DISTRIBUTED ESTIMATION WITH DEPENDENT OBSERVATIONS IN WIRELESS SENSOR NETWORKS

Sung-Hyun Son, Sanjeev R. Kulkarni, and Stuart C. Schwartz

Department of Electrical Engineering, Princeton University
Princeton, NJ 08544 USA
email: {sungson, kulkarni, stuart}@princeton.edu

ABSTRACT

A wireless sensor network with a fusion center is considered to study the effects of dependent observations on the parameter estimation problem. The sensor observations are corrupted by Gaussian noise with *geometric* spatial correlation. From an energy point of view, sending all the local data to the fusion center is the most costly, but leads to optimum performance results since all the dependencies are taken into account. From an estimation accuracy point of view, sending only parameter estimates is the least accurate, but is the most parsimonious in terms of communication costs. Hence, this tradeoff between the energy efficiency and the estimation accuracy is explored by comparing the performance of maximum likelihood estimator (MLE) and the sample average estimator (SAE) under various topologies and communication protocols. We start by reviewing the results from the one-dimensional case and continue by extending those results to various two-dimensional topologies. Surprisingly, we discover a class of regular polygon topologies where the MLE under spatial correlation reduces to the SAE.

1. INTRODUCTION

The canonical wireless sensor network (WSN) consists of a fusion center and a set of low-cost sensors with limited dynamic range, power, resolution, and wireless communication capabilities. The role of the fusion center is to consolidate information and aid the sensors when necessary. The job of each sensor is to collect local data and transmit relevant information to the fusion center. With limited communication and computation capabilities, spatially distributed sensors are expected to coordinate, communicate, and, in our case, estimate a parameter of interest from the environment (e.g., monitoring temperature readings or toxicity levels of a chemical agent in a region).

In the present setting of the distributed estimation problem, each sensor collects observations based on a parameter of interest. Either the observations or sufficient statistics, if available, are shared amongst the sensors or passed to the fusion center. To further reduce the energy cost for communication, the observations may be quantized before transmission. Ultimately, the estimate of the parameter is obtained by optimizing a non-linear function based on the received observations (e.g., maximum likelihood or minimum mean-square error).

Deriving the maximum likelihood estimator (MLE) in a WSN setting has been studied under various contexts. The simplest approach is to send the full set of unprocessed observations to the fusion center where the MLE can be computed. This approach is not feasible for WSNs due to the high communication cost, but provides a benchmark for accuracy performance. Another approach is to develop procedures that take into account the power and bandwidth constraints. In [1], the focus is on finding a class of MLEs that attain a variance that is close to the optimal MLE when the observations are quantized to one bit. In [2, 3], the distributed estimation scheme takes the quantization idea one step further by requiring each sensor to send a message whose length is determined by the local SNR. The proposed scheme is shown to be within a constant factor of the optimal estimator.

While most of the results on distributed estimation assume that the sensor observations are conditionally independent, less is known about the broader, more difficult problem in which the sensor observations are *conditionally dependent*. In many practical applications, a large number of sensors are deployed over a finite region. Hence, some spatial correlation most likely exists among the sensor observations. The issue of distributed estimation with dependent observations was studied in [4], where the authors implemented suboptimal estimates to show that their scheme outperforms procedures which neglect dependency.

In this paper, we consider a deterministic mean location parameter estimation problem with dependent observations. In the presence of independent observations, deriving the MLE is straightforward and highly energy efficient. The full observation set from all the sensors does not need to be present at the fusion center when the parameter estimate is calculated. Instead, only specific quality measures based on local sensor observations are necessary. Hence, if we consider a WSN with N sensors randomly placed on a unit grid where all the sensors only send quality measures to the fusion center, then $O(N)$ bit-meters of transport energy cost is required. Furthermore, the specific quality measures from each sensor can be passed sequentially from sensor to sensor and still incur no loss of information. Given the individual sensor quality measures, a cumulative sum can be computed where each sensors adds its own local contribution to the previous cumulative sum. This sequential communication protocol requires $O(\sqrt{N})$ bit-meters of transport energy cost. Refer to [5] for details.

On the other hand, deriving the MLE for the dependent observation case is, in general, not analytically feasible. In addition, the MLE under dependent observations requires a centralized communication protocol in which all data observations are sent to the fusion center, with an associated $O(NM)$ bit-meters in terms of transport cost, where M is the number of local sensor observations. To address these issues, we start by summarizing the results for the one-dimensional, linear topology with fixed and proximity spacing found in [6]. This simple topology allows the MLE and the corresponding Cramer-Rao lower bound (CRLB) to have explicit analytic forms. Then, given a specific topology, a comparison is made between the transport cost needed to implement the communication protocol and the accuracy of the estimate produced by the communication protocol. Specifically, the difference between the variance of MLE under dependent observations and the variance of the MLE assuming independent observations is observed to see the improvement in accuracy by incorporating dependency in the estimate. Then, we extend these results to the two-dimensional grid topology. In the general $N \times N$ grid case, simulations provide insight on the role of dependency since an analytical solution, to the best of our knowledge, has yet to be found. Also, we consider scenarios where the sensors are placed randomly on a unit square and compared with the grid topology case. Finally, a class of two- dimensional topologies are found where MLE under dependent observations reduces to the SAE. Under this class of topologies, the sequential procedure can be implemented without any loss in estimation accuracy.

The organization of this paper is as follows. In Section 2, the problem formulation as well as background material on the MLE and the CRLB for any general covariance matrix are presented. In Section 3, a one-dimensional topology is presented. Two cases are considered where as the number of sensors increases, either the area the sensors cover grows, (i.e., fixed spacing), or the sensors are confined to a fixed area and get closer together, (i.e., proximity spacing). In Section 4, a two-dimensional fixed grid spacing is considered. Various topologies are shown which include fixed grid spacing, proximity grid spacing, random grid spacing and regular polygon spacing. Section 5 provides the concluding remarks.

2. PROBLEM FORMULATION

Parameter	Description
M	# of measurements per sensor
N	# of sensors
i	measurement index: $i = 1, \ldots, M$
j	sensor index : $j = 1, \ldots, N$
θ	scalar parameter of interest
$x_{i,j}$	i^{th} observation from j^{th} sensor
$\boldsymbol{x}_i$	i^{th} observation vector from all sensors

Consider a WSN comprised of N sensors where each sensor collects M measurements. While we assume that the sensor observations are independent from measurement to measurement, they are not necessarily independent from sensor to sensor. Hence, the observations at time index i of N sensors are modeled as

$$\boldsymbol{x}_i = \theta \mathbf{1} + \boldsymbol{w}_i \tag{1}$$

where

$$f(\boldsymbol{w}_i|\Sigma) = \frac{1}{(2\pi)^{\frac{N}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\boldsymbol{w}_i^T \Sigma^{-1} \boldsymbol{w}_i\right) . \tag{2}$$

We let $f(\boldsymbol{w}_i|\Sigma)$ denote the noise probability density function (Gaussian) with covariance matrix Σ. We assume θ is fixed but unknown. This is the standard deterministic mean location parameter estimate formulation.

If all observations from the WSN, $\{\boldsymbol{x}_i\}_{i=1}^M$, are available, the MLE of θ is given by

$$\widehat{\theta}_{MLE} = \frac{1}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} \frac{1}{M} \sum_{i=1}^{M} \mathbf{1}^T \Sigma^{-1} \boldsymbol{x}_i . \tag{3}$$

The variance of the estimator in (3) is $\mathrm{Var}\,(\widehat{\theta}_{MLE}) = 1/\big(M(\mathbf{1}^T\Sigma^{-1}\mathbf{1})\big)$ while the Fisher information is $\mathrm{I}\,(\widehat{\theta}_{MLE}) = M\left(\mathbf{1}^T\Sigma^{-1}\mathbf{1}\right)$. Hence, the CRLB is achieved by the estimator.

The estimator in (3) represents the most general form of the MLE of θ. Depending on the structure of Σ, simplifications can be made in the form of the MLE. The simplest case imposes spatial independence on the sensor observations. If in addition, $\sigma_j^2 = \sigma^2$ for all j, then $\Sigma = \sigma^2 \mathbf{I}$. The MLE of θ is given by

$$\widehat{\theta}_{SAE} = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} x_{i,j} , \tag{4}$$

henceforth referred to as the sample average estimator (SAE).

From the SAE in (4), it is evident that a sequential procedure can be implemented wherein each sensor passes only certain statistics of their own data from sensor to sensor. As the statistics traverse the WSN, each sensor updates the current statistic value based on their own observation data. For independent observations in the Gaussian noise case, in which the covariance matrix $\Sigma = \sigma^2\mathbf{I}$, the sequential procedure only requires the sample average, $\widehat{\mu}_j$, to be computed and passed from sensor to sensor, where $\widehat{\mu}_j = \frac{1}{M}\sum_{i=1}^{M} x_{i,j}$.

The case with independent observations is the motivation behind our sequential procedure. Without any loss of accuracy in the MLE estimate of θ, the transport cost is reduced by implementing a sequential procedure over a centralized procedure. Now, we adopt a similar approach and consider various topologies with dependent observations.

3. ONE-DIMENSIONAL TOPOLOGY

3.1 Fixed Line Spacing

Assume that the sensor nodes are equally spaced and each individual sensor has the same variance. As more sensors are added, the area the sensors cover grows accordingly. The elements of the covariance matrix have a geometric form, $\Sigma_{i,j} = \sigma^2 \rho^{|i-j|}$, where ρ is the correlation coefficient. Then, for the one-dimensional sensor array, the covariance matrix will be

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{N-1} \\ \rho & 1 & \rho & \dots & \rho^{N-2} \\ \rho^2 & \rho & 1 & \dots & \rho^{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{N-1} & \rho^{N-2} & \rho^{N-3} & \dots & 1 \end{bmatrix} . \quad (5)$$

The matrix in (5) is referred to as the Kac-Murdock-Szegö matrix [7] and using the fact that the matrix has a tridiagonal inverse, the MLE of θ is

$$\widehat{\theta}_{MLE} = \frac{1}{M(N(1-\rho)+2\rho)} \times \left[\sum_{i=1}^{M} \sum_{j=1}^{N} x_{i,j} - \rho \sum_{i=1}^{M} \sum_{j=2}^{N-1} x_{i,j} \right] \quad (6)$$

and the corresponding variance of the estimator is

$$\text{Var}(\widehat{\theta}_{MLE}) = \frac{\sigma^2(1+\rho)}{M\big(N(1-\rho)+2\rho\big)} . \quad (7)$$

To implement the MLE, all observations must be sent to the fusion center. A distributed technique is not apparent since correlations exist amongst the observations. Thus, the energy expenditure in terms of transport cost is $O(MN)$ bit-meters. However, if we only calculate the sample average, a sequential procedure can be used that only costs $O(\sqrt{N})$ bit-meters. So the following questions arise. Is it worth all this extra transport cost to achieve the best MLE? What is the accuracy performance gain in terms of the variance of both estimators under dependent observations? To answer these questions, the variance of the SAE needs to be calculated under the noise conditions of (5). The difference between the variance of the MLE and the variance of the SAE provides a measure for the accuracy improvement.

Under the same noise conditions, the variance of the estimator in (4) is found to be

$$\text{Var}\ (\widehat{\theta}_{SAE}) = \frac{\sigma^2}{NM} + \frac{2\sigma^2\rho}{N^2M(1-\rho)} \times \left[(N-1) - \frac{\rho}{1-\rho}\big(1-\rho^{N-1}\big) \right] . \quad (8)$$

Given both analytical forms of the variances, a straightforward computation shows the following proposition.

Proposition 1 *For the covariance matrix given by* (5)*, if* $N \geq 3$ *and* $\rho \in (0,1)$*, we have*

$$\frac{\text{Var}\ (\widehat{\theta}_{\text{SAE}}) - \text{Var}\ (\widehat{\theta}_{\text{MLE}})}{\text{Var}\ (\widehat{\theta}_{\text{MLE}})} = O(N^{-1}) . \quad (9)$$

For the fixed line spacing topology, both communication protocols behave the same asymptotically. This was expected since the area grew as the number of sensors grew. However, a more interesting case arises when the coverage area remains fixed.

3.2 Proximity Line Spacing

A different model arises when the area the sensors cover is fixed. As more sensors are added, the sensors get closer and thus, more correlated. If we add sensors, equally spaced, on a unit straight line, the maximum distance between adjacent sensors is $d = 1/(N-1)$. Then, the elements of the covariance matrix are $A_{i,j} = \sigma^2 \rho^{|i-j|d}$, while the covariance matrix has the form

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & \rho^{\frac{1}{N-1}} & \rho^{\frac{2}{N-1}} & \dots & \rho \\ \rho^{\frac{1}{N-1}} & 1 & \rho^{\frac{1}{N-1}} & \dots & \rho^{\frac{N-2}{N-1}} \\ \rho^{\frac{2}{N-1}} & \rho^{\frac{1}{N-1}} & 1 & \dots & \rho^{\frac{N-3}{N-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho & \rho^{\frac{N-2}{N-1}} & \rho^{\frac{N-3}{N-1}} & \dots & 1 \end{bmatrix} . \quad (10)$$

The MLE for the covariance matrix in (10) is

$$\widehat{\theta}_{MLE} = \frac{1}{M\left(N(1-\rho^{\frac{1}{N-1}}) + 2\rho^{\frac{1}{N-1}}\right)} \times \left[\sum_{j=1}^{N} \widehat{\mu}_j - \rho^{\frac{1}{N-1}} \sum_{j=2}^{N-1} \widehat{\mu}_j \right] \quad (11)$$

with

$$\text{Var}\ (\widehat{\theta}_{MLE}) = \frac{\sigma^2(1+\rho^{\frac{1}{N-1}})}{M\big(N(1-\rho^{\frac{1}{N-1}}) + 2\rho^{\frac{1}{N-1}}\big)} . \quad (12)$$

Note, as $N \to \infty$, the variance of the MLE approaches $\frac{2\sigma^2}{M\left(2-\log\rho\right)}$.

Following the steps used for the fixed spacing case, the variance of the SAE under the same covariance matrix is

$$\text{Var}\ (\widehat{\theta}_{SAE}) = \frac{\sigma^2}{NM} + \frac{2\sigma^2\rho^{\frac{1}{N-1}}}{N^2M(1-\rho^{\frac{1}{N-1}})} \times \left[(N-1) - \frac{\rho^{\frac{1}{N-1}}}{1-\rho^{\frac{1}{N-1}}}\big(1-\rho\big) \right] . \quad (13)$$

Given the analytical expressions for the variances of each estimator, we have the following propositions about the absolute and relative difference.

Proposition 2 *For the covariance matrix given by* (10)*, if* $N \geq 3$ *and* $\rho \in (0,1)$*, we have*

$$\sup_N \left[\text{Var}\ (\widehat{\theta}_{\text{SAE}}) - \text{Var}\ (\widehat{\theta}_{\text{MLE}}) \right] = -\frac{2\sigma^2}{M} \left[\frac{1}{\log\rho} + \frac{1-\rho}{(\log\rho)^2} + \frac{1}{2-\log\rho} \right] , \quad (14)$$

and

$$\sup_{N,\rho}\left[\mathrm{Var}\,(\widehat{\theta}_{\mathrm{SAE}})-\mathrm{Var}\,(\widehat{\theta}_{\mathrm{MLE}})\right]\leq 0.072\frac{\sigma^2}{M}\,. \tag{15}$$

Proposition 3 *For the covariance matrix given by* (10)*, if* $N \geq 3$ *and* $\rho \in (0,1)$*, we have*

$$\sup_{N}\left[\frac{\mathrm{Var}\,(\widehat{\theta}_{\mathrm{SAE}})-\mathrm{Var}\,(\widehat{\theta}_{\mathrm{MLE}})}{\mathrm{Var}\,(\widehat{\theta}_{\mathrm{MLE}})}\right]= \left(1-\frac{2}{\log\rho}\right)\left(1+\frac{1-\rho}{\log\rho}\right)-1\,, \tag{16}$$

and

$$\sup_{N,\rho}\left[\frac{\mathrm{Var}\,(\widehat{\theta}_{\mathrm{SAE}})-\mathrm{Var}\,(\widehat{\theta}_{\mathrm{MLE}})}{\mathrm{Var}\,(\widehat{\theta}_{\mathrm{MLE}})}\right]\leq 0.14\,. \tag{17}$$

Propositions 2 and 3 claim that for any sensor number $N \geq 3$, the absolute and relative performance losses of the SAE compared to the performance of the MLE are bounded by a function of ρ, as seen by the right hand side of (14) and (16), respectively. Also, for all N and ρ, the bound on absolute performance loss is found to be $7.2\% \times \frac{\sigma^2}{M}$, while the bound on the relative performance loss is found to be 14%. Therefore, given the linear proximity spacing topology, if the WSN application can tolerate accuracy performance degradations within these bounds, a sequential communication protocol can be implemented. On the other hand, if the accuracy performance loss is too much to sustain, then the centralized protocol with higher transport cost can be used.

4. TWO-DIMENSIONAL TOPOLOGY

4.1 Fixed and Proximity Grid Spacing

To extend the framework to a more realistic WSN scenario, two-dimensional topologies are considered. For the fixed and proximity spacing cases, the sensors form an equally spaced grid across a square coverage area. From the covariance matrix, an analytical expression for the MLE and its corresponding variance has yet to be found. Therefore, simulations were conducted for large N to see the asymptotic trend of the variance of SAE relative to the variance of the MLE. Both figures represent the relationship between the variance of both estimators and ρ for various N. For the fixed spacing topology shown in Fig. (1), as N increases, the gap between the variance of both estimators decreases. Also, for any fixed N, the gap between variance of both estimators is relatively small. In Fig. (2), the effects of random placement in the proximity grid setting were observed. The sensors were placed by a uniform distribution over a square grid. It is evident from Fig. (2) that as N increases, the gap between the variance of the grid proximity spacing and the random proximity spacing decreases. For the case where $N = 32^2 = 1024$, the variance curves between the grid proximity spacing and the random proximity spacing are virtually indistinguishable.

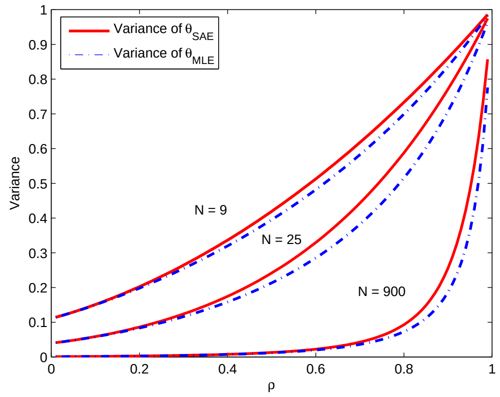

Figure 1: Plot of the variance versus the correlation coefficient ρ for the MLE and the SAE for the cases where $N = 9, 25, 900$. The sensors are in a fixed spaced, grid topology. The entries of the covariance matrix are given by $\Sigma_{i,j} = \sigma^2\rho^{|i-j|}$, ($M = 10, \sigma^2 = 10$).

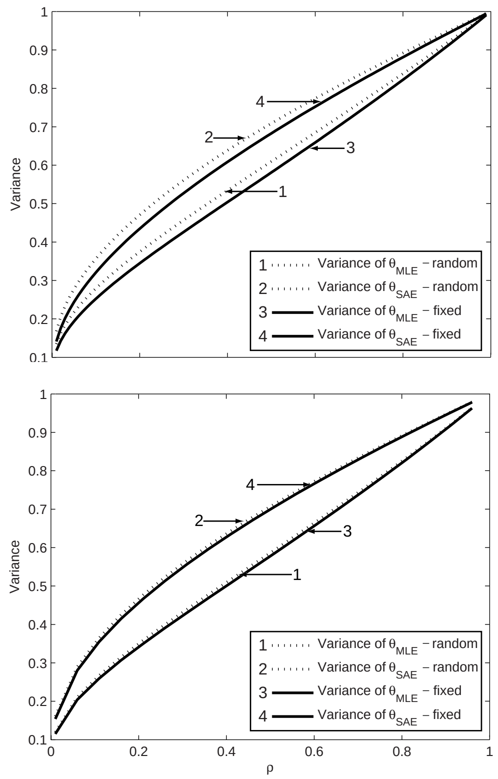

Figure 2: Plot of the variance versus the correlation coefficient ρ for the MLE and the SAE, fixed and random, for the cases where $N = 100$ (top) and $N = 1024$ (bottom). For the random placement, 500 realizations were considered. The sensors are in a random proximity spaced, two-dimensional topology. The entries of the covariance matrix are given by $\Sigma_{i,j} = \sigma^2\rho^{\frac{|i-j|}{N-1}}$, ($M = 10, \sigma^2 = 10$).

4.2 Regular Polygon Spacing

Another approach considers various two-dimensional topologies which allow the calculation of the MLE under dependent observations to be analytically feasible. Interesting results arise from topologies that form regular polygons. If each vertex of a regular polygon is the location of a sensor, then, with the proper labeling, the covariance matrix induced by the topology is circulant. Then, by using the properties of circulant matrices, the following proposition can be stated.

Proposition 4 *For any regular polygon topology, if the elements of the covariance matrix are given by $\Sigma = \sigma^2\rho^\ell$, where ℓ is the distance between the sensors, then*

$$\widehat{\theta}_{MLE} = \widehat{\theta}_{SAE} \ . \tag{18}$$

The claim that $\widehat{\theta}_{MLE} = \widehat{\theta}_{SAE}$ also holds for any three-dimensional regular polyhedron (e.g., triangular pyramid, cube). If the sensors are placed in a regular polygon formation, then the transport cost in communication is significantly reduced and the estimation accuracy incurs no loss in performance. Thus, the regular polygon topology eliminates the effect of dependency in the calculation of the MLE, and hence, reduces the MLE to the SAE.

Proof. To prove Proposition 4, it suffices to show that

$$\frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N} x_{i,j} = \frac{1}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\frac{1}{M}\sum_{i=1}^{M}\mathbf{1}^T\Sigma^{-1}\boldsymbol{x}_i \ .$$

First, given a regular polygon topology, index any arbitrary sensor as sensor 1. Then, in a clockwise or counterclockwise fashion, label the other sensors, in consecutive order, relative to sensor 1. Afterwards, construct the covariance matrix using the fact that the elements are generated by the expression $\Sigma = \sigma^2\rho^\ell$. Due to the rotational symmetry of a regular polygon, Σ is a circulant matrix. A circulant matrix is any matrix, $\text{circ}(a_0, \ldots, a_{N-1}) \in \mathbb{R}^{N\times N}$, of the form

$$\text{circ}(a_0, \ldots, a_{N-1}) \triangleq \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{N-2} & a_{N-1} \\ a_{N-1} & a_0 & a_1 & \cdots & a_{N-3} & a_{N-2} \\ a_{N-2} & a_{N-1} & a_0 & \cdots & a_{N-4} & a_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_0 & a_1 \\ a_1 & a_2 & a_3 & \cdots & a_{N-1} & a_0 \end{bmatrix} .$$

Then, the following properties of circulant matrices are invoked.

(P1) If a matrix A is nonsingular and circulant, then A^{-1} is circulant.

(P2) If a matrix A is circulant, then all the rows and columns have equal sum.

Let C equal the sum of any column in Σ^{-1}. Then,

$$\frac{1}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} = \frac{1}{NC} \ ,$$

and

$$\frac{1}{M}\sum_{i=1}^{M}\mathbf{1}^T\Sigma^{-1}\boldsymbol{x}_i = \frac{1}{M}C\left[\sum_{i=1}^{M}\sum_{j=1}^{N} x_{i,j}\right] .$$

By combining the above results, the proof is complete. ■

5. CONCLUSION

For the covariance matrices studied in this paper, the performance of the SAE is close to that of the optimal MLE. More precisely, for the one-dimensional case, we showed that if the dependent noise structure has the form in (5), the SAE is asymptotically equivalent to the MLE. Also, if the noise covariance has the form in (10), we found numerical and analytical bounds on the performance loss. While we incur a small loss in performance by using the SAE instead of the optimal MLE, there are considerable energy savings due to the sequential nature of calculating the SAE. We also found promising simulation results that show the asymptotic trends extend the the two-dimensional case. Furthermore, for proximity spacing, we showed that random placement on a square performs as well as the grid placement case. Finally, we proved that for regular polygon topologies, the MLE reduces to the SAE.

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