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Extending Laplace and *z* Transform Domains

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ABSTRACT
A generalisation of the Dirac-delta function and its family of derivatives recently proposed as a means of introducing impulses on the complex plane in Laplace and *z* transform domains is shown to extend the applications of Bilateral Laplace and z transforms. Transforms of two-sided signals and sequences are made possible by extending the domain of distributions to cover generalized functions of complex variables. The domains of Bilateral Laplace and *z* transforms are shown to extend to two-sided exponentials and fast-rising functions, which, without such generalized impulses have no transform. Applications include generalized forms of the sampling theorem, a new type of spatial convolution on the *s* and z planes and solutions of differential and difference equations with two-sided infinite duration forcing functions and sequences.

1. INTRODUCTION

Generalized functions have expanded considerably the domain of existence of the Fourier transform [1]-[5]. Weighted spectra leading to impulses on the complex Laplace and z transform planes have been proposed for the exponential decomposition of finite duration signals [6], [7]. The decomposition of infinite duration complex exponential continuous-time and discrete-time signals leads in general to diverging integrals and summation. Generalizing the Dirac-delta impulse has for objective to define transforms for a class of functions which leads to integrals that are not absolutely convergent. In this paper, the distribution theoretic basis of the generalization is presented, followed by properties of the new distributions and the resulting Bilateral transforms. Applications of the bilateral transforms to the solution of differential and difference equations are illustrated through examples.

2. COMPLEX-DOMAIN DISTRIBUTIONS

A generalised distribution $G(s)$, associated with Laplace transform complex domain, as a generalised function of a complex variable $s=\sigma+j\omega$, may be defined as an integral along a straight line contour in the s plane extending from a point $s=\sigma-j\infty$ to $s=\sigma+j\infty$ of the product of $G(s)$ with a test function $\Phi(s)$. For convenience we refer to this integral by the symbol $I_G[\Phi(s)]$, or simply $I_G[\Phi]$, and use the notation

$$I_G\left[\Phi(s)\right] = \left< G(s),\Phi(s)\right>_{\Re[s]=\sigma} = \int_{\sigma-j\infty}^{\sigma+j\infty} G(s)\Phi(s)\,ds.$$

The test function $\Phi(s)$ has derivatives of any order along straight lines in the s plane going through the origin, and tends to zero more rapidly than any power of $|s|$. For example, if the generalised distribution is the generalised impulse $\xi(s)$ [8], [9] we may write

$$I_G\left[\Phi(s)\right] = \left< \xi(s),\Phi(s)\right>_{\Re[s]=\sigma} = \int_{\sigma-j\infty}^{\sigma+j\infty} \xi(s)\Phi(s)\,ds = \begin{cases} j\Phi(0), & \sigma=0 \\ 0, & \sigma\neq 0. \end{cases}$$

Basic Properties
In the following a selection of basic properties of generalized distributions in the context of the

continuous-time domain and Laplace transform is included due to their importance in evaluating transforms.

Shift in s Plane

Letting $s_0 = \sigma_0 + j\omega_0$ we may erite

$$< G(s-s_0), \Phi(s) >_{\Re[s]=\sigma} = \int_{\sigma-j\infty}^{\sigma+j\infty} G(s-s_0)\Phi(s)ds.$$

Letting $s - s_0 = y, \quad ds = dy$ we obtain

$$< G(s-s_0), \Phi(s) >_{\Re[s]=\sigma} \\ =< G(y), \Phi(y+s_0) >_{\Re[y]=\sigma-\sigma_0} .$$

Scaling

Let $\gamma \neq 0$ be a real constant. We can write

$$< G(\gamma s), \Phi(s) >_{\Re[s]=\sigma} = \int_{\sigma-j\infty}^{\sigma+j\infty} G(\gamma s)\Phi(s)ds.$$

Letting $\gamma s = y, \quad \gamma ds = dy$ we obtain

$$< G(\gamma s), \Phi(s) >_{\Re[s]=\sigma} \\ = \frac{1}{|\gamma|} < G(y), \Phi(y/\gamma) >_{\Re[y]=\gamma\sigma} .$$

Product with an Ordinary Function

Consider the product $G(s)F(s)$. We can write

$$< G(s)F(s), \Phi(s) >_{\Re[s]=\sigma} \\ =< G(s), F(s)\Phi(s) >_{\Re[s]=\sigma}$$

if $F(s)\Phi(s) \in C$, the class of test functions.

Convolution

Denoting by $G_1(s) * G_2(s)$ the convolution of two generalised distributions, with $y = \Sigma + j\Omega$, we may write

$$I =< G_1(s) * G_2(s), \Phi(s) >_{\Re[s]=\sigma} \\ =< \int_{\Sigma-j\infty}^{\Sigma+j\infty} G_1(y) G_2(s-y)dy, \Phi(s) >_{\Re[s]=\sigma}$$

$$I =< G_1(y), \int_{\sigma-j\infty}^{\sigma+j\infty} G_2(s-y)\Phi(s)ds >_{\Re[y]=\Sigma}$$

the integral on the right, being in the form of a convolution with a test function, belongs to the class of test functions.

Differentiation

$$< G'(s), \Phi(s) >_{\Re[s]=\sigma} = \int_{\sigma-j\infty}^{\sigma+j\infty} G'(s)\Phi(s)ds.$$

Integrating by parts we obtain

$$< G'(s), \Phi(s) >_{\Re[s]=\sigma} = -< G(s), \Phi'(s) >_{\Re[s]=\sigma}$$

and, by repeated differentiation,

$$< G^{(n)}(s), \Phi(s) >_{\Re[s]=\sigma} \\ = (-1)^n < G(s), \Phi^{(n)}(s) >_{\Re[s]=\sigma} .$$

Multiplication of the Derivative Times an Ordinary Function

Consider the product $G'(s)F(s)$. We can write

$$< G'(s)F(s), \Phi(s) >_{\Re[s]=\sigma} \\ = \int_{\sigma-j\infty}^{\sigma+j\infty} G'(s)F(s)\Phi(s)ds.$$

Integrating by parts we obtain

$$< G'(s)F(s), \Phi(s) >_{\Re[s]=\sigma} \\ = -< G(s), F(s)\Phi'(s) >_{\Re[s]=\sigma} \\ -< G(s), F'(s)\Phi(s) >_{\Re[s]=\sigma} .$$

3. DISCRETE TIME DOMAIN

A distribution $G(z)$ may be defined as the value of the integral, denoted $I_G[\Phi(z)]$, of its product with a test function $\Phi(z)$. Symbolically, we write

$$I_G[\Phi(z)] = < G(z), \Phi(z) >_{|z|=r} \\ = \oint_{|z|=r} G(z)\Phi(z)dz$$

where the contour of integration is a circle of radius r centred at the origin in the z plane. Similar properties to those of the continuous time domain are encountered in the discrete-time domain.

4. THE GENERALISED DIRAC_DELTA IMPULSE IN THE *s* DOMAIN

The generalised Dirac-delta impulse denoted $\xi(s)$ [8], [9] may be defined by the relation

$$< \xi(s), \Phi(s) >_{\Re[s]=\sigma} \\ = \begin{cases} \int_{-j\infty}^{j\infty} \xi(s)\Phi(s)ds = j\Phi(0), & \sigma = 0 \\ 0, & \sigma \neq 0. \end{cases}$$

If $F(s)$ is analytic at $s = 0$ then

$$< \xi(s), F(s) >_{\Re[s]=\sigma} = \begin{cases} \int_{-j\infty}^{j\infty} \xi(s) F(s)\, ds = jF(0), & \sigma = 0 \\ 0, & \sigma \neq 0. \end{cases}$$

Some important properties are summarized in the following.

Differentiation

Letting $s_0 = \sigma_0 + j\omega_0$ we may erite

$$< G(s-s_0), \Phi(s) >_{\Re[s]=\sigma} = \int_{\sigma-j\infty}^{\sigma+j\infty} G(s-s_0)\Phi(s) ds$$

$$< \xi^{(n)}(s-s_0), \Phi(s) >_{\Re[s]=\sigma} = \begin{cases} (-1)^n\, j\, \Phi^{(n)}(s_0), & \sigma = \sigma_0 \\ 0, & \sigma \neq \sigma_0. \end{cases}$$

Convolution

$$\xi(s-a) * \xi(s-b) = j\xi\left[s-(a+b)\right].$$

Convolution with an Ordinary Function

$$\xi(s-s_0) * F(s) = jF(s-s_0).$$

Multiplication of an Impulse Times an Ordinary Function

$$\xi(s-a) F(s) = F(a)\xi(s-a).$$

Multiplication by the n^{th} derivative of the Impulse

Applying the property of the derivative times an ordinary function we obtain

$$\xi'(s)F(s) = F(0)\xi'(s) - F'(0)\xi(s).$$

More generally we obtain

$$F(s)\xi^{(n)}(s) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} F^{(k)}(0)\xi^{(n-k)}(s).$$

5. APPLICATION TO DISCRETE-TIME GENERALISED IMPULSES

The discrete-time domain generalised impulse will be denoted by the symbol $\psi(z)$ and is equivalent to the symbol $\zeta(z-1)$ used earlier [8] , that is,

$$\psi(z) = \zeta(z-1)$$

$$< \psi(z), \Phi(z) >_{|z|=r} = \begin{cases} j\Phi(1), & r = 1 \\ 0, & r \neq 1. \end{cases}$$

If $X(z)$ is analytic at $z = 1$ then

$$\oint_{|z|=r} \psi(z) F(z) dz = \begin{cases} jF(1), & r = 1 \\ 0, & r \neq 1. \end{cases}$$

Differentiation

$$< \psi^{(n)}(z),\ \Phi(z) >_{|z|=r} = \begin{cases} j(-1)^n \Phi^{(n)}(1), & r = 1 \\ 0, & r \neq 1. \end{cases}$$

6. DIFFERENTIAL AND DIFFERENCE EQUATIONS WITH TWO-SIDED FORCING FUNCTIONS

In what follows the steady state solution of differential and difference equations are evaluated with non-causal infinite duration two sided functions, i.e. extending from $-\infty$ to $+\infty$.

Example 1

To find the steady state solution of the differential equation

$$y''' + 3y'' + 3y' + y = 2t + 8.$$

Applying the Laplace transform we have

$$\left(s^3 + 3s^2 + 3s + 1\right) Y(s) = -4\pi\xi'(s) + 16\pi\xi(s)$$

$$Y(s) = \frac{-4\pi\xi'(s) + 16\pi\xi(s)}{\left(s^3 + 3s^2 + 3s + 1\right)}$$

i.e.

$$= \frac{-4\pi\xi'(s)}{\left(s^3 + 3s^2 + 3s + 1\right)} + 16\pi\xi(s).$$

Let

$$F(s) = \frac{1}{\left(s^3 + 3s^2 + 3s + 1\right)} = \frac{1}{(s+1)^3}$$

$$\begin{aligned} Y(s) &= -4\pi F(s)\xi'(s) + 16\pi\xi(s) \\ &= -4\pi\left\{F(0)\xi'(s) - F'(0)\xi(s)\right\} + 16\pi\xi(s) \\ &= -4\pi\xi'(s) + 4\pi(-3)\xi(s) + 16\pi\xi(s). \end{aligned}$$

$$y(t) = 2t + 2.$$

Example 2

To evaluate the steady state solution of the difference equation

$$y[n] - by[n-1] = x[n]$$

with

$$x[n] = a^n.$$

Applying the z transform

$$Y(z)\left(1 - bz^{-1}\right) = 2\pi\psi\left(a^{-1}z\right)$$

$$Y(z) = \frac{2\pi\psi\left(a^{-1}z\right)}{\left(1 - bz^{-1}\right)}.$$

Let

$$F(z) = \frac{2\pi}{(1-bz^{-1})},$$

$$Y(z) = F(z)\psi(a^{-1}z)$$

$$= F(a)\psi(a^{-1}z) = \frac{2\pi}{(1-ba^{-1})}\psi(a^{-1}z).$$

Hence

$$y[n] = \frac{a^{n+1}}{a-b}.$$

Example 3
To obtain the steady state solution of the same difference equation with

$$x[n] = n.$$

Applying the z transform to both sides of the equation we have

$$Y(z)(1-bz^{-1}) = -2\pi z\psi'(z)$$

$$Y(z) = \frac{-2\pi z\psi'(z)}{(1-bz^{-1})}.$$

Writing

$$F(z) = \frac{-2\pi z}{(1-bz^{-1})}$$

we have

$$Y(z) = F(z)\psi'(z).$$

Now

$$F(z)\psi'(z) = F(1)\psi'(z) - F'(1)\psi(z),$$

$$F'(z) = \frac{-2\pi(1-2bz^{-1})}{(1-bz^{-1})^2}$$

Hence

$$Y(z) = \frac{-2\pi}{(1-b)}\psi'(z) + \frac{2\pi(1-2b)}{(1-b)^2}\psi(z),$$

$$y[n] = \frac{n-1}{1-b} + \frac{1-2b}{(1-b)^2} = \frac{(1-b)n-b}{(1-b)^2}.$$

7. GENERALIZED SAMPLING IMPULSE TRAIN

Generalized sampling impulse trains which grow or decay exponentially or as a special case are the common uniform constant level train can be transformed by the extended Bilateral Laplace and z transforms.

Fig.1 shows the Laplace transform of an exponentially rising impulse train. The sampling theorem can be readily written as a convolution in the *s* plane

Let

$$\rho_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

be a sampling train chosen for simplicity as a the usual uniform train. A signal $f(t)$, ideally sampled, is given by

$$f_s(t) = f(t)\rho_T(t).$$

The sampled signal spectrum may be directly written in the form

$$f(t)\rho_T(t) \leftrightarrow \frac{1}{2\pi}F(s) * \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\xi\left(s - jn\frac{2\pi}{T}\right)$$

$$f(t)\rho_T(t) \leftrightarrow \frac{1}{T}\sum_{n=-\infty}^{\infty}F\left(s - jn\frac{2\pi}{T}\right)$$

which is possible since now the Laplace transform of the impulse train exists. Similarly, sampling by a growing or decaying impulse train may be effecting in the s plane.

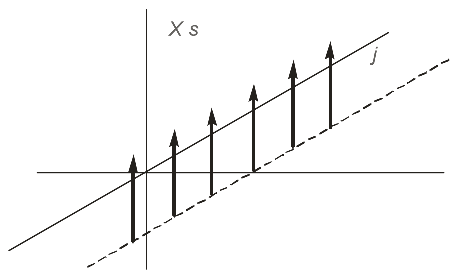

Fig.1 The Laplace transform of an exponentially rising impulse train.

8. NEW EXTENDED TRANSFORMS

Tables 1 and 2 in the Appendix list basic new Laplace and *z* transforms. As can be seen from these tables, thanks to the generalised new distributions, the domains of existence of Laplace and *z* transform are extended to functions that had to date no transform. The Fourier transform, if it exists, can be directly obtained as a special case from the Laplace and z transform, even for functions that lead to impulses on the imaginary axis.

9. CONCLUSION

Bilateral transforms domains are extended by the generalization of distributions to complex variables. In particular the generalization of the Dirac delta impulse in both Laplace and z domains is shown to extend these existence of these transforms and their applications to a large class of two-sided signals.

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Appendix

Table 1 Extended Laplace Transform of Basic Functions

$x_c(t)$	Extended Laplace Transform
1	$2\pi\xi(s)$
e^{at}	$2\pi\ \xi(s-a)$
$\cosh(at)$	$\pi\{\xi[s-a]+\xi[s+a]\}$
$\cosh(j\beta t)$	$\pi\{\delta[\omega-\beta]+\xi[\omega+\beta]\}$
$u(t)$	$1/s+\pi\xi(s)$
$e^{at}u(t)$	$1/(s-a)+\pi\xi(s-a)$
$e^{\alpha t}\cos(\beta t)$	$\pi\{\xi[s-(\alpha+j\beta)]+\xi[s-(\alpha-j\beta)]\}$
$e^{\alpha t}\cos\beta t\ u(t)$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}+\pi\{\xi[s-(\alpha+j\beta)]$ $+\xi[s-(\alpha-j\beta)]\}$
t	$-2\pi d\xi(s)/ds$
t^n	$(-1)^n 2\pi\xi^{(n)}(s)$
$t^n u(t)$	$n!/s^{n+1}+(-1)^n\pi\xi^{(n)}(s)$

Table 2 Extended *z* Transforms of Basic Sequences

$x[n]$	Extended z Transform $X(z)$
1	$2\pi\psi(z)$
a^n	$2\pi\psi(z/a)$
$u[n]$	$\frac{1}{1-z^{-1}}+\pi\psi(z)$
$a^n u[n]$	$\frac{1}{1-az^{-1}}+\pi\psi(z/a)$
$n^r u[n]$	$(-1)^r\sum_{i=1}^{r}S(r,i)\frac{(-1)^i i!}{(z-1)^{i+1}}z^i$ $+\pi\sum_{i=1}^{r+1}(-1)^{i+1}S(r+1,i)\psi^{(i-1)}(z)$
n^r	$2\pi\sum_{i=1}^{r+1}(-1)^{i+1}S(r+1,i)\psi^{(i-1)}(z)$