14th European Signal Processing Conference (EUSIPCO 2006), Florence, Italy, September 4-8, 2006, copyright by EURASIP

DESIGNING LINEAR-PHASE DIGITAL DIFFERENTIATORS

A NOVEL APPROACH

Ewa Hermanowicz

Multimedia Systems Department, Faculty of Electronics, Telecommunications and Informatics, Gdansk University of Technology, ul. Narutowicza 11/12, 80-952 Gdansk, Poland
phone: + (48) 058 3472870, fax: + (48) 058 3472870, email: hewa@.eti.pg.gda.pl

ABSTRACT

In this paper two methods of designing efficiently a digital linear-phase differentiator of an arbitrary degree of differentiation are proposed. The first one utilizes a symbolic expression for the coefficients of a generic fractional delay filter and is based on a fundamental relationship between the coefficients of a digital differentiator and the coefficients of the generic FD filter. The second profits from one of the crucial attributes of the structure invented by Farrow for FD filters. It lies in an alternate symmetry and anti-symmetry of sub-filters which are linear-phase differentiators. The proposed design methods are illustrated by examples. Some practical remarks concerning the usage of the Farrow structure are also included.

1. INTRODUCTION

Digital linear-phase differentiators play an important role in many up-to-date applications such as radar, sonar, image processing, biomedical engineering and others. Often it is necessary to obtain not only first order, but also higher order derivatives, e.g., of biomedical data. Therefore there is a need for feasible and efficient methods of designing digital filters offering different degrees of differentiation. This is also reflected by a constant progress in the development of new techniques of designing digital differentiators. An example is a recent publication [1] where Tseng derived a closed-form solution (see (52) in [1]) for the coefficients of a Lagrangian digital differentiator performing differentiation of second degree. This is an FIR filter maximally accurate at direct component (DC), thus especially suited for precise measurement applications. Tseng's result can be considered as an extension of the results presented in [2] where also a closed form formula for the coefficients of such a differentiator (see (10) in [2]) but for first degree differentiation was presented. Both these differentiators have their complex frequency approximation error $E_{k,N}(e^{j\omega})$ maximally flat at DC. This error is defined as the difference between an ideal k-th degree differentiator (DDk) frequency response

$$H_k(e^{j\omega}) = (j\omega)^k \exp(-j\omega N/2), \quad |\omega| < \pi \tag{1}$$

where k=1,2,… and the frequency response

$$H_{k,N}(e^{j\omega}) = \sum_{n=0}^{N} h_{k,N}[n] e^{-j\omega n}, \quad |\omega| < \pi \tag{2}$$

of the approximating FIR of order N, i.e. $E_{k,N}(e^{j\omega}) = H_k(e^{j\omega}) - H_{k,N}(e^{j\omega})$. In (1) N/2 stands for the "transport" delay inevitable in the target FIR.

Obviously, one can continue the process of deriving closed form formulae for the DDk coefficients with higher degree of differentiation, i.e. for k>2. But the value of such formulae would be rather of theoretical nature. This is because with k increasing these formulae become more and more complicated. As a result the accuracy of coefficients' computation becomes problematic, especially for wideband applications, thus for high FIR order N. Also, the process of coding becomes more and more laborious. The aim of this paper is to propose two efficient approaches to designing a DDk of an arbitrary order k on the basis of a fractional delay filter but without limit computation, thus simpler than in [1].

2. MAIN RESULTS

The first approach to designing a DDk efficiently is based on the following fundamental formula [14], [15]

$$h_{k,N}[n] = h_{k,N,p}[n]\Big|_{p=0} = (-1)^k \left.\frac{\mathrm{d}^k h_{0,N,p}[n]}{\mathrm{d}D^k}\right|_{\substack{D=N/2+p \\ p=0}} \tag{3}$$

relating the coefficients $h_{k,N}[n]$, n=0,1, … N of the linear-phase DDk to the coefficients $h_{0,N,p}[n]$ of a generic fractional delay (FD) FIR filter having k=0, whose fractional delay $p \in [-0.5, 0.5]$ covers a sample interval of duration T conveniently normalized here to unity and whose total delay is D=N/2+p. The same relationship obeys for the frequency responses of the above mentioned filters, i.e. (2) and

$$H_{0,N,p}(e^{j\omega}) = \sum_{n=0}^{N} h_{0,N,p}[n] e^{-j\omega n} \tag{4}$$

as well as for their ideal counterparts: (1) for DDk and

$$H_{0,p}(e^{j\omega}) = \exp(-j\omega(N/2+p)), \quad |\omega| < \pi \tag{5}$$

for the FD filter. A proof for the latter is the following:

$$\begin{aligned} H_k(e^{j\omega}) &= (-1)^k [\frac{\mathrm{d}^k}{\mathrm{d}D^k} H_{0,p}(e^{j\omega})|_{p=D-N/2}]|_{p=0} \\ &= (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}D^k} \exp(-j\omega D)|_{D=N/2} \\ &= (j\omega)^k \exp(-j\omega N/2) \end{aligned}$$

On the basis of (3) one can design a linear-phase DDk by differentiating the impulse response of a FD filter k times over D and substituting $D=N/2$, which means $p=0$. In case when one has at disposal directly a closed form formula for the FD filter coefficients which normally is much simpler in form than that for a differentiator (see, e.g., [3] for a Lagrangian FD filter, thus maximally flat around DC), the coefficients of the linear-phase DDk can be calculated readily and accurately using Symbolic Toolbox in MATLAB. An exemplary code was given in [4]. Similar code can be generated also for any other than Lagrangian method of approximating the ideal frequency response of a FD filter, either an FIR or IIR, although by nature of the IIR filter in the latter case the target DDk does not have its phase response linear. Also, this approach can be found useful when the FD filter is designed in an iterative manner, as it is normally done, e.g., in Chebyshev approximation using cremez.m in MATLAB.

The second approach proposed here derives from that the Farrow structure [5], very popular for implementing variable FD filters efficiently [6] ÷ [14], is comprised of sub-filters which are linear-phase differentiators of different orders [8] whose coefficients are fixed, thus independent on p. Therefore, in the Farrow structure of a FD filter of order N having N+1 sub-filters, we have in hand the impulse responses of digital differentiators having the degree of differentiation k=1,2, … , N. (A proof of the Farrow sub-filters' phase linearity is given in [8].) Thus again the problem resolves to designing the generic FD filter. The specific of this second approach lies however not in differentiating the FD filter impulse response but in rearranging the generic FD filter transfer function

$$H_{0,N,p}(z) = \sum_{n=0}^{N} h_{0,N,p}[n] z^{-n} \tag{6}$$

(cf. (4)) from the polynomial in the unit delay z^{-1} as in (6) to the polynomial in fractional delay p, as in [5] or [4], and then extracting from it the DDk impulse response for the desired value of k. This way, for example, the two-rate technique given in [9] for designing wideband FD filters can be readily adopted and exploited for designing linear-phase differentiators of different orders.

Obviously, both approaches considered here lead to the same results.

However, there is an important point concerning the Farrow structure that should not be overlooked in order to derive the desired linear-phase DDk successfully. Linear-phase response means that the set of coefficients of the DDk (i.e. the set of elements of the DDk impulse response) is either symmetric or anti-symmetric. It depends on the degree of differentiation. This symmetry is an attribute of the true Farrow structure (see Table I in [5]). Namely, the sets of the coefficients of the Farrow sub-filters are alternately symmetric and anti-symmetric. However, in the literature, there exist a number of publications, where their authors attribute to Farrow a structure at first sight very similar to that symmetric, perfect one, but which is a vitiation of Farrow's invention in that the parameter for controlling the variable delay is changed from p to $D=N/2+p$ relative to that in [5]. (In the Farrow original work [5] the fractional delay parameter is called α, which is an equivalent to p used here.) It means that the transfer function $H_{0,N,p}(z)$ (6) is mistakenly rearranged into a polynomial in D rather than p. A result of this change, while the frequency response is kept unchanged, is an asymmetry of sub-filters. Consequently, the asymmetric sub-filters, which are differentiators of different orders, are not linear-phase filters. The desired phase-linearity of sub filters, very important for an efficient implementation, is lost. To sum up, the correct Farrow structure should be arranged as presented in Fig. 1 for a successful derivation of the symmetric/anti-symmetric DDk thus having the linear-phase response as desired.

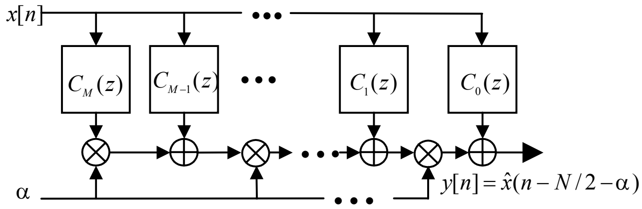

Fig. 1. The Farrow structure for a generic FD FIR filter, with fractional delay $\alpha \equiv p$, having total delay $D=N/2+\alpha$.

In Fig. 1 $C_m(z) = \sum_{n=0}^{N} c_m[n] z^{-n}$, $m = 0,1,\ldots,M$ are the transfer functions of the M+1 Farrow sub-filters, where M can be equal to or different than N, and $c_m[n]$, $m = 0,1,\ldots,M$ are the impulse responses of these sub-filters. The overall generic FD filter transfer function after the above mentioned rearrangement of (6) is given by

$$H_{0,N,p}(z)\Big|_{p=\alpha} = \sum_{m=0}^{M}\sum_{n=0}^{N} c_m[n] z^{-n} \alpha^m = \sum_{m=0}^{M} C_m(z) \alpha^m \tag{7}$$

The coefficients of digital differentiators, DDk, are obtained in the following way

$$h_{k,N}[n] = (-1)^k (k)! c_k[n];\ n = 0,1,\ldots,N,\ k = 0,1,\ldots,M \quad (8)$$

3. EXAMPLES

Example 1. The coefficients of the Lagrangian FD filter, maximally flat at DC, are given by the formula [3]

$$h_{0,N,p}[n] = \prod_{\substack{k=0\\k\neq n}}^{N} \frac{N/2+p-k}{n-k} = \prod_{\substack{k=0\\k\neq n}}^{N} \frac{D-k}{n-k} \quad (9)$$

for $n = 0,1,\ldots,N$. From this formula we obtain for N=3

$$h_{0,3,p}[0] = (-D^3 + 6D^2 - 11D + 6)/6$$
$$= (-p^3 + 3p^2/2 + p/4 - 3/8)/6$$
$$h_{0,3,p}[1] = (D^3 - 5D^2 + 6D)/2$$
$$= (p^3 - p^2/2 - 9p/4 + 9/8)/2$$
$$h_{0,3,p}[2] = (-D^3 + 4D^2 - 3D)/2$$
$$= (-p^3 - p^2/2 + 9p/4 + 9/8)/2$$
$$h_{0,3,p}[3] = (D^3 - 3D^2 + 2D)/6$$
$$= (p^3 + 3p^2/2 - p/4 - 3/8)/6$$

where D=3/2+p. The transfer function of this FD filter is

$$H_{0,3,p}(z) = \sum_{n=0}^{3} h_{0,3,p}[n] z^{-n} \quad (10)$$

In the first approach we differentiate $h_{0,3,p}[n]$ N times over D in accordance with (3) and set D=3/2. This way we obtain the coefficients (8) of the Lagrangian differentiators with the degree of differentiation k=0,1,2,3. These coefficients are shown in Table 1. Notice their alternate symmetry and anti-symmetry.

Table 1. The coefficients of FIR Lagrangian differentiators of order N=3 odd and degree of differentiation k=0,1,2,3, maximally accurate at DC.

n	$h_{0,3}[n]$ k=0	$h_{1,3}[n]$ k=1	$h_{2,3}[n]$ k=2	$h_{3,3}[n]$ k=3
0	-1/16	-1/24	1/2	1
1	9/16	9/8	-1/2	-3
2	9/16	-9/8	-1/2	3
3	-1/16	1/24	1/2	-1

In the second approach we rearrange (10) in agreement with (7). This way for M=N we obtain

$$H_{0,3,p}(z) = \sum_{m=0}^{3} C_m(z) p^m \quad (11)$$

where

$$C_m(z) = \sum_{n=0}^{3} c_m[n] z^{-n} \quad (12)$$

and

$$C_0(z) = -1/16 + 9z^{-1}/16 + 9z^{-2}/16 - z^{-3}/16$$
$$C_1(z) = 1/24 - 9z^{-1}/8 + 9z^{-2}/8 - z^{-3}/24$$
$$C_2(z) = 1/4 - z^{-1}/4 - z^{-2}/4 + z^{-3}/4$$
$$C_3(z) = -1/6 + z^{-1}/2 - z^{-2}/2 + z^{-3}/6$$

The coefficients $c_m[n]$ of respective linear-phase Farrow sub-filters are gathered in Table 2.

Table 2. The coefficients of Farrow sub-filters of order N=3 for a variable FD filter maximally flat around DC.

n	$c_0[n]$ m=0	$c_1[n]$ m=1	$c_2[n]$ m=2	$c_3[n]$ m=3
0	-1/16	1/24	1/4	-1/6
1	9/16	-9/8	-1/4	1/2
2	9/16	9/8	-1/4	-1/2
3	-1/16	-1/24	1/4	1/6

Using (8) and Table 2 with k=m we arrive at the coefficients of linear-phase digital differentiators from Table 1.

For the sake of comparison Table 3 shows the coefficients of FIR Lagrangian differentiators of even order N=2.

Opposite to the case of order odd, for order even the zero degree differentiator (see column two in Table 3) is an integer delayer, thus a “constant system” (cf. [6]), while for order even (see column two in Table 1) it is a half-sample delayer. The latter is an equivalent to that considered, e.g., in [6], i.e. having the same transfer function, but with the fractional delay parameter (here denoted by p) set to zero.

Table 3. The coefficients of FIR Lagrangian differentiators of order N=2 even and degree of differentiation k=0,1,2, maximally accurate at DC.

n	$h_{0,2}[n]$ k=0	$h_{1,2}[n]$ k=1	$h_{2,2}[n]$ k=2
0	0	1/2	1
1	1	0	-2
2	0	-1/2	1

Example 2. Here we consider a differentiator with first degree of differentiation, obtained on the basis of a two-stage variable FD filter designed in [9]. The generic wideband optimal (mini-max) variable FD filter in [9] is composed of a half-band symmetric Nyquist [14] FIR filter of order 58 with 16 nonzero coefficients of different values and a FD filter of order 6 with four sub-filters whose alternately symmetric and

anti-symmetric coefficients are given in [9] in Figure 4. Using the above mentioned approach this filter was redesigned into a DDk with k=1 (differentiation of first degree).

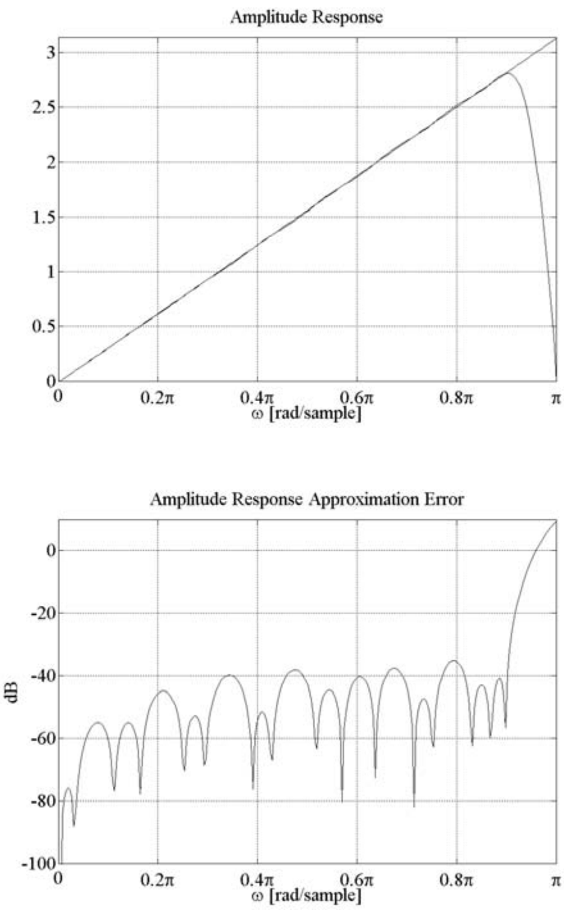

Fig. 2. The amplitude response and amplitude response approximation error for a differentiating FIR filter having 19 coefficients of different values in total.

The amplitude response and the amplitude response approximation error in dB for this linear-phase differentiating filter are shown in Fig. 2. Straight line in the upper part of Fig. 2 represents the ideal DD1.

4. CONCLUSIONS

In this paper two methods of designing a digital linear-phase differentiator of an arbitrary degree of differentiation efficiently have been proposed. The first one utilizes a symbolic expression for the coefficients of a generic fractional delay filter and is based on a fundamental relationship between the coefficients of a digital differentiator and the coefficients of the generic FD filter. The second profits from one of the principal attributes of the structure invented by Farrow for FD filters. It lies in an alternate symmetry and anti-symmetry of sub-filters which are linear-phase differentiators. The proposed design methods are illustrated by examples. Some practical remarks concerning usage of the true Farrow structure with symmetric/anti-symmetric sub-filters, as opposed to the asymmetric one often met in the literature are also included.

REFERENCES

[1] C. C. Tseng, "Digital differentiator design using fractional dely filter and limit computation", *IEEE Trans. on Circuits and Systems – I: Regular Papers*, pp. 2248-2259, vol. 52, No. 10, October 2005.

[2] E. Hermanowicz and M. Rojewski, "Design of FIR first order digital differentiators of variable fractional sample delay using maximally flat error criterion", *Electronics Letters*, vol. 30, No. 1, pp. 17-18, 1994.

[3] E. Hermanowicz, "Explicit formulas for weighting coefficients of maximally flat tunable FIR delayers", *Electronics Letters*, vol. 28, No. 20, pp. 1936-1937, 1992.

[4] E. Hermanowicz, " On designing a wideband fractional delay filter using the Farrow approach", *Proc. EUSIPCO 2004*, Vienna, Austria, September 6-10, 2004 (on CD).

[5] C. W. Farrow, "A continuously variable digital delay element", in *Proc. IEEE Int. Symp. Circuits and Systems*, Espoo, Finland, June 1988, vol. 3, pp. 2641-2645.

[6] S. Samadi, M. O. Ahmad and M. N. S. Swamy, "Results on maximally flat fractional delay systems", *IEEE Trans. Circuits and Systems I: Regular Papers*, vol. 51, pp.2271-2286, November 2004.

[7] F. Harris, *Multirate Signal Processing for Communication Systems*. Prentice Hall PTR, Upper Saddle River, New Jersey, 2004.

[8] C. C. Tseng, "Design of variable fractional FIR filter using differentiator bank", in *Proc. IEEE Int. Symp. Circuits and Systems*, Phoenix, Arizona, USA, May 2002, vol. IV, pp. 421-424.

[9] E. Hermanowicz and H. Johansson, "On designing minimax adjustable wideband fractional delay FIR filters using two-rate approach", in *Proc. 2005 European Conference on Circuit Theory and Design*, Cork, Ireland, 28 August – 2 September, 2005, vol. III, pp. 437-440.

[10] T. I. Laakso, V. Valimaki, M. Karjalainen and U. K. Laine, "Splitting the unit delay. Tools for fractional delay filter design", *IEEE Signal Processing Magazine*, pp. 30-60, January 1996.

[11] V. Valimaki and T. I. Laakso, "Principles of fractional delay filters", *Proc. IEEE Int. Conf. on Acoustics, Speech and Signal Processing*, Istanbul, Turkey, May 2000, pp. 3870-3873.

[12] F. Marvasti, *Nonuniform Sampling. Theory and Practice*. Kluwer Academic/Plenum Publishers, New York, 2001.

[13] S. K. Mitra, *Digital Signal Processing. A Computer Based Approach*. McGraw-Hill, 2004, 2nd ed.

[14] E. Hermanowicz, *Digital Special Filters and Applications*. EXIT, Warsaw 2005, www.exit.pl.

[15] E. Hermanowicz, "Novel complex FIR k-th degree differentiator of variable delay", *Proc. EUSIPCO-94*, Edinburgh, Sept. 1994, pp. 1042-1045.