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ML approach to radial acceleration estimation and CRLB computation

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1 Introduction

A radar echo is characterized by a Doppler shift depending on the target radial motion. Usually only ambiguous radial speed is extracted from Moving Target Detector with an accuracy depending on the bank filters width; then the ambiguity may be eliminated employing several algorithms. The radial acceleration may be estimated by the polynomial phase evolution of the collected target echoes. Several work has been done about the computation of the Cramer Rao Lower Bound (CRLB) of estimation [1]; the aim of this paper is to illustrate an algorithm, based on Maximum Likelihood Estimator (MLE), to estimate also radial acceleration. The accuracy of the estimation has been analyzed theoretically by means of the CRLB (Cramer Rao Lower Bound) and by means of Monte Carlo simulation. This kinematic parameter of the target may be exploited, for instance, to discriminate an ABT (Air Breathing Target) vs. a BT (Ballistic Target).

2 Model of the data

Consider a radar system operating with M bursts, each characterized by N pulses, Pulse Repetition Time (PRT) T_j, wavelength λ_j, with j=1,…,M. The signal received from the i-th transmitted pulse of the j-th burst has the following expression:

$$z_{ij} = A_{ij} e^{j\varphi_{ij}} + n_{ij} \tag{1}$$

with i=1,…,N and j=1,…,M. The complex amplitude is represented by A_{ij}, n_{ij} is the noise sample, φ_{ij} is the phase of the received signal, whose value depends on an initial phase and the target movement. Indicate range as ρ, the relationship between the range and the phase is:

$$\varphi_{ij} = \varphi_{0j} + \frac{4\pi}{\lambda_j}\left(\dot{\rho}(iT_j) + \ddot{\rho}\frac{(iT_j)^2}{2}\right) \tag{2}$$

where φ_{0j} is the initial phase which is maintained during each burst; it changes every burst. The range derivative with respect to the time, indicated by $\dot{\rho}$, gives the radial speed; $\ddot{\rho}$ is the second derivative of the range with respect to the time, i.e.: the radial acceleration. The thermal noise samples are Gaussian, independent, with zero mean and variance σ^2 which is considered constant during all the bursts. Assume a Swerling 1 target model, the amplitude is considered constant during each burst, so $A_{1j} = \cdots = A_{Nj} = A_j$; in this model the initial phase of each burst, φ_{0j}, is accounted by the complex value of the amplitude A_j. The probability density function (p.d.f.) of the received signal z_{ij}, conditioned to A_j, $\dot{\rho}$ and $\ddot{\rho}$, is:

$$p_{z_{ij}}(z_{ij} / A_j, \dot{\rho}, \ddot{\rho}) = \frac{1}{\pi\sigma}\exp\left\{-\frac{\left|z_{ij} - A_j e^{j\varphi_{ij}}\right|^2}{\sigma^2}\right\} \tag{3}$$

The received signals and the noise samples can be collected respectively into the matrices $\mathbf{Z} = [\mathbf{z}_1 \ \cdots \ \mathbf{z}_M]$ and $\mathbf{N} = [\mathbf{n}_1 \ \cdots \ \mathbf{n}_M]$. Let $\mathbf{A}$ be the vector containing the amplitudes at each burst and $\mathbf{S} = [\mathbf{s}_1 \ \cdots \ \mathbf{s}_M]$ be the contribution of phase exponential, then the p.d.f. of the vector $\mathbf{z}_j$, conditioned to A_j, $\dot{\rho}$ and $\ddot{\rho}$, is:

$$p_{\mathbf{z}_j}(\mathbf{z}_j / A_j, \dot{\rho}, \ddot{\rho}) = \frac{1}{\pi^N|\mathbf{M}|} e^{-[\mathbf{z}_j - A_j\mathbf{s}_j]^H \mathbf{M}^{-1}[\mathbf{z}_j - A_j\mathbf{s}_j]} \tag{4}$$

where the matrix $\mathbf{M} = \sigma^2\mathbf{I}_N$ is the noise covariance matrix ($\mathbf{I}_N$ is the N-dim identity matrix). Being the received signals $\mathbf{z}_j$ statistically independent, the p.d.f. of matrix $\mathbf{Z}$ can be expressed as the product of the p.d.f. of eq. (4), for j=1,…,M:

$$p_{\mathrm{Z}}(\mathrm{Z}/\mathrm{A}, \dot{\rho}, \ddot{\rho}) = \left(\frac{1}{\pi^N|\mathrm{M}|}\right)^M \prod_{j=1}^{M} e^{-[\mathrm{z}_j - A_j\mathrm{s}_j]^H \mathrm{M}^{-1}[\mathrm{z}_j - A_j\mathrm{s}_j]} =$$

$$= \left(\frac{1}{\pi^N|\mathrm{M}|}\right)^M e^{-\sum_{j=1}^{M}[\mathrm{z}_j - A_j\mathrm{s}_j]^H \mathrm{M}^{-1}[\mathrm{z}_j - A_j\mathrm{s}_j]} \tag{5}$$

3 Characterization of radial acceleration

The Doppler shift due to a relative radial movement of the target is $f_d = 2\dot{\rho}/\lambda$; it is unambiguous if $|f_d| < PRF/2$. By replacing f_d the following limitation is obtained:

$$|\dot{\rho}| < \frac{\lambda \cdot PRF}{4} \tag{6}$$

The Doppler shift due to a relative radial acceleration of the target is $f_d = 2\ddot{\rho}T/\lambda$. The corresponding limitation is:

$$|\ddot{\rho}| < \frac{\lambda \cdot PRF^2}{4} \tag{7}$$

Eq.s (6) and (7) are the ambiguity limitations respectively for the value of $\dot{\rho}$ and $\ddot{\rho}$. The ambiguity problem limits the estimation capabilities of the system: most of the targets of interest have ambiguous speed, so only radial acceleration is correctly estimated. For example, consider a radar working at frequency f=9.1 GHz, with T=50 μs, then the ambiguity limitations are $|\dot{\rho}| < 164 m/s$ and $|\ddot{\rho}| < 206062 m/s^2$. It is evident from the numerical example that there are several targets which are characterized by ambiguous speed, while there isn't any kind of target with ambiguous acceleration. The estimators described in the following sections estimate the ambiguous value of $\dot{\rho}$ referred to the interval given by eq. (6) and the true non ambiguous value of $\ddot{\rho}$.

4 MLE algorithm in case of one burst

Consider the case M=1; the $\ddot{\rho}$ estimation, joint to the unknown parameters ($\dot{\rho}$ and real and complex parts of amplitude, A_R and A_I), is obtained by the maximization of the likelihood function, i.e. the logarithm of p.d.f. of eq. (4):

$$\left(\hat{\dot{\rho}}, \hat{\ddot{\rho}}, \hat{A}_R, \hat{A}_I\right) = \arg\max_{\dot{\rho},\ddot{\rho},A}\left\{\ln\left\{p_{z_1\cdots z_N}(z_1,\ldots,z_N / A,\dot{\rho},\ddot{\rho})\right\}\right\} =$$
$$= \arg\min_{\dot{\rho},\ddot{\rho},A}\left\{\sum_{i=1}^{N}\left|z_i - Ae^{j\varphi_i}\right|^2\right\} = \arg\min_{\dot{\rho},\ddot{\rho},A}\left\{[\mathrm{z} - A\mathrm{s}]^H[\mathrm{z} - A\mathrm{s}]\right\} =$$
$$= \arg\min_{\dot{\rho},\ddot{\rho},A}\left\{Q(\dot{\rho},\ddot{\rho},A)\right\} \quad \textbf{(8)}$$

In the minimization of $Q(\dot{\rho},\ddot{\rho},A)$ the problem of ambiguity of the radial speed must be accounted. The estimation, which is based on the Newton Raphson recursive minimization algorithm, can be separated into four steps as represented in Figure 1:

1. First approximate estimation of amplitude, $\hat{A}_1$:

$$\hat{A}_1 = \frac{1}{N}\sum_{i=1}^{N}|z_i| \quad \textbf{(9)}$$

2. Estimation of ambiguous speed $\hat{\dot{\rho}}_A$ in the hypothesis $\ddot{\rho} = 0$:

$$\hat{\dot{\rho}}_A = \arg\min_{\dot{\rho}}\left\{Q(\dot{\rho},\ddot{\rho},A)\Big|_{\ddot{\rho}=0,\,A=\hat{A}_1}\right\} \quad \textbf{(10)}$$

3. Initialization of the Newton Raphson recursion with the following values:

$$\begin{cases} \hat{A}_{in} = \dfrac{1}{N}\sum z_i \exp\left\{-j\dfrac{4\pi}{\lambda}\left(\hat{\dot{\rho}}_A(iT) + \hat{\ddot{\rho}}_{in}\dfrac{(iT)^2}{2}\right)\right\} \\ \hat{\dot{\rho}}_{in} = \hat{\dot{\rho}}_A \\ \hat{\ddot{\rho}}_{in} = 0 \end{cases} \quad \textbf{(11)}$$

where $\hat{A}_{in}$ is a more accurate estimation of the amplitude.

4. Application of Newton Raphson recursive algorithm to find the minimum of eq. (8).

The Newton Raphson algorithm usually achieves the real value of the acceleration with a reasonable number of iterations when the SNR is enough large.

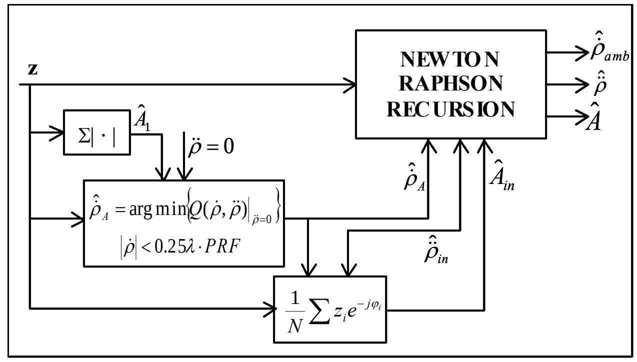

Figure 1: block diagram of the algorithm employed for radial acceleration estimation.

5 CRLB in case of one burst

The computation of Cramer Rao Lower Bound (CRLB) is performed by means of the inversion of the Fisher Information Matrix (FIM). Let $\mathbf{z}$ be the collected data set conditioned to the parameters $\boldsymbol{\xi} = [\xi_1,\ldots,\xi_L]^T$ to be estimated, whose p.d.f. is the Gaussian of eq. (12) where both mean value $\boldsymbol{\mu}(\boldsymbol{\xi})$ and covariance matrix $\mathbf{C}(\boldsymbol{\xi})$ might depend on the estimated parameters:

$$p_{\mathbf{z}}(\mathbf{z}/\boldsymbol{\xi}) = \frac{1}{\pi^N|\mathbf{C}(\boldsymbol{\xi})|}\exp\left\{-[\mathbf{z}-\boldsymbol{\mu}(\boldsymbol{\xi})]^H\mathbf{C}^{-1}(\boldsymbol{\xi})[\mathbf{z}-\boldsymbol{\mu}(\boldsymbol{\xi})]\right\} \quad \textbf{(12)}$$

The i,j element of the FIM J_{ij} is [1]:

$$J_{ij} = \mathrm{Tr}\left\{\mathbf{C}^{-1}(\boldsymbol{\xi})\frac{\partial\mathbf{C}(\boldsymbol{\xi})}{\partial\xi_i}\mathbf{C}^{-1}(\boldsymbol{\xi})\frac{\partial\mathbf{C}(\boldsymbol{\xi})}{\partial\xi_j}\right\} - 2\,\mathrm{Re}\left\{\frac{\partial\boldsymbol{\mu}^H(\boldsymbol{\xi})}{\partial\xi_i}\mathbf{C}^{-1}(\boldsymbol{\xi})\frac{\partial\boldsymbol{\mu}(\boldsymbol{\xi})}{\partial\xi_j}\right\} \quad \textbf{(13)}$$

where Tr indicates the trace of the argument. The FIM has been computed in case of (i) known $\dot{\rho}$, A, and σ^2; (ii) unknown $\dot{\rho}$, known A and σ^2; (iii) unknown $\dot{\rho}$, A, and σ^2. It will be shown analytically that the noise power estimation is totally irrelevant to the target parameters estimation.

5.1 Known radial speed, amplitude and noise power

In this case $\boldsymbol{\xi} = [\ddot{\rho}]$; mean value and covariance matrix of data respectively are $\boldsymbol{\mu}(\boldsymbol{\xi}) = A\mathbf{s}$ and $\mathbf{C} = \sigma^2\mathbf{I}_N$, which doesn't depend on the parameter to be estimated. The FIM reduces to only one term [see *Appendix*]:

$$J = \left(\frac{|A|^2}{\sigma^2}\right)\left(\frac{8\pi^2T^2}{\lambda^2}\right)T^2\sum_{i=1}^{N}i^4 \quad \textbf{(14)}$$

The CRLB is obtained by the inverse of eq. (14). Notice that the ratio $|A|^2/\sigma^2$ coincides with the *SNR*. If the number of pulses N in the time on target T_{onT} is enough large, eq. (14) can be simplified with the following asymptotic expression [see *Appendix*]:

$$\sigma_{\dot{\rho}}^2 \approx \frac{5\lambda^2}{8\pi^2\pi^2(N\cdot SNR_1)(NT)^2} \approx \frac{1}{16}\cdot\frac{\lambda^2}{N\cdot SNR\cdot T_{onT}^4} \tag{15}$$

5.2 Known amplitude and noise power

In this case $\boldsymbol{\xi} = [\dot{\rho}, \ddot{\rho}]^T$. The FIM is [see *Appendix*]:

$$J = \left(\frac{|A|^2}{\sigma^2}\right)\left(\frac{8\pi^2T^2}{\lambda^2}\right)\begin{bmatrix} 4\sum_{i=1}^{N} i^2 & 2T\sum_{i=1}^{N} i^3 \\ 2T\sum_{i=1}^{N} i^3 & T^2\sum_{i=1}^{N} i^4 \end{bmatrix} \tag{16}$$

By the inversion of the FIM and the extraction of the main diagonal elements the CRLB is obtained ($\sigma_{\dot{\rho}}^2 = [J^{-1}]_{1,1}$ and $\sigma_{\ddot{\rho}}^2 = [J^{-1}]_{2,2}$). If N is enough large, CRLB can be simplified with the following asymptotic expressions [see *Appendix*]:

$$\sigma_{\dot{\rho}}^2 \approx \frac{3\lambda^2}{2(N\cdot SNR_1)\pi^2(NT)^2} = \frac{3}{2\pi^2}\cdot\frac{\lambda^2}{N\cdot SNR\cdot T_{onT}^2} \tag{17}$$

$$\sigma_{\ddot{\rho}}^2 \approx \frac{10\lambda^2}{(N\cdot SNR_1)\pi^2(NT)^4} \approx \frac{\lambda^2}{N\cdot SNR\cdot T_{onT}^4} \tag{18}$$

By the comparison of eq.s (15) and (18), it is evident that the estimation of radial speed deteriorates radial acceleration estimation of a factor 4 in term of standard deviation.

5.3 Unknown amplitude and noise power

In this case $\boldsymbol{\xi} = [\dot{\rho}, \ddot{\rho}, A_R, A_I, \sigma^2]^T$. The FIM is [see *Appendix*]:

$$J = \begin{bmatrix} 32\alpha^2\frac{|A|^2}{\sigma^2}\sum_{i=1}^{N} i^2 & 16\alpha^2T\frac{|A|^2}{\sigma^2}\sum_{i=1}^{N} i^3 & \frac{8\alpha}{\sigma^2}\mathrm{Im}\{A^*\}\sum_{i=1}^{N} i & \frac{8\alpha}{\sigma^2}\mathrm{Re}\{A^*\}\sum_{i=1}^{N} i & 0 \\ 16\alpha^2T\frac{|A|^2}{\sigma^2}\sum_{i=1}^{N} i^3 & 8\alpha^2T^2\frac{|A|^2}{\sigma^2}\sum_{i=1}^{N} i^4 & \frac{4\alpha T}{\sigma^2}\mathrm{Im}\{A^*\}\sum_{i=1}^{N} i^2 & \frac{4\alpha T}{\sigma^2}\mathrm{Re}\{A^*\}\sum_{i=1}^{N} i^2 & 0 \\ \frac{8\alpha}{\sigma^2}\mathrm{Im}\{A^*\}\sum_{i=1}^{N} i & \frac{4\alpha T}{\sigma^2}\mathrm{Im}\{A^*\}\sum_{i=1}^{N} i^2 & \frac{2N}{\sigma^2} & 0 & 0 \\ \frac{8\alpha}{\sigma^2}\mathrm{Re}\{A^*\}\sum_{i=1}^{N} i & \frac{4\alpha T}{\sigma^2}\mathrm{Re}\{A^*\}\sum_{i=1}^{N} i^2 & 0 & \frac{2N}{\sigma^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{4N}{\sigma^2} \end{bmatrix} \tag{19}$$

where $\alpha = \pi T/\lambda$. Notice that the elements J_{11}, J_{12}, J_{21}, J_{22} of the matrix are equivalent to the elements of FIM of eq. (16). The estimation of the noise parameter doesn't change the CRLB of the estimation of the target parameters, because the covariance matrix **C** doesn't depend on the target parameters and the mean value $\boldsymbol{\mu}$ doesn't depend on the noise parameter. The estimation of $\dot{\rho}$ and $\ddot{\rho}$ is degraded just only by the joint estimation of the amplitude, because the level of uncertainty has increased. The degradation of the CRLB due to amplitude estimation can be quantified by partitioning the matrix of eq. (19) as follows:

$$J = \begin{bmatrix} \bar{J} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & 4N/\sigma^2 \end{bmatrix} \tag{20}$$

$$\bar{J} = \begin{bmatrix} D & C^T \\ C & B \end{bmatrix} \tag{21}$$

where D is equal to the FIM of eq. (16). The portion of the inverse of the FIM corresponding to D is :

$$\left[\bar{J}^{-1}\right]_D = D^{-1} + D^{-1}C^T(B - CD^{-1}C^T)^{-1}CD^{-1} \tag{22}$$

The term $D^{-1}C^T(B - CD^{-1}C^T)^{-1}CD^{-1}$ quantifies the degradation of the CRLB introduced by the amplitude estimation. When N is enough large the expression of the CRLB can be simplified with the following asymptotic expressions [see *Appendix*]:

$$\sigma_{\dot{\rho}}^2 \approx \frac{12\lambda^2}{2\pi^2(N\cdot SNR_1)(NT)^2} = 4\left(\frac{3}{2\pi^2}\cdot\frac{\lambda^2}{SNR_N T_{onT}^2}\right) \tag{23}$$

$$\sigma_{\ddot{\rho}}^2 \approx \frac{45\lambda^2}{2\pi^2(N\cdot SNR_1)(NT)^4} \approx 2.3\cdot\left(\frac{\lambda^2}{SNR_N T_{onT}^4}\right) \tag{24}$$

Notice that the expressions in brackets correspond exactly to the eq.s (17) and (18); thus the introduction of the amplitude estimation has deteriorated the accuracy (standard deviation) of $\dot{\rho}$ and $\ddot{\rho}$ estimation respectively of a factor 2 and 1.5.

6 MLE algorithm for *M* burst

The likelihood function is the logarithm of p.d.f. of eq. (5):

$$\ln\left[p_{\mathbf{z}}(\mathbf{Z}/\mathbf{A},\dot{\rho},\ddot{\rho})\right] = -M\ln\left[\pi^N|\mathbf{M}|\right] - \sum_{j=1}^{M}\left[\mathbf{z}_j - A_j\mathbf{s}_j\right]^H\mathbf{M}^{-1}\left[\mathbf{z}_j - A_j\mathbf{s}_j\right] \tag{25}$$

The estimation of A, $\dot{\rho}$ and $\ddot{\rho}$ is:

$$\begin{aligned}(\hat{\mathbf{A}},\hat{\dot{\rho}},\hat{\ddot{\rho}}) &= \arg\max_{\mathbf{A},\dot{\rho},\ddot{\rho}}\left\{\ln\{p_{\mathbf{Z}}(\mathbf{Z}/\mathbf{A},\dot{\rho},\ddot{\rho})\}\right\} = \\ &= \arg\min_{\mathrm{A},\dot{\rho},\ddot{\rho}}\left\{\sum_{j=1}^{M}\left[\mathrm{z}_j - A_j\mathrm{s}_j\right]^H\mathrm{M}^{-1}\left[\mathrm{z}_j - A_j\mathrm{s}_j\right]\right\} \\ &= \arg\min_{\mathrm{A},\dot{\rho},\ddot{\rho}}\left\{Q(\mathrm{A},\dot{\rho},\ddot{\rho})\right\}\end{aligned} \tag{26}$$

The estimation accuracy that can be achieved, in case of M burst, is improved just of a factor $\sqrt{M}$; notwithstanding the total number of processed pulses is $N_{tot}=M\cdot N$, every N pulses the observation of the phase is reset because of the changing of the values of transmitted frequency and PRT. The estimation of $\ddot{\rho}$ throughout M bursts jointly processed is equivalent to the case of M parallel estimators (see the block diagram of Figure 1) which processes N pulses.

7 Results

In Figure 2, 3 and 4 the CRLB has been computed for one burst, N=80, N=160, N=240 and T=50 μs, f=9.1 GHz. For the case N=240, also a Monte Carlo simulation of ML algorithm has been performed as described in section 4 by means of 500 independent trials (see asterisks). Figure 2(a) shows the square root of CRLB of $\ddot{\rho}$ estimation vs. SNR in case of known $\dot{\rho}$, A and σ^2; for SNR=0 dB and N=240, the distribution of estimation errors has been reported in Figure 2(b). It is evident that the distribution is unbiased, and its standard deviation (4.6 m/s^2) is very close to the theoretical CRLB. Figure 3(a) and 3(b) show respectively the square root of CRLB of $\dot{\rho}$ estimation respectively in case of known and unknown A. Figure 4(a) and 4(b) show respectively the square root of CRLB of $\ddot{\rho}$ estimation respectively in case of known and unknown A. Notice that in Figures 2, 3 and 4 the asymptotic curves of CRLB have not been reported because they are coincident with the true curves because of the high number of integrated pulses. In Table 1 a summary of the results is reported: the first two columns are pertinent to theoretical accuracies; while the third and the fourth columns are pertinent to simulated data (200 independent trials, M=3, N=80, T_1=50 μs, T_2=52.5 μs, T_3=55 μ s, f_1=9.1 GHz, f_2=9.2 GHz, f_3=9.3 GHz) in case of known and unknown A and $\dot{\rho}$. Simulated accuracy is very close to the CRLB obtained for one burst (i.e.: N=80) multiplied by a factor $\sqrt{3}$.

SNR [dB]	$\sqrt{CRLB_{\ddot{\rho}}}$ (N=240)		Simulated $\sigma_{\ddot{\rho}}$ (M=3, N=80)	
	known A $\dot{\rho}$	Unknown A $\dot{\rho}$	known A $\dot{\rho}$	unknown A $\dot{\rho}$
-10	11.69	70.56	~106	
0	3.7	22.31	~33	~205
10	1.17	7.06	~11	~65
20	0.37	2.23	~3.4	~20
30	0.12	0.71	~1	~6.5

Table 1: simulated accuracy obtained for *M*=3 and *N*=80, compared to CRLB vs. SNR per single pulse

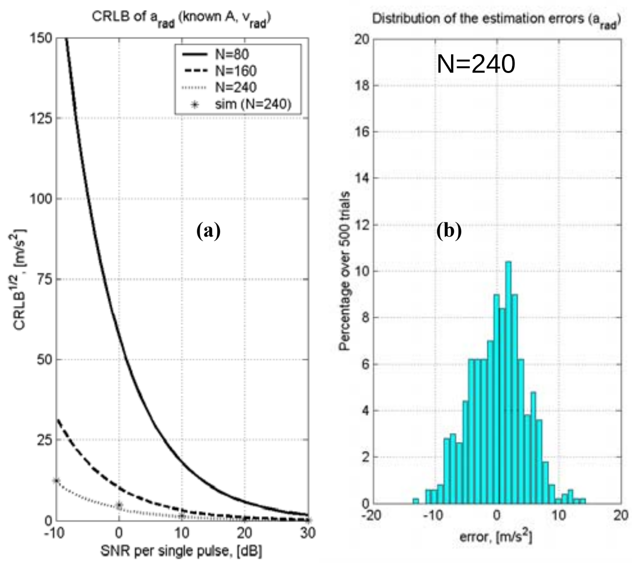

Figure 2: (a) CRLB of radial acceleration estimation, in case of known amplitude and radial speed, for N=80, N=160 and N=240; (b) distribution of estimation errors of radial acceleration over 500 independent Monte Carlo trials, for SNR=0 dB and N=240.

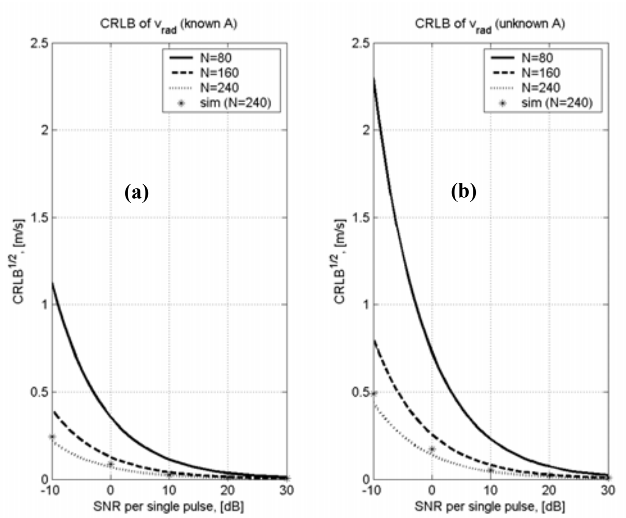

Figure 3: CRLB of estimated radial speed in case of known and unknown amplitude, for *N*=80, *N*=160 and *N*=240.

8 Conclusions

The paper analyzes a method to estimate radial acceleration jointly to other target parameters (i.e.: $\dot{\rho}$ and A). The estimator performance has been analyzed in case of one burst and M burst, in both analytical and simulated ways. To have an exploitable estimate of $\ddot{\rho}$ an accuracy of at least 10 m/s^2 is needed. To achieve this level of accuracy very high SNR is needed.

To improve the estimation, some a-priori knowledge about $\dot{\rho}$ and A may be exploited. Every burst an estimate of radial speed from the Moving Target Detector (MTD), $\hat{\dot{\rho}}_{MTD}$ and of amplitude from extractor, $\hat{\mathbf{A}}_{ext}$, are available, respectively with the accuracies σ_{ext} and σ_{MTD}. The could be inserted into the functional of eq.s (8) (26), which would be minimized only with respect to $\ddot{\rho}$. Unfortunately the uncertainty on $\dot{\rho}$ may determine an uncertainty on the phase which is larger than the phase contribution due to the radial acceleration which is very low: In this case the estimation of $\ddot{\rho}$ radial acceleration is meaningless. Also the introduction of the estimation of amplitude from extractor is, as before, quite destructive for the estimation of $\ddot{\rho}$. Notice that from the

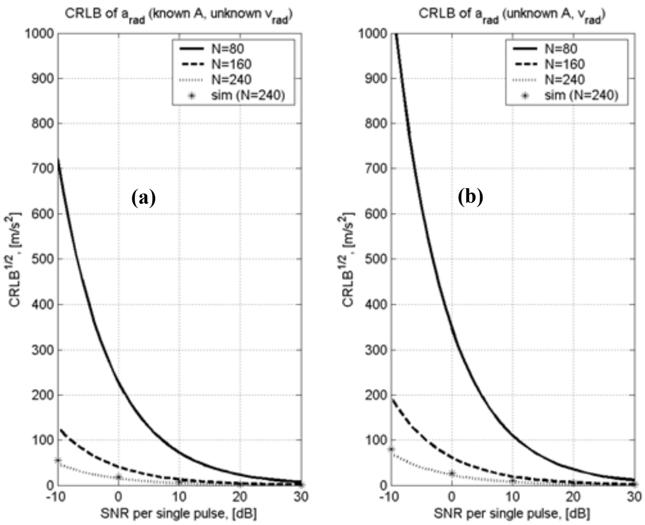

Figure 4: CRLB of estimated radial acceleration (with unknown radial speed) in case of known and unknown amplitude, for *N*=80, *N*=160 and *N*=240.

extractor only an information on the level of the signal is available; i.e.: $|A|$; so the real and complex parts of amplitude, which are necessary for the estimator, are in practice unknown. A simple MLE of radial acceleration is not "able" to exploit directly the information from MTD and extractor; an analytical way to exploit this information consists of *constrained* MLE. The constraints to apply to the estimator are a mathematical way to quantify some a-priori information. The estimator becomes:

$$\begin{cases} \left(\hat{\mathbf{A}}, \hat{\dot{\rho}}, \hat{\ddot{\rho}}\right) = \underset{\mathbf{A},\dot{\rho},\ddot{\rho}}{\arg\max}\left\{\ln\left(p_{\mathbf{Z}}\left(\mathbf{Z}/\mathbf{A},\dot{\rho},\ddot{\rho}\right)\right)\right\} \\ |\mathbf{A}| \propto \aleph\left(|\mathbf{A}_{true}|, \sigma_{extractor}\right) \\ \dot{\rho} \propto \aleph\left(\dot{\rho}_{true}, \sigma_{MTD}\right) \end{cases} \quad \textbf{(27)}$$

The constrained MLE estimates four parameters, but considers the A and $\dot{\rho}$ as Gaussian variable with mean value coincident with their true value and the variance is given by the extractor and MTD. This estimator accounts and describes into an analytical way the fact that the information from extractor and MTD is not perfectly correct. This approach will be surely characterized by a performance which is better than the one obtained in case of unknown A and $\dot{\rho}$, while it will be surely worse than the case of known A and $\dot{\rho}$. The performance of this estimator may be the object of a following paper

9 References

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10 Appendix

The derivatives of eq. (13) are:

$$\frac{\partial \boldsymbol{\mu}}{\partial \ddot{\rho}} = j\frac{4\pi}{\lambda}TA\left[e^{j\frac{4\pi}{\lambda}\left(\dot{\rho}T+\ddot{\rho}\frac{T}{2}\right)} \quad \cdots e^{j\frac{4\pi}{\lambda}\left(\dot{\rho}iT+\ddot{\rho}\frac{(iT)}{2}\right)} \cdots \quad e^{j\frac{4\pi}{\lambda}\left(\dot{\rho}NT+\ddot{\rho}\frac{(NT)}{2}\right)} \right]^T \quad \textbf{(28)}$$

$$\frac{\partial \boldsymbol{\mu}}{\partial \dot{\rho}} = j\frac{4\pi}{\lambda}TA\left[e^{j\frac{4\pi}{\lambda}\left(\dot{\rho}T+\ddot{\rho}\frac{T}{2}\right)} \quad \cdots e^{j\frac{4\pi}{\lambda}\left(\dot{\rho}iT+\ddot{\rho}\frac{(iT)}{2}\right)} \cdots \quad e^{j\frac{4\pi}{\lambda}\left(\dot{\rho}NT+\ddot{\rho}\frac{(NT)}{2}\right)} \right]^T \quad \textbf{(29)}$$

$$\frac{\partial \boldsymbol{\mu}}{\partial A_R} = \mathbf{s} = \left[e^{j\frac{4\pi}{\lambda}\left(\dot{\rho}T+\ddot{\rho}\frac{T}{2}\right)} \quad \cdots e^{j\frac{4\pi}{\lambda}\left(\dot{\rho}iT+\ddot{\rho}\frac{(iT)}{2}\right)} \cdots \quad e^{j\frac{4\pi}{\lambda}\left(\dot{\rho}NT+\ddot{\rho}\frac{(NT)}{2}\right)} \right]^T \quad \textbf{(30)}$$

$$\frac{\partial \boldsymbol{\mu}}{\partial A_I} = j\mathbf{s} = j\left[e^{j\frac{4\pi}{\lambda}\left(\dot{\rho}T+\ddot{\rho}\frac{T}{2}\right)} \quad \cdots e^{j\frac{4\pi}{\lambda}\left(\dot{\rho}iT+\ddot{\rho}\frac{(iT)}{2}\right)} \cdots \quad e^{j\frac{4\pi}{\lambda}\left(\dot{\rho}NT+\ddot{\rho}\frac{(NT)}{2}\right)} \right]^T \quad \textbf{(31)}$$

$$\frac{\partial \boldsymbol{\mu}}{\partial \sigma^2} = 0 \quad \textbf{(32)}$$

$$\frac{\partial \mathbf{C}}{\partial \dot{\rho}} = \frac{\partial \mathbf{C}}{\partial \ddot{\rho}} = \frac{\partial \mathbf{C}}{\partial A_R} = \frac{\partial \mathbf{C}}{\partial A_I} = 0 \quad \textbf{(33)}$$

$$\frac{\partial \mathbf{C}}{\partial \sigma} = 2\sigma \mathbf{I}_N \quad \textbf{(34)}$$

Combining the previous derivatives the FIM elements are obtained.

The sums into eq.s (14), (16) and (19), when N is enough large, can be simplified as follows:

$$\sum_1^N i^4 = \frac{1}{5}(N+1)^5 - \frac{1}{2}(N+1)^4 + \frac{1}{3}(N+1) - \frac{N}{30} - \frac{1}{30} \rightarrow \frac{1}{5}N^5 \quad \textbf{(35)}$$

$$\sum_1^N i^2 = \frac{1}{3}(N+1)^3 - \frac{1}{2}(N+1)^2 + \frac{N}{6} + \frac{1}{6} \rightarrow \frac{1}{3}N^3 \quad \textbf{(36)}$$

$$\sum_1^N i^3 = \frac{1}{4}(N+1)^4 - \frac{1}{2}(N+1)^3 + \frac{1}{4}(N+1) \rightarrow \frac{1}{4}N^4 \quad \textbf{(37)}$$

By replacing the sums with the previous simplifications the asymptotic expressions of CRLB are obtained.