization basics and explains how it can be used to construct a rank transformation. Section 4 presents two Manifold Learning methods for codebook re-ordering. Section 5 presents experimental results. Last Section concludes.

2. RANK TRANSFORMATION

In the sequel, we consider the general case of multivariate images. A multivariate image can be represented by the mapping $f : \Omega \subset \mathbb{Z}^l \rightarrow \mathbb{R}^p$ where l is the image dimension and p the number of channels. One way to define an ordering relation between vectors is to use a transform [5] h from \mathbb{R}^p to \mathbb{R}^q followed by the natural ordering on each dimension of \mathbb{R}^q :

$$\text{With } h : \mathbb{R}^p \rightarrow \mathbb{R}^q, \text{ and } \mathbf{x} \rightarrow h(\mathbf{x}) \text{ then} \quad (3)$$

$$\forall (\mathbf{x}_i, \mathbf{x}_j) \in \mathbb{R}^p \times \mathbb{R}^p, \mathbf{x}_i \leq \mathbf{x}_j \Leftrightarrow h(\mathbf{x}_i) \leq h(\mathbf{x}_j)$$

When h is bijective, this corresponds to define a space filling curve that goes through each point of the \mathbb{R}^p space just once and thus induces a total ordering. Therefore, there is an equivalence: (total ordering on \mathbb{R}^p) \Leftrightarrow (bijective application $h : \mathbb{R}^p \rightarrow \mathbb{R}^q$) \Leftrightarrow (space filling curve in \mathbb{R}^p) [6]. Moreover, another equivalence can be considered: (total ordering on \mathbb{R}^p) \Leftrightarrow (rank transformation on \mathbb{R}^p) that means that the transformation h can be seen as a rank transformation. Indeed, to create a total order for building the complete lattice structure for MM operators, the images' values are in fact not important, only the rank position on the lattice structure is relevant [7, 8, 9]. When the complete lattice is created, the MM operators only need to perform comparisons between ranks. Therefore, we can replace each element of a multivariate image by its rank, creating a rank image. This rank image is the lattice representation of the multivalued image according to the ordering strategy and corresponds to a transformation h from \mathbb{R}^p to \mathbb{N} .

Definition 1. A rank transformation $r : \mathbb{R}^p \rightarrow \mathbb{N}$ is a function that associates to a vector $\mathbf{x} \in \mathbb{R}^p$ the value $r(\mathbf{x}) \in \mathbb{N}$ where $r(\mathbf{x})$ is the rank position of \mathbf{x} on a lattice \mathcal{L}^r .

\mathcal{L}^r is the lattice associated to the rank transformation r that creates a given total ordering. The lattice corresponds to an ordered set of multivariate vectors. From this rank transformation, a rank image can be created by associating its rank to each pixel $x \in \Omega$. The advantage is that this rank image

can then be used for classical MM processing (it is a scalar image).

This formulation is general enough to represent all the classical approaches for building a complete lattice [2]. Indeed, with classical approaches, an ordering criterion is proposed to induce the complete lattice. Once such an ordering criterion is available, it is easy to sort all the multivariate values of the image to define the rank transform that corresponds to the complete lattice creation. Moreover, as it will be discussed in Section 5, the rank transformation enables to easily compare different multivariate data ordering criteria.

3. VECTOR QUANTIZATION

Usual approach to mathematical morphology do not explicitly construct the complete lattice: they first define a total ordering relation that induces a complete lattice. In this paper, we take an opposite approach and we first build the complete lattice from a multivariate image. Given the space where multivariate data live, \mathbb{R}^p , the complete lattice is in fact a manifold view of this space. Obviously, creating the lattice directly from a multivariate image is computationally unfeasible. To alleviate this problem, we reduce the amount of data of a multivariate image by Vector Quantization. Vector Quantization (VQ) is a technique used for data compression. VQ maps a vector \mathbf{x} to another vector \mathbf{x}' that belongs to m' prototype vectors the set of which is named a codebook. A codebook D is built from a training set S of size m ($m \gg m'$). A VQ algorithm has to produce a set D of prototypes \mathbf{x}' that minimizes the distortion defined by

$$\frac{1}{m} \sum_{i=1}^m \min_{1 \leq j \leq m'} \|\mathbf{x}_i - \mathbf{x}'_j\|_2 \quad (4)$$

LBG [10] is one algorithm that can build such a codebook. It is an iterative algorithm that produces 2^k prototypes after k iterates. Given a color image f , VQ: $\mathbb{R}^p \rightarrow \mathbb{R}^p$ is applied to construct a color codebook $D: \mathbb{N} \rightarrow \mathbb{R}^p$ and an encoder $E: \mathbb{R}^p \rightarrow \mathbb{N}$. An index image $g: \mathbb{Z}^l \rightarrow \mathbb{N}$ can be deduced from D and E by applying $g(x) = E(f(x))$ to each vector $f(x) = \mathbf{x}$ of the original color image f . The color image can be reconstructed with loss from the index image and the color codebook by $D(g(x))$. The latter color image is an approximation of the initial color image with only 2^k colors. Given this representation, the color codebook corresponds to the complete lattice since all the colors are ordered, and the index image is the result of a rank transformation applied to the original color image. The color codebook is only an approximation of the whole complete lattice living on \mathbb{R}^p and therefore corresponds to a sub-manifold of the complete lattice. However, this is sufficient to perform mathematical morphology operations because they are not intended for denoising but extraction purposes. First row of Figure 1 presents the process of VQ on a color image with, from left to right, the initial image, the quantized reconstructed image with $2^8 = 256$ colors, its associated index image corresponding to the result of a rank transformation and the color codebook corresponding to the complete lattice. The color codebook is presented here as a color image but colors are ordered and form a complete lattice from the top-left to the bottom-right with a raster scan. As it can be seen on Figure 1, there is no real visual difference between the original and the quantized image and the latter is sufficient for morphological processing.

Once a rank transformation has been applied to the original image, the rank image is nothing more than a scalar image. On this image, the classical scalar ordering holds, and classical mathematical morphology can be used (e.g. an erosion $\varepsilon(g(x))$). Then, one can reconstruct the corresponding color image with the use of the color codebook (e.g. $D(\varepsilon(g(x)))$).

This strategy enables simple multivariate mathematical morphology. However, the color codebook obtained by VQ is not necessarily an accurate total ordering of the colors (it can be totally arbitrary) and reordering the colors of the codebook can be interesting to have a more accurate rank image with respect to the content of the color image. This is known in literature as palette reordering [11]. Given any ordering criterion, we can re-order the colors of the color codebook and modify the rank image according to the re-ordering. For instance, the classical ordering criteria used in MM can be used to reorder the color codebook. This will be investigated for comparison purposes between total ordering criteria. Moreover, to enable further accurate morphological processing, the level lines of the rank image (and therefore its basic geometry) has to coincide with the level lines of the vector-valued image level lines [12].

4. CODEBOOK REORDERING BY MANIFOLD LEARNING

To re-order the codebook, we consider Manifold Learning techniques [13]. Indeed, the color codebook being a sub-manifold of the whole complete lattice, Manifold Learning is a good candidate to perform the codebook re-ordering. In the last few years, many unsupervised learning algorithms have been proposed for Manifold Learning which share the use of an eigen-decomposition for obtaining a lower-dimensional embedding of the data. In this paper, we focus only on two Graph-based methods for nonlinear dimensionality reduction: Diffusion Maps and Laplacian Eigenmaps. These Manifold Learning techniques preserve the local proximity between data points by first constructing a graph representation for the underlying manifold with vertices and edges. The vertices represent the data points, and the edges connecting the vertices, represent the similarities between adjacent nodes. After representing the graph with a matrix, the spectral properties of this matrix are used to embed the data points into a lower dimensional space, and gain insight into the geometry of the dataset. Let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_l\} \in \mathbb{R}^p$ be l sample vectors. Dimensionality reduction consists in searching for a new representation $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_l\}$ with $\mathbf{y}_i \in \mathbb{R}^l$. Given a neighborhood graph G associated to these vectors, one considers its adjacency matrix W where weights W_{ij} are given by a Gaussian kernel $W_{ij} = K(\mathbf{x}_i, \mathbf{x}_j) = e\left(-\frac{\|\mathbf{x}_i - \mathbf{x}_j\|^2}{\sigma^2}\right)$. Let D denote the diagonal matrix with elements $D_{ii} = \sum_j W_{ij}$ and L denote the un-normalized Laplacian defined by $L = D - W$. Finally, let $P = WD^{-1}$ denote the transition matrix. In the sequel, we consider that the neighbourhood graph is a fully connected graph. Manifold Learning is achieved by finding the eigenvectors of matrix L (named Laplacian Eigenmaps [14]) or matrix P^l (named Diffusion Maps [15]). For Laplacian Eigenmaps, the manifold representation can be found by solving $Ly = \lambda Dy$. The eigenvectors corresponding to the smallest nonzero eigenvalues form the manifold representation. For Diffusion Maps, one considers $P^l Y = \lambda Y$ and because the graph is fully connected, the largest eigenvalue

is trivial ($\lambda_1 = 1$), and its eigenvector \mathbf{y}_1 is thus discarded. The manifold representation \mathbf{Y} is given by the next eigenvectors. Given the previous relations, Manifold Learning can be obtained by considering the highest eigenvector except the first one for Diffusion Maps or nonzero lowest eigenvectors for Laplacian Eigenmaps. To perform the color codebook re-ordering, a vertex is associated to each color and the graph is fully connected. Then, we consider only the first eigenvector of the obtained Manifold representation and re-arrange the colors increasingly according to their value in the first eigenvector. To have a parameter free algorithm, σ is set to the maximum distance between the colors of the lattice.

5. RESULTS

Figure 1 presents a visual comparison of the proposed methodology (vector quantization of a color image followed by a re-ordering of the color codebook with the associated modification of the rank image). First line presents a color image, its 256 colors quantized version and the corresponding rank and lattice. Then, we re-order the codebook with MM ordering criterion that correspond to total orderings and enable to obtain complete lattices: lexicographic ordering [16], lexicographic ordering in the IHSL color space [4], α -trimmed lexicographic ordering [17], bit-mixing ordering [6], pairwise ordering [18], majority ordering [8], and finally two manifold learning orderings, laplacian eigenmaps [14] and diffusion maps [15]. All these experiments are performed in the *RGB* color space except for the second used ordering criterion [4]. The first element to notice is that the obtained lattices can have large differences from one ordering criterion to another. It is difficult to conclude which ordering criteria is the best one with a visual analysis of the obtained rank images. Therefore, to have a real quantitative comparison of the different lattices, we follow the evaluation procedure used in palette re-ordering [11]. To compare palette re-ordering methods, the assumption that differences of neighbouring pixels of rank images should follow a Laplacian distribution is used. This is in accordance with the JPEG-LS image coding standard, which also assumes a Laplacian model for the prediction residuals and, therefore, the compression ability of the rank image is a measure of performance for re-ordering schemes. We used this principle to compare all the above-mentioned reordering criteria. Three different databases from [11] that contained images quantized in 256 colors have been used: their color codebook has been reordered, the associated rank image modified and compressed with JPEG-LS. Used datasets are composed of images having different geometries: synthetic set contains 6 computer-generated images, natural1 is composed of 92 natural images, natural2 is composed of 12 popular natural images [11]. Figure 2 presents JPEG-LS loss-less compression results in bits per pixel, of the reordered index images (the size of the corresponding codebooks are included in the presented values) on the three datasets and the average compression rate. It is easy to see that the quality of the re-ordering methods can be roughly classified into four groups by decreasing quality: Manifold Learning [14, 15], Lexicographic [16, 17, 4], Pairwise [18], Bit-mixing [6], VQ, Majority [8] and Graph Minimum Spanning Tree [9]. These results are important because : 1) a quantitative way of comparing lattice construction methods is provided, 2) all the recently proposed ordering criteria do not outperform the classical lex-

icographic ordering for lattice construction, 3) lattice construction by Manifold Learning methods does outperform all the other methods proving the quality of our proposal. Finally, Figure 3 presents an example of MM processing with the proposed strategy. The original color image is quantized in 512 colors and its color codebook reordered by Diffusion Maps. Then, MM operations are performed on the rank image and the corresponding color image result is reconstructed with the color codebook. One can see the quality of obtained results. To show that our proposal is easily applicable to multivariate image, Figure 4 presents a sample segmentation result on a 20-channels multispectral image with a 1024 codebook. Segmentation is performed by a watershed on a gradient image obtained from an Alternate Sequential Filter on the rank image.

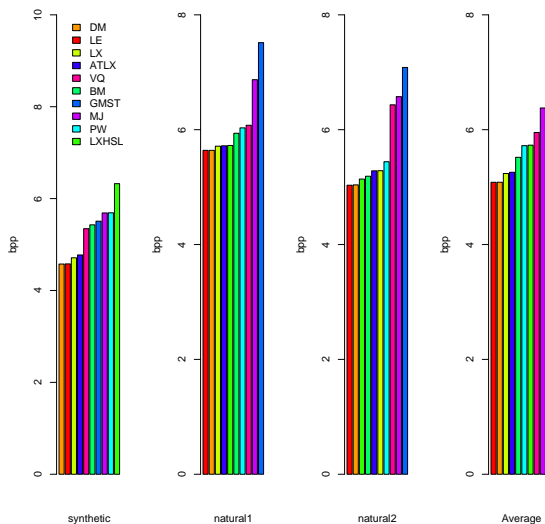


Figure 2: Loss-less compression results, in bits per pixel, applied to the rank images after applying the following re-ordering methods: Laplacian Eigenmaps (LM), Diffusion Maps (DM), Lexicographic (LX), Lexicographic IHSL (LXHSL), BitMixing (BM), Pairwise (PW), Graph Minimum Spanning Tree (GMST), α -trimmed Lexicographic (ATX), Majority (MJ), VQ (Vector Quantization).

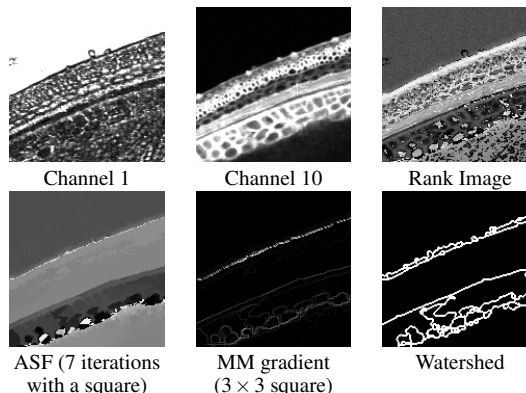


Figure 4: MM Processing example of a 20-channels multispectral barley grain image.

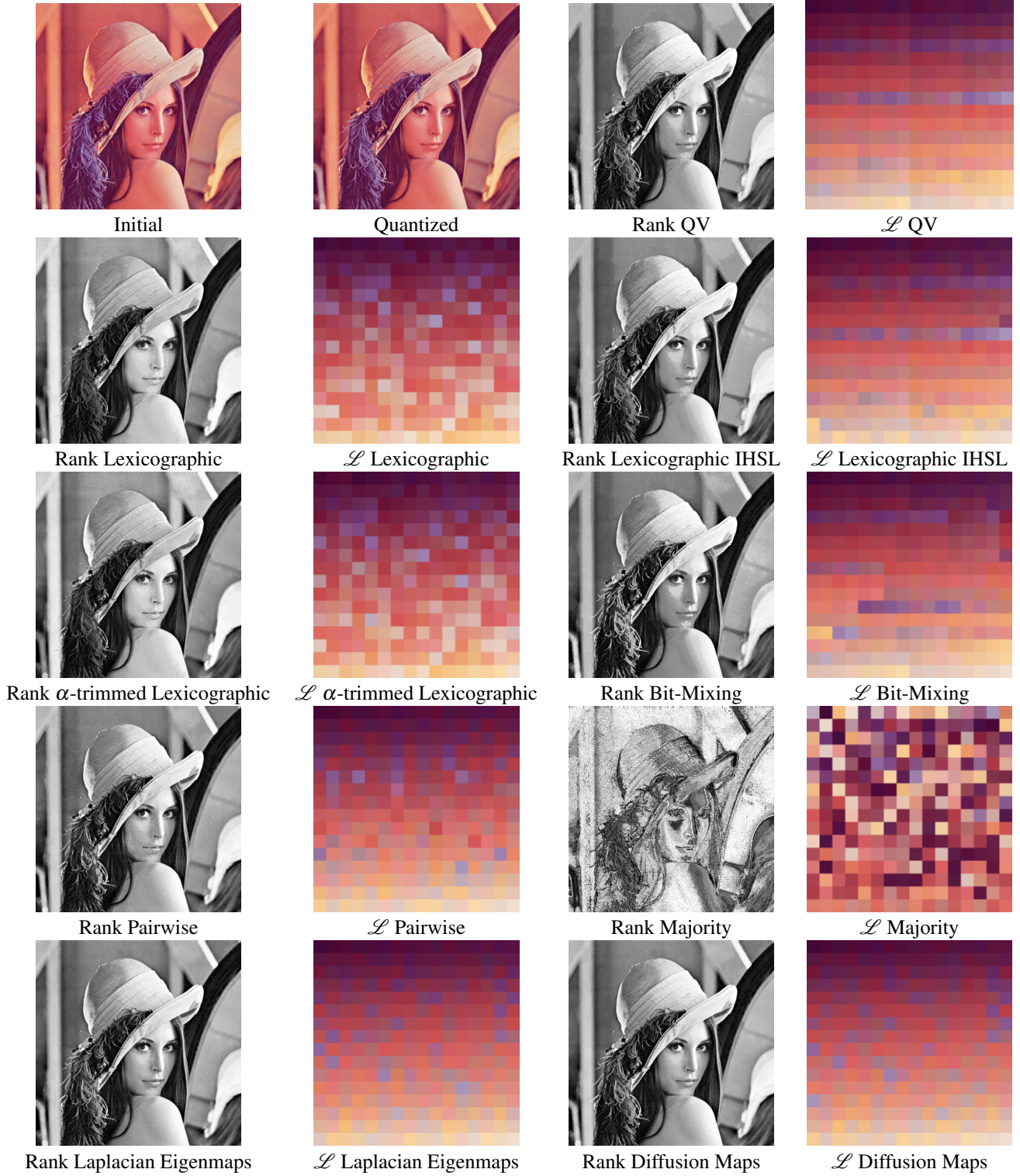


Figure 1: Illustration of color-codebook reordering for lattice construction with the associated rank images (\mathcal{L} designs the obtained lattice, see text for details).

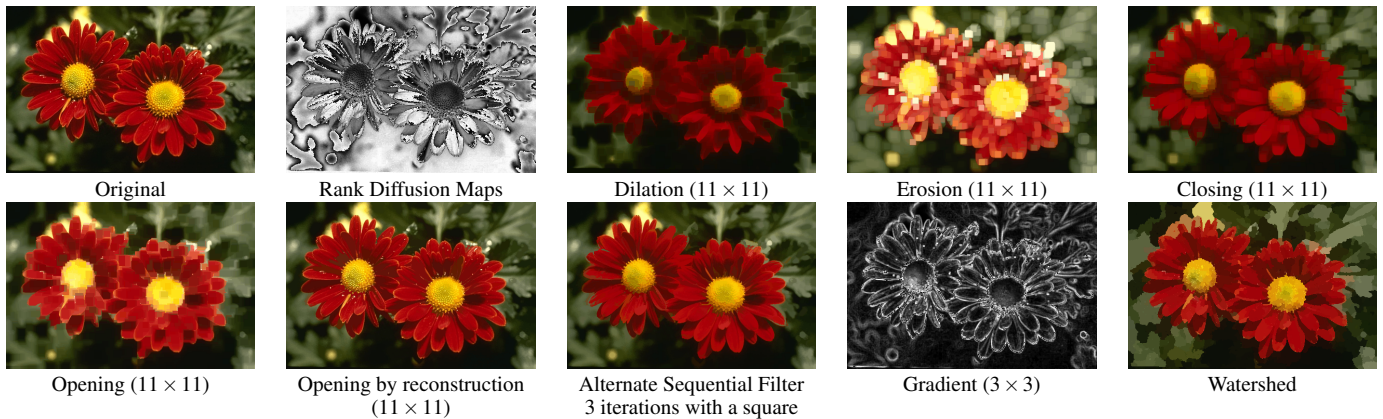


Figure 3: Processing examples with a rank image obtained from Diffusion Maps with a 512 colors Vector Quantization.

6. CONCLUSION

In this paper, a new way of performing MM processing on multivariate images has been proposed. Instead of defining a new ordering criterion that induces a complete lattice without its explicit construction, we construct the complete lattice by reordering a codebook obtained by vector quantization. Moreover, we have shown that Manifold Learning is the best candidate to perform the lattice construction. However, since the learned lattice is made on a codebook, it is only a submanifold of the original color manifold. Future works will address that issue.

REFERENCES

- [1] C. Ronse, “Why mathematical morphology needs complete lattices,” *Signal Processing*, vol. 21, no. 2, pp. 129–154, 1990.
- [2] E. Aptoula and S. Lefèvre, “A comparative study on multivariate mathematical morphology,” *Pattern Recognition*, vol. 40, no. 11, pp. 2914–2929, 2007.
- [3] P. Soille, *Morphological Image Analysis: Principles and Applications*, Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2003.
- [4] J. Angulo, “Unified morphological color processing framework in a lum/sat/hue representation,” in *Proceedings of the 7th International Symposium on Mathematical Morphology*, 2005, pp. 387–396.
- [5] J. Goutsias, H.J.A.M. Heijmans, and K. Sivakumar, “Morphological operators for image sequences,” *Computer Vision and Image Understanding*, vol. 62, no. 3, pp. 326–346, 1995.
- [6] J. Chanussot and P. Lambert, “Bit mixing paradigm for multivalued morphological filters,” in *International Conference on Image Processing and Its Applications*, 1997, vol. 2, pp. 804 – 808.
- [7] A. Garcia, C. Vachier, and J.-P. Vallée, “Multivariate mathematical morphology and bayesian classifier application to colour and medical images,” in *Image Processing: Algorithms and Systems VI*. SPIE, 2008, vol. 6812, pp. 03–11.
- [8] A. Ledda and W. Philips, “Majority ordering and the morphological pattern spectrum,” in *Advanced Concepts for Intelligent Vision Systems*, 2005, pp. 356–363.
- [9] O. Lezoray, C. Meurie, and A. Elmoataz, “Mathematical morphology in any color space,” in *IAPR/IEEE International Conference on Image Analysis and Processing, Computational Color Imaging Workshop*. IEEE, 2007.
- [10] A. Gersho and R.M. Gray, *Vector Quantization and Signal Compression*, Kluwer Academic, 1991.
- [11] A.J. Pinho and A.J.R. Neves, “A survey on palette reordering methods for improving the compression of color-indexed images,” *IEEE Transactions on Image Processing*, vol. 13, no. 11, pp. 1411–1418, 2004.
- [12] Do Hyun Chung and G. Sapiro, “On the level lines and geometry of vector-valued images,” *Signal Processing Letters, IEEE*, vol. 7, no. 9, pp. 241–243, 2000.
- [13] L. Saul, K. Weinberger, J. Ham, F. Sha, and D. Lee, “Spectral methods for dimensionality reduction,” in *Semi-Supervised Learning*, O. Chapelle, B. Scholkopf, and A. Zien, Eds., pp. 279–294. MIT PRESS, 2006.
- [14] M. Belkin and P. Niyogi, “Laplacian eigenmaps for dimensionality reduction and data representation,” *Neural Computation*, vol. 15, no. 6, pp. 1373–1396, 2003.
- [15] S. Lafon, Y. Keller, and R.R. Coifman, “Data fusion and multicue data matching by diffusion maps,” *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 28, no. 11, pp. 1784–1797, 2006.
- [16] E. Aptoula and S. Lefèvre, “On lexicographical ordering in multivariate mathematical morphology,” *Pattern Recognition Letters*, vol. 29, no. 2, pp. 109–118, 2008.
- [17] E. Aptoula and S. Lefèvre, “Alpha-trimmed lexicographical extrema for pseudo-morphological image analysis,” *Journal of Visual Communication and Image Representation*, vol. 19, no. 3, pp. 165–174, 2008.
- [18] A. N. Evans, “Nonlinear operations for colour images based on pairwise vector ordering,” in *Digital Image Computing: Techniques and Applications*, 2005, pp. 61–66.