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ACCURACY OF GAUSS-LAGUERRE POLAR MONOPULSE RECEIVER

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ABSTRACT

In this paper the CRB accuracy of a monopulse receiver parametrized by two (off-boresight and revolution) angles obtained by combining polar-separable and angularly periodic Gauss-Laguerre directivity patterns is calculated and compared to the CRB of monopulse receivers based on cartesian separable beams.

1. INTRODUCTION

Besides the well known applications in the radar field [1], [2], in recent years, Amplitude Comparison Monopulse (ACM) techniques, have found interesting applications also in the telecommunications field [3]. Let us cite for instance tracking control of antennas for data links between a moving platform and a satellite.

To estimate the source off-bore sight angles, in both azimuth and elevation, Cartesian ACM (C-ACM) estimators employ two nested pairs of sum-difference beams, one for each angular coordinate [1]. In particular, four cartesian separable beams are arranged on a rectangular grid, whose sides are aligned along the azimuth (horizontal) and the elevation (vertical) directions, respectively. One channel sums all of the four beams. Other two channels process the difference of the signals from the azimuth beams and from the elevation beams respectively [1].

The Polar Amplitude Comparison Monopulse (P-ACM) configuration adopts polar separable beams instead of cartesian difference beams. Gauss-Laguerre (GL) polar functions [4] do constitute suitable antenna patterns for P-ACM. In fact, it is shown here that the P-ACM based on the lowest order set of GL functions achieves a theoretical accuracy comparable to the one of the C-ACM having the same sum beam aperture. The scope of this work is to provide theoretical comparative results.

2. THE GAUSS-LAGUERRE MONOPULSE RECEIVER

2.1 Gauss-Laguerre functions

The GL functions of radial order $k = 0,1,2,\ldots$ and angular order $n = 0,\pm1,\pm2,\ldots$ are defined in usual polar coordinates (r,φ) as [4]:

$$L_{k,n}(r,\varphi) = (-1)^k \, 2^{(|n|+1)/2} \pi^{|n|/2} \sqrt{\frac{k!}{(|n|+k)!}} \times r^{|n|} P_{k,n}\left(2\pi r^2\right) e^{-\pi r^2} e^{jn\varphi} \tag{1}$$

where:

$$P_{k,n}(x) = \sum_{h=0}^{k} (-1)^h \binom{n+k}{k-h} \frac{x^h}{h!} \tag{2}$$

is the generalized Laguerre polynomial of orders k,n. In particular, for $k=0$, $P_{0,n}(x)=1$ and:

$$L_{0,n}(r,\varphi) \propto r^{|n|} e^{-\pi r^2} e^{jn\varphi} \tag{3}$$

Therefore, for any order $n \ge 0$

$$R_n(r,\varphi) \propto \frac{L_{0,n+1}(r,\varphi)}{L_{0,n}(r,\varphi)} \propto r e^{j\varphi} \tag{4}$$

This ratio does suggest a simple and natural solution to the problem of bi-dimensional monopulse measurement. In fact, any two consecutive GL functions with $k=0$ can be adopted as beams. The above ratio allows estimating the *revolution angle* φ around the bore-sight axis, with $0 \le \varphi < 2\pi$, and the *off-boresight angle* $\vartheta \ge 0$, having posed $\vartheta \propto r$ [1].

[1] More elaborate schemes are devised using several GL functions within a general array processing framework.

2.2 Gauss Laguerre Polar Amplitude Comparison Monopulse (GLP-ACM) receiver

In particular, let us consider here for simplicity only the case for $n=0$ using $m=3$ beams. Referring to the normalized off-bore-sight angle $\gamma = \vartheta/\sqrt{2}\beta$, where β denotes the angular width of sum beam, possible polar separable beampatterns are:

$$G_1(\gamma) = \mathrm{e}^{-\gamma^2} \quad (\textit{sum beam}^2) \tag{5.a}$$

$$G_2(\gamma,\varphi) = \mathrm{Re}\left[\gamma \mathrm{e}^{-\gamma^2} \mathrm{e}^{j\varphi}\right] = \gamma \mathrm{e}^{-\gamma^2} \cos\varphi \quad (\textit{in-phase beam}) \tag{5.b}$$

$$G_3(\gamma,\varphi) = \mathrm{Im}\left[\gamma \mathrm{e}^{-\gamma^2} \mathrm{e}^{j\varphi}\right] = \gamma \mathrm{e}^{-\gamma^2} \sin\varphi \quad (\textit{quadrature beam}) \tag{5.c}$$

Beams (5) can be approximated by using Butler matrices or multi-modal circular waveguides [5].

2.3 Maximum Likelihood GLP-ACM

The derivation of the Maximum Likelihood (ML) coherent receiver for the GLP-ACM is conducted following the classical approach of [1], which is briefly recalled here. The complex envelopes of the received waveforms associated to beams after matched filtering generate a sufficient statistic and obey the model [1]:

$$y_i(t) = Ae^{j\psi}G_i(\vartheta,\varphi) + n_i(t) \quad \text{for } i=1,2,3 \text{ and } t=1,2,\ldots. \tag{6}$$

where $A \geq 0$ is the unknown amplitude of the signal reflected by a point source; ψ is the unknown carrier phase rotation.

It is assumed that each receiver is affected by a zero-mean complex, circular AWGN process[3] $n_i(t)$ ($i=1,2,3$), that we assume independent from channel to channel. The variance of one real noise component is denoted as σ_N^2.

Each function $G_i(\vartheta,\varphi)$ describes the i-th polar beampattern. Therefore the receiver *steering vector* is $\mathbf{g}^T = \left[G_1(\vartheta,\varphi),\ G_2(\vartheta,\varphi),\ G_3(\vartheta,\varphi)\right]$, where $(.)^T$ denotes transposition. In the foregoing it is assumed that a single temporal snapshot (6) is processed by the GLP-ACM.

Under the hypotheses made above, the log-likelihood ratio $\mathfrak{L}(A,\psi,\varphi,\vartheta)$ is expressed in the form [1]:

$$\sigma_N^2 \mathrm{L}(A,\psi,\varphi,\vartheta) = A\left(2\mathbf{g}^T\mathbf{u} - A\|\mathbf{g}\|^2\right) \tag{8}$$

where $\mathbf{u}^T = [u_1 \;\; u_2 \;\; u_3]$ and $u_i = \mathrm{Re}\left[y_i e^{-j\psi}\right]$ for $i=1,2,3$. $\mathfrak{L}(A,\psi,\varphi,\vartheta)$ in (8) has to be maximized with respect to the unknowns A, ψ, φ and $\vartheta \geq 0$.

In particular, the quantities A and ψ, related to the source signal, are *nuisance parameters* for the angle estimator. In fact, it can be shown, as it happens in other relevant direction finding problems, that for large SNR the estimates of A and ψ are statistically uncorrelated from those of φ and ϑ, since the Fisher Information Matrix is block diagonal [6]. Therefore, these nuisance parameters can be eliminated from the source angle estimator by first maximizing (8) with respect to A and ψ.

By setting the partial derivative of (8) with respect to A equal to zero, the optimum value $\hat{A} = \dfrac{\mathbf{g}^T\mathbf{u}}{\|\mathbf{g}\|^2}$ is obtained. By back-substituting $\hat{A}$ into (8), the log-likelihood ratio becomes:

$$\sigma_N^2 \cdot \mathfrak{L}(\psi,\varphi,\vartheta) = \frac{\left(\mathbf{g}^T\mathbf{u}\right)^2}{\|\mathbf{g}\|^2} \leq \|\mathbf{u}\|^2 \tag{9}$$

because of the Schwartz inequality applied to the real-valued vectors $\mathbf{g}$ and $\mathbf{u}$. The quantity $\|\mathbf{u}\|^2$ appearing in (9) can be written as the sum of two addenda:

$$\begin{aligned} \|\mathbf{u}\|^2 &= \sum_{i=1}^{3}\left\{\mathrm{Re}\left[y_i e^{-j\psi}\right]\right\}^2 \\ &= \frac{1}{2}\sum_{i=1}^{3}|y_i|^2 + \frac{1}{2}\mathrm{Re}\left[e^{-2j\psi}\sum_{i=1}^{3} y_i^2\right] \end{aligned} \tag{10}$$

The second term contains ψ and becomes maximum for the choice:

$$\hat{\psi} = \frac{1}{2}\arg\left[\sum_{i=1}^{m} y_i^2\right] \tag{11}$$

Therefore the concentrated log-likelihood ratio (8) takes the form:

$$\sigma_N^2 \mathfrak{L}(\varphi,\vartheta) = \frac{\left(\mathbf{g}^T\mathbf{u}\right)^2}{\|\mathbf{g}\|^2} \leq \frac{1}{2}\sum_{i=1}^{m}|y_i|^2 + \frac{1}{2}\left|\sum_{i=1}^{m} y_i^2\right| \tag{12}$$

From the Schwartz inequality applied to (12), the two ML estimates of the source angles are obtained when the real vector $\mathbf{u}$ is proportional to the real vector $\mathbf{g}$, i.e. when:

$$G_i(\vartheta,\varphi) = k\,\mathrm{Re}\left[y_i e^{-j\hat{\psi}}\right] \text{ for } i=1,2,3. \tag{13}$$

[2] The pattern of the sum beam satisfies $G_1(0)=1$ and $\int_{-\infty}^{+\infty} G_1(\gamma)\,d\gamma = \beta\sqrt{2\pi}$.

[3] AWGN: Additive White Gaussian Noise

Given the explicit expression of the three beam gains, the two angular coordinates of the point source can be estimated by solving the equation system (13). For the GLP-ACM, the ML estimates of the angles are obtained as:

$$\hat{\varphi}_{ML} = \arctan 2\left(\frac{M_3}{M_2}\right)^4 \quad \text{and} \tag{14.a}$$

$$\hat{\gamma}_{ML} = \frac{1}{M_1}\sqrt{M_2^2 + M_3^2} \tag{14.b}$$

where

$$M_i = \mathrm{Re}\left\{y_i e^{-j\hat{\psi}}\right\} \text{ for } i = 1,2,3\,. \tag{15}$$

For small values of the off-bore-sight angle $\vartheta \geq 0$, i.e. when the source is within the antenna beamwidth the conventional *error angle in azimuth* α and the *error angle in elevation* λ between the line of sight axis and the source are approximately related as:

$$\alpha \cong \vartheta \sin\varphi\,;\ \lambda \cong \vartheta \cos\varphi\,. \tag{16}$$

From (14) and (16) and small ϑ, the approximated expression of the errors α and λ of the source are:

$$\alpha \cong G_3(\gamma,\varphi)/G_1(\gamma) = \beta M_3/M_1 \tag{17.a}$$

$$\lambda \cong G_2(\gamma,\varphi)/G_1(\gamma) = \beta M_2/M_1 \tag{17.b}$$

3. ACCURACY

For large SNR, the ML estimates of the source angles are unbiased and asymptotically achieve the Cramer-Rao bound (CRB) [1]:

$$\mathrm{CRB}_{\gamma\varphi} = \frac{1}{\rho}\begin{bmatrix} \|\mathbf{g}_\gamma\|^2 & \mathbf{g}_\gamma^T\mathbf{g}_\varphi \\ \mathbf{g}_\gamma^T\mathbf{g}_\varphi & \|\mathbf{g}_\varphi\|^2 \end{bmatrix}^{-1} \tag{18}$$

where

$$\rho = \frac{A^2}{\sigma_N^2};\quad \mathbf{g}_\gamma = \left[\mathbf{I} - \frac{\mathbf{g}\mathbf{g}^T}{\|\mathbf{g}\|^2}\right]\frac{\partial\mathbf{g}}{\partial\gamma};\quad \mathbf{g}_\varphi = \left[\mathbf{I} - \frac{\mathbf{g}\mathbf{g}^T}{\|\mathbf{g}\|^2}\right]\frac{\partial\mathbf{g}}{\partial\varphi} \tag{19}$$

In the GLP-ACM configuration (5), the real-valued vector of the beam gains becomes:

$$\mathbf{g}^T = e^{-\gamma^2}\left[1 \quad \gamma\cos\varphi \quad \gamma\sin\varphi\right] \tag{20}$$

A straightforward computation of (18) shows that the off-boresight angle γ and revolution angle φ estimates are asymptotically (e.g., for large SNR) uncorrelated because $\mathbf{g}_\lambda^T\mathbf{g}_\varphi = 0$ and that their variances are lower bounded by the CRB:

$$\sigma_\gamma^2 \geq \left(\frac{A^2}{\sigma_N^2}\right)^{-1} \mathrm{e}^{2\gamma^2}\left(1+\gamma^2\right) \quad \text{and} \tag{21}$$

$$\sigma_\varphi^2 \geq \left(\frac{A^2}{\sigma_N^2}\right)^{-1} \mathrm{e}^{2\gamma^2}\,\frac{1}{\gamma^2} \tag{22}$$

As it can be expected from the polar symmetry of the beams, the asymptotic variances (21) and (22) do not depend on the source revolution angle around the bore-sight axis.

For small values of the off-boresight angle ϑ, the variances of the elevation and azimuth coordinates (17a) and (17b) are approximated as

$$\sigma_\alpha^2 \approx 2\beta^2\left(\gamma^2\sigma_\varphi^2\cos^2\varphi + \sigma_\gamma^2\sin^2\varphi\right) \tag{23.a}$$

$$\sigma_\lambda^2 \approx 2\beta^2\left(\gamma^2\sigma_\varphi^2\sin^2\varphi + \sigma_\gamma^2\cos^2\varphi\right) \tag{23.b}$$

while the covariance is approximated as

$$Cov(\alpha\lambda) \approx 2\beta^2\sin\varphi\cos\varphi\left(\sigma_\gamma^2 - \gamma^2\sigma_\varphi^2\right) \tag{23.c}$$

From (21) and (22) it follows that on the line-of-sight we have

$$\lim_{\vartheta\to 0}\sigma_\alpha^2 = \lim_{\vartheta\to 0}\sigma_\lambda^2 = 2\beta^2\left(\frac{A^2}{\sigma_N^2}\right)^{-1} \tag{24}$$

4. COMPARISON BETWEEN THE GLP-ACM AND THE C-ACM ESTIMATOR

The GLP-ACM estimator is compared with the C-ACM based on three nested beams and analyzed in [1]. This configuration employs four product beams arranged on a rectangular grid; the elementary beams are Gaussian shaped with $P_1(\alpha) = e^{-\alpha^2/2\beta_C^2}$ and $P_1(\lambda) = \mathrm{e}^{-\lambda^2/2\beta_C^2}$, where β_{DM} denotes the beamwidth. The theoretical beampatterns are:

$$\begin{aligned} G_{P1}(\alpha,\lambda) &= P_1(\alpha-\alpha_0)P_2(\lambda-\lambda_0), \\ G_{P2}(\alpha,\lambda) &= P_1(\alpha-\alpha_0)P_2(\lambda+\lambda_0), \\ G_{P3}(\alpha,\lambda) &= P_1(\alpha+\alpha_0)P_2(\lambda-\lambda_0), \\ G_{P4}(\alpha,\lambda) &= P_1(\alpha+\alpha_0)P_2(\lambda+\lambda_0). \end{aligned} \tag{25}$$

Azimuth and elevation estimates are obtained by applying the ML estimator (13) to the following three orthogonal beams:

$$\begin{bmatrix} G_{1C}(\alpha,\lambda) \\ G_{2C}(\alpha,\lambda) \\ G_{3C}(\alpha,\lambda) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} G_{P1}(\alpha,\lambda) \\ G_{P2}(\alpha,\lambda) \\ G_{P3}(\alpha,\lambda) \\ G_{P4}(\alpha,\lambda) \end{bmatrix} \tag{26}$$

With reference to Figure 2, the beampatterns of the two ACMs have been selected to obtain the same width of the sum beams; this condition is reached, for example, for $\beta = 1.0$ deg., $\beta_C = 0.8$ deg. and for an offset angle between the beam pairs, both in azimuth and elevation, of $\alpha_0 = \lambda_0 = 0.7\beta_C$.

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[4] The arctan2 function return angle arguments between 0 and 2π radians.

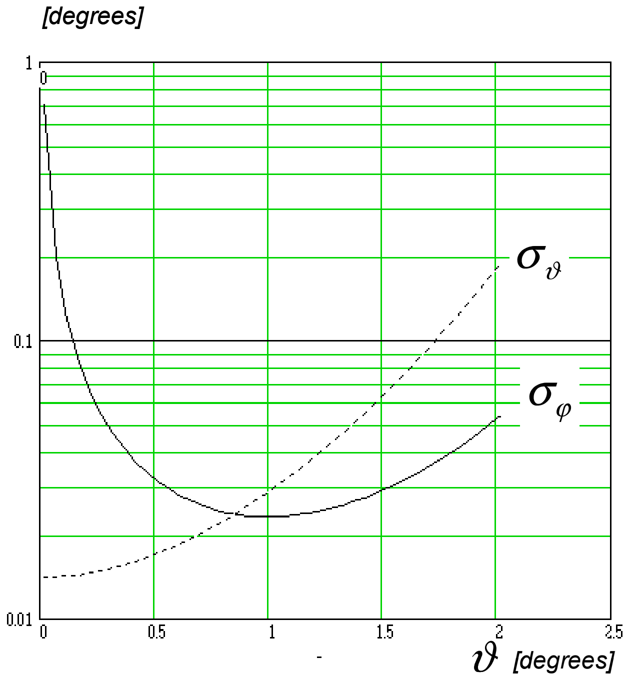

Figure 1. Asymptotic standard deviation $\sigma_\vartheta = \sqrt{2}\beta\sigma_\gamma$ of the off-boresight angle estimate and σ_φ of the revolution angle estimates as a function of the off-boresight angle of the source. $\beta = 1.0$ degree; $\rho = 40$ dB.

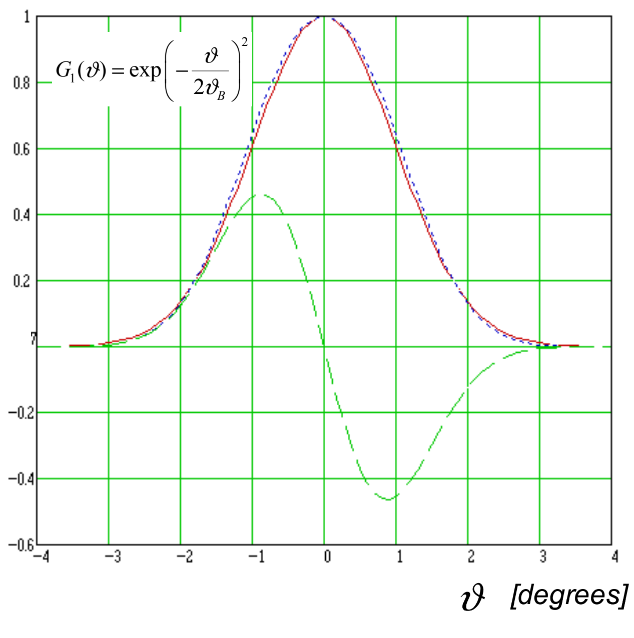

Figure 2: Comparison of the beams – Beamwidth of the GLP-ACM estimator: $\beta = 1.0$ deg.; beam-width of the C-ACM monopulse estimator $\beta_C = 0.8$ deg; beam displacement $\alpha_0 = \gamma_0 = 0.7\beta_C$.

Taking into account (23.a) and (23.b), the CRBs (21) and (22) of the GLP-ACM ML estimator were used to lower bound the variance of the azimuth and elevation angles σ_α^2 and σ_λ^2. They can be compared with the corresponding CRBs, in azimuth and elevation of the C-ACM (25) [1].

In order to compare the two estimators, we analyse the case when the source describes a radial trajectory $\{(\vartheta,\varphi) : \varphi = \varphi_0, 0 \le \vartheta \le 2\beta\}$ characterized by a constant value $\varphi = \varphi_0$ of the revolution angle.

Figures 3, 4 and 5 compare the standard deviations of the two error angle estimators in azimuth $\sigma_{\alpha GLP}$ and $\sigma_{\alpha C}$ when the revolution angle is respectively equal to $\varphi_0 = 0$ deg., $\varphi_0 = 45$ deg. and $\varphi_0 = 90$ deg.. The standard deviation $\sigma_{\alpha C}$ of the C-ACM [1], is lower-bounded by the CRBs obtained from the general formulas (18) and (19). A SNR $\rho = 40$ dB has been assumed.

When the radial trajectory of the source is horizontal (see Fig. 3), i.e. when the error angle in elevation is zero, the wavefront impinges on the azimuth difference beams of the C-ACM at their maximum gain and the error in azimuth is statistically minimum. Under these conditions, the CRBs in azimuth of the C-ACM and the GLP-ACM ML estimators are comparable.

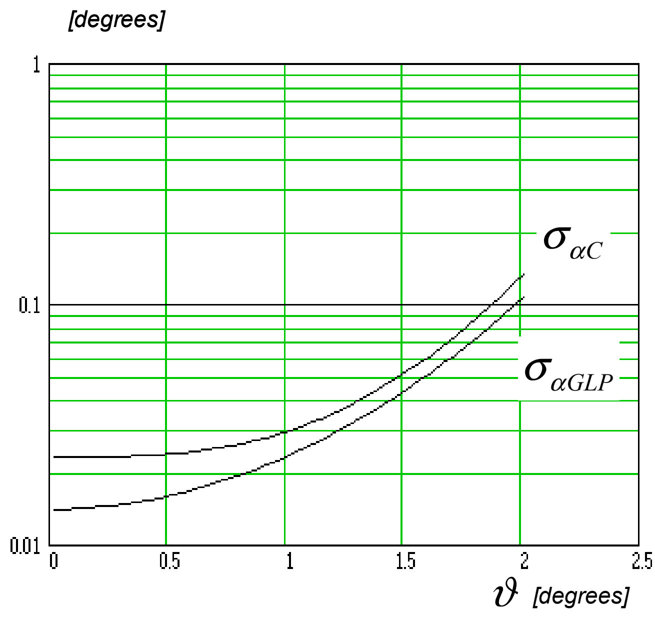

Figure 3: Asymptotic standard deviation of the error angle in azimuth for the radial trajectory $\varphi_0 = 0$ degrees. $\rho = 40$ dB.

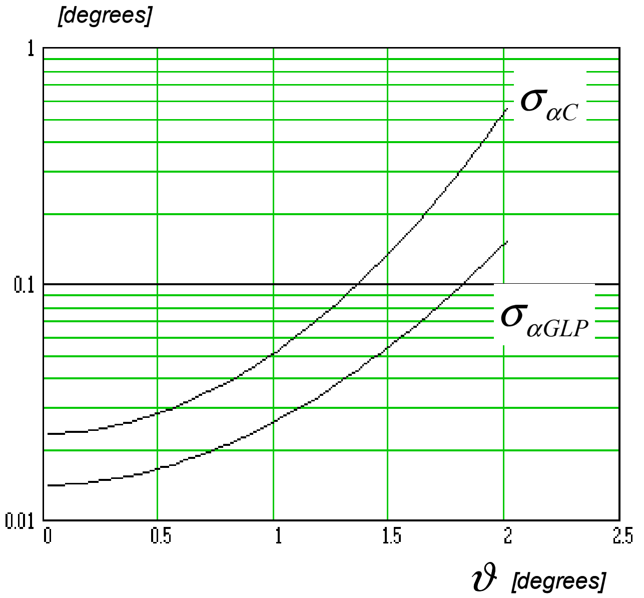

Figure 4: Asymptotic standard deviation of the error angle in azimuth for the radial trajectory $\varphi_0 = 45$ **degrees.** $\rho = 40$ **dB.**

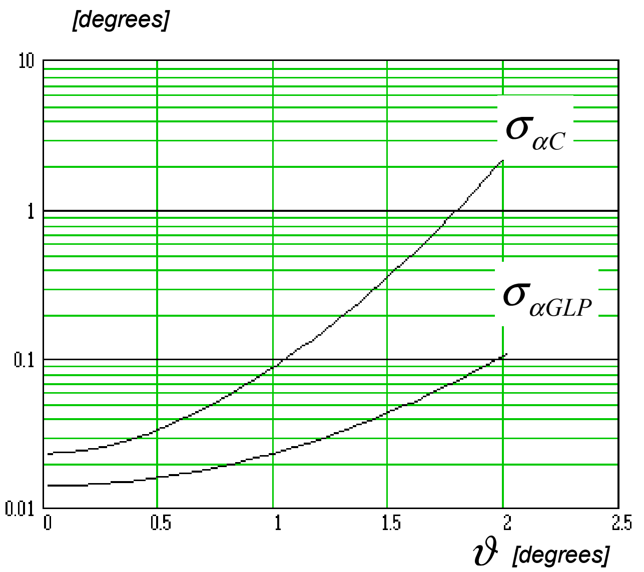

Figure 5: Asymptotic standard deviation of the error angle in azimuth for the radial trajectory $\varphi_0 = 90$ **degrees.** $\rho = 40$ **dB.**

When the angular trajectory of the source shows a significant value of the elevation error (see Figs. 4, 5), due to the rectangular configuration of its beams, the CRB in azimuth of the C-ACM increases. In this respect, the CRB in azimuth of the GLP-ACM estimator is smaller in all three cases and its performance degradation, as the source off-bore-sight angle increases, is much less sensitive to the revolution angle φ than the C-ACM.

5. CONCLUSION

The Gauss-Laguerre functions constitute a valid basis for building efficient ACM systems. In particular, the GLP-ACM solution, for an equivalent antenna aperture, exhibits a CRB less sensitive to the revolution angle of the source with respect to the C-ACM.

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