14th European Signal Processing Conference (EUSIPCO 2006), Florence, Italy, September 4-8, 2006, copyright by EURASIP

MIMO-ISI CHANNEL EQUALIZATION – WHICH PRICE WE HAVE TO PAY FOR CAUSALITY

Holger Boche and Volker Pohl

Technische Universität Berlin, Heinrich Hertz Chair for Mobile Communications
Werner-von-Siemens Bau, Einsteinufer 25, 10587 Berlin, Germany
phone: +49-30-314-28459, fax: +49-30-314-28320, email: {holger.boche, volker.pohl}@mk.tu-berlin.de

ABSTRACT

In the investigation of equalizers and precoders for multiple-input multiple-output systems with intersymbol interference, completely new phenomena appear if the causality of theses filters is required. Both, for transmit as well as for receive filters, the stability norm is an important performance measure which is connected to several performance criteria in communications. The paper shows that the optimal causal precoder with minimal stability norm is linear but time variant, in general. It is time invariant only if the channel is flat fading. Moreover, it is discussed that there exist causal precoders or equalizers for which the stability norm grows exponential with the minimum number of transmit and receive antennas of the MIMO system, whereas the stability norm of the optimal non-causal inverse is always independent from the dimensions of the MIMO system.

1. INTRODUCTION

The use of multiple antennas at both sides of a wireless link is considered as a promising way to achieve high data rates in future communication systems. If the channel is frequency selective, appropriated equalizing techniques have to be used to mitigate the intersymbol interference (ISI). There exist several approaches for this equalization (or pre-equalization) of the multiple-input multiple-output (MIMO) channel. Most popular are multicarrier techniques like orthogonal frequency division multiplexing (OFDM) [1]. In single carrier systems, block transmission [2] or the equalization via finite impulse response (FIR) filters [3] are typically used. In all these approaches the causality constraint is bypassed in a certain way, such that the determination of the optimal receive or transmit filter is a comparatively simple task. Under causality constraints, on the other hand, we have a completely different situation with totally new phenomena which are unknown in the non-causal case. Such phenomena are discussed and investigated in this paper and compared with the better known non-causal case. The understanding of these phenomena will also be important for the design of non-linear receives such as decision-feedback equalizers, since also in these equalizers linear filters are employed.

2. MOTIVATION AND PROBLEM STATEMENT

2.1 System model

Consider a time discrete, linear, frequency selective MIMO system $\mathscr{S}$ with N inputs and M outputs. The input-output relation of $\mathscr{S}$ can be written in general as

$$y_n = \sum_{k=-\infty}^{\infty} \mathbf{H}_k x_{n-k} + \nu_n \ . \tag{1}$$

Therein, $\{x_k\}_{k\in\mathbb{Z}}$ and $\{y_k\}_{k\in\mathbb{Z}}$ are the sequences of the input and output symbols, respectively. Each $x_k \in \mathbb{C}^N$ and $y_k \in \mathbb{C}^M$ is an element of the common N- and M-dimensional Euclidean vector space over the complex numbers $\mathbb{C}$, respectively. The sequence $\{\nu_k\}_{k\in\mathbb{Z}}$ with $\nu_k \in \mathbb{C}^M$ describes additional disturbances (e.g. noise) at the receiver. Finally, the sequence $\{\mathbf{H}_k\}_{k\in\mathbb{Z}}$ of complex $M\times N$ matrices (i.e. $\mathbf{H}_k \in \mathbb{C}^{M\times N}$ for all k) is called the *impulse response* of the linear system $\mathscr{S}$. The system $\mathscr{S}$ is said to be *causal* if $\mathbf{H}_k = 0$ for all $k<0$.

Equation (1) can be expressed equivalently in the frequency domain

$$y(e^{i\omega}) = \mathbf{H}(e^{i\omega})x(e^{i\omega}) + \nu(e^{i\omega}) \ , \quad \omega \in [-\pi,\pi) \tag{2}$$

in which $\mathbf{H}(e^{i\omega})$ is the *matrix transfer function* (MTF) of $\mathscr{S}$ given by the discrete *Fourier transform* of the channel impulse response $\{\mathbf{H}_k\}_{k\in\mathbb{Z}}$

$$\mathbf{H}(e^{i\omega}) = \sum_{k=-\infty}^{\infty} \mathbf{H}_k e^{ik\omega} \ . \tag{3}$$

Conversely, the single elements $\mathbf{H}_k$ of the impulse response are equal to the *Fourier coefficients* of $\mathbf{H}(e^{i\omega})$ given by

$$\mathbf{H}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{H}(e^{i\omega}) e^{-ik\omega}\, \mathrm{d}\omega \ . \tag{4}$$

Similarly, the vector functions $x_k(e^{i\omega})$, $y_k(e^{i\omega})$, and $\nu_k(e^{i\omega})$ are given by the Fourier transform of the corresponding sequences $\{x_k\}$, $\{y_k\}$, and $\{\nu_k\}$, respectively.

2.2 Signal and operator norms

Let $1 \le p \le \infty$, then $L^p(\mathbb{C}^N)$ denotes the space of all measurable vector functions $x : \mathbb{T} \to \mathbb{C}^N$ defined on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ with values in $\mathbb{C}^N$ for which the norm

$$\|x\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left\|x(e^{i\omega})\right\|_{\mathbb{C}^N}^p \mathrm{d}\omega\right)^{1/p} \ , \quad p < \infty \tag{5}$$

or

$$\|x\|_\infty = \sup_{\omega\in[-\pi,\pi]} \|x(e^{i\omega})\|_{\mathbb{C}^N}$$

is finite. Therein, $\|\cdot\|_{\mathbb{C}^N}$ denotes the usual euclidean norm in $\mathbb{C}^N$. Of particular interest will be the space $L^2(\mathbb{C}^N)$ since it is Hilbert space under the scalar product

$$\langle x,y\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\langle x(e^{i\omega}), y(e^{i\omega})\right\rangle_{\mathbb{C}^N} \mathrm{d}\omega \, .$$

Again, $\langle x(e^{i\omega}), y(e^{i\omega})\rangle_{\mathbb{C}^N} = y^H(e^{i\omega})\, x(e^{i\omega})$ denotes the usual scalar product in $\mathbb{C}^N$. Note that the energy of a signal vector can be identified with the square of its $L^2(\mathbb{C}^N)$-norm.

Throughout this paper, it is always assumed that the signal vectors x, y and the noise v in (2) have finite energy, i.e. that they belong to the corresponding L^2-space. Then, the linear MIMO system $\mathscr{S}$ represents an operator $\mathscr{S} : L^2(\mathbb{C}^N) \rightarrow L^2(\mathbb{C}^M)$. It is clear that these operators can be identified with matrix transfer functions $\mathbf{H}(e^{i\omega})$ on $\mathbb{T}$ with values in the space of complex $M \times N$ matrices, and the adjoint operator $\mathscr{S}^*$ can be identified with the complex-conjugate, transposed matrix $\mathbf{H}^*(e^{i\omega})$ for all $\omega \in \mathbb{T}$. The L^2-norm of the signals induces a corresponding (energy) norm for the operators $\mathscr{S}$ by

$$\|\mathscr{S}\|_E := \sup_{x \in L^2(\mathbb{C}^N)} \frac{\|\mathscr{S}x\|_2}{\|x\|_2} . \tag{6}$$

This norm describes the maximal amplification of the signal energy by the linear system $\mathscr{S}$. It is called the *stability norm of* $\mathscr{S}$ and can be expressed as the supremum norm of the MTF $\mathbf{H}$ of $\mathscr{S}$, i.e. it holds $\|\mathscr{S}\|_E = \|\mathbf{H}\|_\infty$ with

$$\|\mathbf{H}\|_\infty = \operatorname*{ess\,sup}_{\omega \in [-\pi,\pi)} \sqrt{\lambda_{max}\left[\mathbf{H}^H(e^{i\omega})\mathbf{H}(e^{i\omega})\right]} . \tag{7}$$

The space of all $M \times N$ matrix transfer functions $\mathbf{H}$ for which $\|\mathbf{H}\|_\infty < \infty$ is denoted by $L^\infty(\mathbb{C}^{M\times N})$.

Every function $\mathbf{H}$ in $L^p(\mathbb{C}^N)$ or $L^\infty(\mathbb{C}^{M\times N})$ with arbitrary dimensions N and M has a Fourier series representation (3) with the Fourier coefficients given by (4). The subspaces of $L^p(\mathbb{C}^N)$ and $L^\infty(\mathbb{C}^{M\times N})$ of all functions for which the Fourier coefficients with negative index $k < 0$ are zero are denoted by $H^p(\mathbb{C}^N)$ and $H^\infty(\mathbb{C}^{M\times N})$, respectively. Each element in these spaces may be interpreted as an analytic function inside the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with values in $\mathbb{C}^N$ or $\mathbb{C}^{M\times N}$, respectively:

$$\mathbf{H}(z) = \textstyle\sum_{k=0}^\infty \mathbf{H}_k z^k .$$

Conversely, each analytic function in $\mathbb{D}$ can be identified with its boundary values $\mathbf{H}(e^{i\omega})$ since $\lim_{r\to 1} \mathbf{H}(re^{i\omega}) = \mathbf{H}(e^{i\omega})$ almost everywhere $\omega \in [-\pi,\pi)$ [4]. The space $H^\infty(\mathbb{C}^{M\times N})$ can be identified with the space of all linear operators $\mathscr{S} : H^2(\mathbb{C}^N) \rightarrow H^2(\mathbb{C}^M)$. Similarly as above, the L^2-norm of the signals in $H^2(\mathbb{C}^N)$ induces an operator norm (6), but know the supremum has to be taken over all $x \in H^2(\mathbb{C}^N)$.

2.3 Problem statement

Assume that a causal and stable linear MIMO system $\mathscr{S}$ with N inputs and $M \leq N$ outputs and with the transfer function $\mathbf{H} \in H^\infty(\mathbb{C}^{M\times N})$ is given. The input output relation is given by (2), in which $x \in H^2(\mathbb{C}^N)$ and $y \in H^2(\mathbb{C}^N)$. We consider the following three problems:

Problem 1 (LTI - Inverses): We are looking for the transfer function $\mathbf{G} \in H^\infty(\mathbb{C}^{N\times M})$ of a linear, time-invariant (LTI), stable, and causal right inverse which satisfies

$$\mathbf{H}(e^{i\omega})\,\mathbf{G}(e^{i\omega}) = \mathbf{I}_M \quad \text{for all } \omega \in [-\pi,\pi) .$$

It is known that such a right inverse exists, if $\mathbf{H}$ satisfies the condition

$$\mathbf{H}(z)\,\mathbf{H}^H(z) \geq \delta^2 \mathbf{I}_M \quad \text{for all } |z| < 1 . \tag{8}$$

If such a right inverse is known, it can be used as a pre-equalizer for the linear system with MTF $\mathbf{H}$ which determines the necessary transmit signal $x = \mathbf{G}u$ such that the desired signals $u \in H^2(\mathbb{C}^M)$ are received at the output of the channel $y = \mathbf{H}\mathbf{G}\,u + v = u + v$. Practically, the transmit power is upper bounded by a certain value P_{max}, i.e. $\|x\|_2^2 \leq P_{max}$. But this transmit power restriction yields a reduction in the effective receive power, because the relation

$$\|x\|_2^2 = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} u^H(e^{i\omega})\mathbf{G}^H(e^{i\omega})\mathbf{G}(e^{i\omega})u(e^{i\omega})\,\mathrm{d}\omega \leq P_{max}$$

implies that $\|u\|_2^2 \leq P_{max}/\|\mathbf{G}\|_\infty^2$, which shows that the effective receive power $\|u\|_2^2$ decreases as the H^∞-norm of $\mathbf{G}$ increases. Since there exists in general more that one right inverse, it is desirable to find a right inverse with the least supremum norm. Such an optimal right inverse is denoted by $\widehat{\mathbf{G}}$, and the minimal norm by

$$CLTI(\mathbf{H}) = \|\widehat{\mathbf{G}}\|_\infty = \inf \|\mathbf{G}\|_\infty$$

where the infimum is taken over all possible right inverses $\mathbf{G} \in H^\infty(\mathbb{C}^{N\times M})$. Moreover, it can be shown that there actually exists a causal LTI inverse $\widehat{\mathbf{G}}$ for which the infimum on the right hand side is achieved.

Problem 2 (Linear Inverses): In the second problem, we look for linear right inverses $\mathbf{R}_I : H^2(\mathbb{C}^M) \rightarrow H^2(\mathbb{C}^N)$ such that $\mathbf{H}\,\mathbf{R_I} = \mathbf{I}_{H^2(\mathbb{C}^M)}$. Thus, compared to Problem 1 we now do not require that the right inverse is time invariant. Again, we are interested in the right inverse with the least stability norm, which will be denoted by $\widehat{\mathbf{R}}_I$, and the minimal norm is denoted by

$$ON(\mathbf{H}) = \|\widehat{\mathbf{R}_I}\|_\infty = \inf \|\mathbf{R}_I\|_\infty$$

in which the infimum is again be taken over all possible linear right inverse $\mathbf{R_I} : H^2(\mathbb{C}^M) \rightarrow H^2(\mathbb{C}^N)$. In Section 3 the optimal solution $\widehat{\mathbf{R}}_I$ will be given explicitly, which will show that the infimum is actually achieved.

Problem 3 (Non-linear Inverses): Finally, we drop also the requirement of linearity on the right inverses. Then, to every required receive signal $u \in H^2(\mathbb{C}^M)$, we ask for the corresponding transmit signal $x \in H^2(\mathbb{C}^N)$ such that $u = \mathbf{H}x$ and which has minimal energy. Thus, to a given $u \in H^2(\mathbb{C}^M)$, we look for an $x_u \in H^2(\mathbb{C}^N)$ with minimal $H^2(\mathbb{C}^N)$-norm, i.e. such that

$$\|x_u\|_2 = \inf\left\{ \|x\|_2 \;:\; x \in H^2(\mathbb{C}^N) \,,\; u = \mathbf{H}x \right\}$$

The corresponding optimal operator norm is then given by

$$OI(\mathbf{H}) = \sup\left\{ \|x_u\|_{H^2(\mathbb{C}^N)} \;:\; \|u\|_{H^2(\mathbb{C}^M)} \leq 1 \right\} .$$

From the above three problems, it is immediately clear

$$OI(\mathbf{H}) \leq ON(\mathbf{H}) \leq CLTI(\mathbf{H}) .$$

Thus, the norm of the LTI-inverse is always larger (or equal) as the norm of the non-linear inverse, since the set of all LTI inverses is a subset of all (possibly non-linear) inverses.

2.4 Riesz projection and Toeplitz operators

This section, shortly reviews some known fact from functional analysis and gives some important operators with some of their properties which are needed subsequently.

Assume that $(\mathscr{H}_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathscr{H}_2, \langle\cdot,\cdot\rangle_2)$ are two Hilbert spaces with a corresponding scalar product and let $L : \mathscr{H}_1 \to \mathscr{H}_2$ be a linear operator with the domain $D(L)$. Recall, that the *adjoint* L^* of L is uniquely defined by the equation

$$\langle Lx, y\rangle_2 = \langle x, L^* y\rangle_1 \quad \text{for all} \quad x \in D(L) .$$

Let $x \in L^2(\mathbb{C}^N)$ be of the form $x(e^{i\omega}) = \sum_{k=-\infty}^{\infty} x_k e^{ik\omega}$. The orthogonal projection $P_+ : L^2(\mathbb{C}^N) \to H^2(\mathbb{C}^N)$ is defined as $(P_+ x)(e^{i\omega}) = \sum_{k=0}^{\infty} x_k e^{ik\omega}$ and it is called the *Riesz projection*. As every orthogonal projection, P_+ is self-adjoint, i.e. $\langle P_+ x, y\rangle = \langle x, P_+ y\rangle$ for all $x, y \in L^2(\mathbb{C}^N)$. In the same way, P_- denotes the orthogonal projection from $L^2(\mathbb{C}^N)$ onto the orthogonal complement of $H^2(\mathbb{C}^N)$: $(P_- x)(e^{i\omega}) = \sum_{k=-\infty}^{-1} x_k e^{ik\omega}$.

Let $\mathbf{\Phi} \in L^\infty(\mathbb{C}^{M\times N})$. Then the *multiplication operator* $M_{\mathbf{\Phi}} : L^2(\mathbb{C}^N) \to L^2(\mathbb{C}^M)$ is given by

$$(M_{\mathbf{\Phi}} x)(e^{i\omega}) := \mathbf{\Phi}(e^{i\omega}) \cdot x(e^{i\omega}) , \quad \omega \in [-\pi, \pi) . \tag{9}$$

The adjoint operator is obviously given by $M^*_{\mathbf{\Phi}} = M_{\mathbf{\Phi}^*}$, and the operator norm is $\|M_{\mathbf{\Phi}}\| = \|\mathbf{\Phi}\|_\infty$.

Let $\mathbf{\Phi} \in L^\infty(\mathbb{C}^{M\times N})$, then the *Toeplitz operator* (with symbol $\mathbf{\Phi}$) $T_{\mathbf{\Phi}} : H^2(\mathbb{C}^N) \to H^2(\mathbb{C}^M)$ is defined by

$$T_{\mathbf{\Phi}} x := P_+(M_{\mathbf{\Phi}} x) .$$

The adjoint operator is given by $T^*_{\mathbf{\Phi}} = T_{\mathbf{\Phi}^*} = P_+ M_{\mathbf{\Phi}^*}$, and for the operator norm it can be shown that $\|T_{\mathbf{\Phi}}\| = \|\mathbf{\Phi}\|_\infty$.

3. INVERSES

3.1 Causal inverses

This section investigates the three problems formulated in Section 2.3. Given a causal and stable MTF $\mathbf{H} \in H^\infty(\mathbb{C}^{M\times N})$. Without loss of generality, we always assume that $\|\mathbf{H}\|_\infty \leq 1$. Using the notations of Section 2.4, Problem 3 can be reformulated as: To every $u \in H^2(\mathbb{C}^M)$ we look for an $x_u \in H^2(\mathbb{C}^N)$ such that

$$T_{\mathbf{H}} x_u = u \tag{10}$$

and such that x_u has the minimal norm among all $x \in H^2(\mathbb{C}^N)$ which satisfy (10). Since the spaces $H^2(\mathbb{C}^N)$ and $H^2(\mathbb{C}^M)$ are Hilbert spaces, the non-linear Problem 3 can be lead back to a linear optimization problem. It has a solution, if and only

$$\inf_{\|u\|_{H^2(\mathbb{C}^M)}=1} \|P_+(\mathbf{H}^* u)\|_{H^2(\mathbb{C}^N)} = \delta_c > 0 . \tag{11}$$

If this condition is satisfied, the solution of (10) with minimal norm is given by

$$x_u = (T_{\mathbf{H}})^\dagger u = T^*_{\mathbf{H}}(T_{\mathbf{H}} T^*_{\mathbf{H}})^{-1} u . \tag{12}$$

Thereby, the operator $(T_{\mathbf{H}})^\dagger$ is called the *generalized inverse* or *pseudo-inverse* of the operator $T_{\mathbf{H}}$. This solution is well known, and a strict proof may be found e.g. in [5, Chapter 8]. Here, we shortly give only a formal way, how this solution may obtained. Thereby, the interrseting point will be that the behavior of $T^*_{\mathbf{H}}$ is directly related to the condition (11). Therefore (11) can be used to analyze the impact of the restriction to causal inverses.

It is known that the range of $T^*_{\mathbf{H}}$ is equal to the orthogonal complement of the null-space of $T_{\mathbf{H}}$: $R(T^*_{\mathbf{H}}) = N(T_{\mathbf{H}})^\perp$. For a fixed $u \in H^2(\mathbb{C}^M)$, the pre-image x_u with minimal norm is an element of the $N(T_{\mathbf{H}})^\perp$ and therefore $x_u \in R(T^*_{\mathbf{H}})$. But this means that there exists a $w_u \in H^2(\mathbb{C}^M)$ such that $x_u = T^*_{\mathbf{H}} w_u$ and consequently $u = T_{\mathbf{H}} T^*_{\mathbf{H}} w_u$. Since, because of (11)

$$\langle T_{\mathbf{H}} T^*_{\mathbf{H}} u, u\rangle = \langle T^*_{\mathbf{H}} u, T^*_{\mathbf{H}} u\rangle = \|T^*_{\mathbf{H}} u\|_{H^2(\mathbb{C}^N)} > \delta . \tag{13}$$

The operator $T_{\mathbf{H}} T^*_{\mathbf{H}}$ is a bijective mapping from $H^2(\mathbb{C}^M)$ onto $H^2(\mathbb{C}^M)$. Therefore, its inverse exists, and one gets $w_u = (T_{\mathbf{H}} T^*_{\mathbf{H}})^{-1} u$ and finally (12).

So, (12) gives the solution to the non-linear Problem 3. It is notable that the optimal solution is given by a *linear* mapping $(T_{\mathbf{H}})^\dagger$. Consequently, (12) solves also Problem 2 in Section 2.3, and the optimal operator norms are equal in both cases: $OI(\mathbf{H}) = ON(\mathbf{H})$. Moreover, from (12) follows

$$\begin{aligned} \|x_u\|_{H^2(\mathbb{C}^N)} &\leq \left\|T^*_{\mathbf{H}}\right\| \cdot \left\|(T_{\mathbf{H}} T^*_{\mathbf{H}})^{-1} u\right\|_{H^2(\mathbb{C}^M)} \qquad (14) \\ &\leq \left\|(T_{\mathbf{H}} T^*_{\mathbf{H}})^{-1}\right\|_{H^2(\mathbb{C}^M)\to H^2(\mathbb{C}^M)} \cdot \|u\|_{H^2(\mathbb{C}^M)} \\ &= \sup_{\|u\|_{H^2(\mathbb{C}^M)=1}} \frac{1}{\|T^*_{\mathbf{H}} u\|_{H^2(\mathbb{C}^N)}} \end{aligned}$$

where for the last line (13) was used. It can be shown that this bound is sharp. Therefore one gets

$$\begin{aligned} OI(\mathbf{H}) = ON(\mathbf{H}) &= \sup_{\|u\|_{H^2(\mathbb{C}^M)=1}} \frac{1}{\|T^*_{\mathbf{H}} u\|_{H^2(\mathbb{C}^N)}} \\ &= \frac{1}{\inf_{\|u\|_{H^2(\mathbb{C}^M)=1}} \|T^*_{\mathbf{H}} u\|_{H^2(\mathbb{C}^N)}} = \frac{1}{\delta_c} . \end{aligned}$$

However, even though the solution (12) is linear, it is time variant. Thus, it is not a solution of Problem 1 in Section 1. Only in the case of a flat fading channel, (12) is time invariant. Moreover, it can be shown that $CLTI(\mathbf{H}) = ON(\mathbf{H}) = OI(\mathbf{H})$, in general. However, this is not trivial. Thus, the H^∞-norm of an optimal right inverse $\widehat{\mathbf{G}}$ is determined by the constant δ_c.

3.2 Comparison with non-causal inverses

If no causality on the inverse is required, the problem (10) would be written as

$$M_{\mathbf{H}} x_u = u \tag{15}$$

in which $M_{\mathbf{H}} : L^2(\mathbb{C}^N) \to L^2(\mathbb{C}^M)$ is simply the multiplication operator (9). Similar as in the causal case, we obtain the optimal x_u which satisfies (15) and has minimal norm by the generalized inverse of $M_{\mathbf{H}}$

$$x_u = (M_{\mathbf{H}})^\dagger u = M^*_{\mathbf{H}}(M_{\mathbf{H}} M^*_{\mathbf{H}})^{-1} u . \tag{16}$$

The operator norm of this non-causal inverse is given by

$$\left\|(M_{\mathbf{H}})^\dagger\right\|_{L^2(\mathbb{C}^M)\to L^2(\mathbb{C}^N)} = \frac{1}{\delta_{nc}} .$$

with

$$\delta_{nc} = \inf_{\|u\|_{L^2(\mathbb{C}^M)=1}} \|M^*_{\mathbf{H}} u\|_{L^2(\mathbb{C}^N)} \tag{17}$$

It can be shown that δ_{nc} is the largest constant for which (8) holds. Next, we will show that this constant δ_{nc}, which determines the norm of the generalized inverse is always larger or equal than the corresponding constant δ_c in the causal case. For every $u \in H^2(\mathbb{C}^M)$ holds

$$\begin{aligned} \|M^*_{\mathbf{H}} u\|^2_{L^2} &= \|M_{\mathbf{H}^*} u\|^2_{L^2} = \|P_+(M_{\mathbf{H}^*} u) + P_-(M_{\mathbf{H}^*} u)\|^2_{L^2} \\ &= \|P_+(M_{\mathbf{H}^*} u)\|_{L^2} + \|P_-(M_{\mathbf{H}^*} u)\|^2_{L^2} \\ &\geq \|P_+(M_{\mathbf{H}^*} u)\|^2_{L^2} = \|T_{\mathbf{H}^*} u\|^2_{H^2} \end{aligned} \tag{18}$$

where for the second line the Pythagoras formula was applied. Therewith, it follows immediately from (11) and (17) that $\delta_c \leq \delta_{nc}$ and consequently that $\|(T_{\mathbf{H}})^\dagger\| \geq \|(M_{\mathbf{H}})^\dagger\|$. Thus, the operator norm of the non-causal pseudoinverse is always smaller or equal than the norm of the causal pseudoinverse. Of course, this is what we would expect. However the considerations in (18) show where this loss in the causal case comes from. Due to the projection onto the causal part in the operator $T^*_{\mathbf{H}}$, the anti-causal part $\|P_-(M_{\mathbf{H}^*} u)\|^2_{L^2}$ is completely lost compared to the non-causal part. This truncation of the non-causal part of the signal reduce the constant δ_c and therefore increases the norm of the right inverse.

The symbol u is causal. However, due to the multiplication with the anti-causal MTF $\mathbf{H}^*$, some parts of u are transformed to the negative part of the time axis. This anti-causal part is cut of by the projection P_+. The corresponding signal energy can not be used. The loss due the causality constrain is obviously as larger as more signal energy is shifted on the anti-causal part by the non-causal MTF.

Finally, we will have a closer look on the causal pseudoinverse (12). As it was shown above, the operator $(T_{\mathbf{H}} T^*_{\mathbf{H}})^{-1}$ is an one-to-one mapping of the whole signal space $H^2(\mathbb{C}^M)$ onto itself. However, this operator is responsible for the enhancement of the operator norm compared to the non-causal case. Because the subsequent multiplication with $T^*_{\mathbf{H}}$ in (12) is only a mapping onto the "larger" space $H^2(\mathbb{C}^N)$. But since $\|T^*_{\mathbf{H}}\| = 1$, this mapping changes not the signal energy (cf. (14)).

3.3 Structure of the left- and right inverse

There exist several different approaches to study the behavior of the causal left and right inverses. Here some of these approaches are considered for left inverses. Let $\mathbf{H} \in H^\infty(\mathbb{C}^{M\times N})$ be an MTF with $M \geq N$ and for which there exists a constant $\delta_{nc} > 0$ such that

$$\mathbf{H}^*(z)\mathbf{H}(z) \geq \delta^2_{nc} \mathbf{I}_N \ , \quad \text{for all } |z| < 1 \tag{19}$$

where δ_{nc} is understood to be the largest constant for which (19) holds. Note, that because of the assumed normalization $\|\mathbf{H}\|_\infty \leq 1$, the constant δ_{nc} is always smaller or equal than 1. It is known that there always exists a so called *inner-outer factorization* of $\mathbf{H}$

$$\mathbf{H} = \mathbf{H}_{in} \cdot \mathbf{H}_{out}$$

in which $\mathbf{H}_{out} \in H^\infty(\mathbb{C}^{N\times N})$ is an invertible matrix which causes no problems, in general. Because from (19) follows immediately that $\mathbf{H}^*_{out}(z)\mathbf{H}_{out}(z) \geq \delta^2_{nc} \mathbf{I}_N$ for all $|z| < 1$, and therefore $\|\mathbf{H}^{-1}_{out}\|_{H^\infty} \leq 1/\delta_{nc}$. So only the behavior of the inner factor $\mathbf{H}_{in}$ has to investigated. Therefore, $\mathbf{H}$ can be assumed to be inner in the following.

1) One method to investigate the behavior of the left inverses is based on a parametrization of all possible left inverses via an extension matrix of $\mathbf{H}$ [6, 7]: If condition (19) is satisfied, then there exist matrices $\mathbf{E} \in H^\infty(\mathbb{C}^{M\times(M-N)})$ such that the extended matrix $\mathbf{H}_E := [\mathbf{H}\ \mathbf{E}] \in H^\infty(\mathbb{C}^{M\times M})$ is invertible. Note that the matrix $\mathbf{E}$ can be chosen to be inner. However, the resulting $\mathbf{H}_E$ is not unitary. The inverse

$$\mathbf{H}^{-1}_E = \begin{bmatrix} \mathbf{G} \\ \mathbf{R} \end{bmatrix}$$

gives a complete parametrization of all left inverses $\mathbf{G} \in H^\infty(\mathbb{C}^{N\times M})$. Every left inverse defines a projector $\mathbf{P}_G = \mathbf{HG}$, and the complementary projector $\mathbf{Q}_G = \mathbf{I} - \mathbf{P}_G$ is given by $\mathbf{Q}_G = \mathbf{ER}$. Thus, it holds $\mathbf{HG} + \mathbf{ER} = \mathbf{I}_M$.

2) Also the second method for the investigation of the left inverses is based on an extension of the given matrix $\mathbf{H}$ [8]. Thereby $\mathbf{H}^T$ is analyzed. If condition (19) is satisfied, the null space $N(\mathbf{H}^T)$ can be parametrized by an inner matrix $\mathbf{V} \in H^\infty(\mathbb{C}^{M\times(M-N)})$ such that

$$N(\mathbf{H}^T(z)) = \mathbf{V}(z)\ \mathbb{C}^{M-N} \tag{20}$$

for all $|z| < 1$ and $|z| = 1$ almost everywhere. Because of the representation (20) of the null space, one has

$$N(\mathbf{H}^T(z)) = \left(\overline{\mathbf{V}}(z)\ \mathbb{C}^{M-N}\right)^\perp .$$

Therewith the unitary matrix $\mathbf{H}_V(z) = [\mathbf{H}(z)\ \overline{\mathbf{V}}(z)]$ can be defined. Note that $\mathbf{H}_V$ does not belong to $H^\infty(\mathbb{C}^{M\times M})$. The matrix $\mathbf{H}_V$ is invertible, and therefore we have

$$\mathbf{H}^{-1}_V \mathbf{H}_V = \begin{bmatrix} \mathbf{G} \\ \mathbf{G}_V \end{bmatrix} \cdot [\mathbf{H}\ \overline{\mathbf{V}}] = \begin{bmatrix} \mathbf{GH} & 0 \\ 0 & \mathbf{G}_V\overline{\mathbf{V}} \end{bmatrix} = \mathbf{I}\ . \tag{21}$$

It should be noted that there exists a direct relation between the extension by $\mathbf{V}$ and the method described under 1) using the projectors $\mathbf{P}_G$ and $\mathbf{Q}_G$. It can be shown that the null space of $\mathbf{H}^T$ is parametrized by $\mathbf{P}^T_G$. Moreover, if the inner function of $\mathbf{P}^T_G$ is used, one obtains the extension matrix $\mathbf{V}$.

Interesting is now that this extension of $\mathbf{H}$ by $\mathbf{V}$ can be used to get a better understanding of the constant δ_c in (11) which determines the operator norm of the optimal causal inverse. In [8], it was shown that the norm of the best left inverse of $\mathbf{H}$ is equal to the norm of the best left inverse of $\mathbf{V}$. There, the case $M = N+1$ was considered, and the considerations in [8] show that the constant δ_c has a completely different behavior for $M > N$ than for the quadratic case $M = N$. In the case $M = N+1$, $\mathbf{V}$ is a column vector with $N+1$ entries. If $\mathbf{V}$ would also satisfy a condition like (19):

$$\mathbf{V}^*(z)\mathbf{V}(z) = \textstyle\sum_{k=1}^{N+1} |V_k(z)|^2 \geq \delta^2_{nc} \ , \quad \text{for all } |z| < 1$$

then it would be possible to give an upper bound on the norm of the left inverse. However, due to the construction (20) of $\mathbf{V}$, it is not possible to control the infimum of $\mathbf{V}^*(z)\mathbf{V}(z)$. Moreover, in [8] examples where given such that

$$\textstyle\inf_{|z|<1} \sum_{k=1}^{N+1} |V_k(z)|^2 \sim \delta^{2N}_{nc} \tag{22}$$

with a constant $0 < \delta_{nc} < 1$. Thus the coercive constant decreases exponential with the dimension N. For *all* $\mathbf{V}$ obtained from the construction (20) holds $1 = \mathbf{G}_V\mathbf{V}(z)$, by (21). Using Cauchy-Schwartz inequality gives therefore

$$\sum_{n=1}^{N+1} |[G_V(z)]_n|^2 \cdot \sum_{n=1}^{N+1} |V_n(z)|^2 \geq 1 \ .$$

With the example (22), one gets

$$\sum_{n=1}^{N+1} |[G_V(z)]_n|^2 \geq \frac{1}{\delta_{nc}^{2N}} \ .$$

As mentioned, it was shown in [8] that the minimal norm among all left inverses $\mathbf{G}$ is equal to the minimal norm among all left inverses of the corresponding extension matrices $\mathbf{V}$. Therefore, the last inequality shows that there exist $(N+1)\times N$ channel matrices $\mathbf{H}$ for which the stability norm $\|\mathbf{G}\|_\infty$ of all left inverses is proportional to δ_{nc}^N in which δ_{nc} is the smallest constant for which (19) holds. Of course, similar examples can be constructed for any case with $M > N$.

3.4 Consequences and discussions

1) Thus, if frequency selective MIMO systems are considered with more outputs than inputs ($M > N$). Then, there always exist causal and stable matrix transfer functions $\mathbf{H} \in H^\infty(\mathbb{C}^{M\times N})$ such that for the stability norm of all causal left inverses hold $\|\mathbf{G}\|_\infty \sim \delta_{nc}^{-N}$. Thus, the stability norm of the optimal left inverse depends on the dimension of the MIMO system, in general and grows exponential with the smallest dimension $\min(N,M)$. In the quadratic case, i.e. if $M = N$, the norm of the inverse is independent on the dimension N but depends only on the coercive constant δ_{nc} in (19): $\|\mathbf{G}\|_\infty \sim 1/\delta_{nc}$. Moreover, compare these results again with the non-causal inverses. There, the optimal left inverse with minimal norm is simply the generalized inverse (16) with the operator norm $\|\mathbf{H}^\dagger\|_\infty \sim 1/\delta_{nc}$ independent on the dimension N and M.

In *random matrix theory*, often the ratio $K = M/N$ of the (flat fading) matrix transfer functions $\mathbf{H}$ is held fixed, and then the limit $N \to \infty$ is considered (e.g. [9]). In this limit and under the causality constrain, it is possible to find infinite many MTF's $\mathbf{H}$ which satisfy (19), but for which no causal left inverse exists. However, since these MTF's satisfy (19) a non-causal inverse (16) will exist. It should be noted, that also the introduction of an arbitrary delay $d > 0$ such that $\mathbf{G}(z)\mathbf{H}(z) = z^d\mathbf{I}$, will not eliminate this problem. Also in this case no stable and causal left inverse exists. Also the restriction to finite impulse response (FIR) systems $\mathbf{H}$, will not resolve this problem. If such an FIR-MTF satisfies (19), there will exist a causal left inverse $\mathbf{G}$, but its norm $\|\mathbf{G}\|_\infty$ may become arbitrary large, and even if M and N are finite this norm may still becomes very large, dependent on M and N.

2) Above, we always assumed that the MTF $\mathbf{H}$ belongs to $H^\infty(\mathbb{C}^{M\times N})$. The question is, whether it is possible to find a subspace of "good behaving" function such that the norms of the inverses does not show a dependency on the dimensions of the MTF as in H^∞. To this end, the authors considered spaces of smooth transfer functions [10, 11], in particular the space $A_\alpha(\mathbb{C}^{M\times N})$ of *Hölder continuous MTF of order* α for which

$$\|\mathbf{H}\|_\alpha := \|\mathbf{H}\|_\infty + \sup_{\substack{z_1,z_2\in\mathbb{T}\\ z_1\neq z_2}} \frac{\|\mathbf{H}(z_1)-\mathbf{H}(z_2)\|}{|z_1-z_2|^\alpha} < \infty.$$

where considered. It can be shown that for $1/2 < \alpha < 1$ the norm of the optimal causal inverse does not depend on the dimension of the MTF. This means, that there exists a constant $C_1(\alpha)$ such that for all $M,N \in \mathbb{N}$ with $M > N$ and for all $\mathbf{H} \in A_\alpha(\mathbb{C}^{M\times N})$ with $\|\mathbf{H}\|_\infty \leq 1$ always $CLTI(\mathbf{H}) \leq C_1(\alpha)/\delta_c$ holds, where the right hand side does not depend explicitly on the dimensions M,N. The technique to show this can not be used for the case that $0 < \alpha \leq 1/2$. So it is an interesting question, whether or not there exist dimensional effects for $0 < \alpha \leq 1/2$. However, this question is still open.

REFERENCES

[1] H. Bölcskei, D. Gesbert, and A. J. Paulrayj, “On the capacity of OFDM-based spatial multiplexing systems,” *IEEE Trans. Commun.*, vol. 50, no. 2, pp. 225–234, Feb. 2002.

[2] A. Scaglione, G. Giannakis, and S. Barbarossa, “Redundant Filterbank Precoders and Equalizers, Part I and II,” *IEEE Trans. Signal Processing*, vol. 47, no. 7, pp. 1988–2022, July 1999.

[3] S.-Y. Kung, Y. Wu, and X. Zhang, “Bezout Space-Time Precoders and Equalizers for MIMO Channels,” *IEEE Trans. Signal Processing*, vol. 50, no. 10, pp. 2499–2514, Oct. 2002.

[4] J. B. Garnett, *Bounded Analytic Functions*. New York: Academic Press, 1981.

[5] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*. New York: John Wiley & Sons, 1974.

[6] V. Tolokonnikov, “Extension Problem to an Invertible Matrix,” *Proc. Amer. Math. Soc.*, vol. 117, no. 4, pp. 1023–1030, Apr. 1993.

[7] H. Boche and V. Pohl, “Geometrical Characterization of the Optimal Causal Linear MIMO-Channel Inverse,” in *Proc. 8-th IEEE Intern. Symp. on Wireless Personal Commun. (WPMC)*, Aalbourg, Denmark, Sept. 2005.

[8] S. Treil, “Lower Bounds in the Matrix Corona Theorem and the Codimension One Conjecture,” *Geom. Funct. Anal.*, vol. 14, pp. 1118–1133, 2004.

[9] A. M. Tulino and S. Verdú, “Random Matrix Theory and Wireless Communications,” *Found. and Trends in Commun. and Inform. Theory*, vol. 1, no. 1, pp. 1–182, 2004.

[10] H. Boche and V. Pohl, “General Characterization of Bezout Space-Time Equalizers and Precoders,” in *Proc. IEEE Intern. Zurich Seminar on Commun. (IZS)*, Zurich, Switzerland, Feb. 2006, pp. 30–33.

[11] ——, “MIMO - ISI Channels: Inner-Outer Factorization and Applications to Equalization,” in *Intern. Wireless Commun. and Mobile Computing Conf. (IWCMC)*, Vancouver, Canada, July 2006.