STRUCTURAL EQUIVALENCES IN WAVE DIGITAL SYSTEMS BASED ON DYNAMIC SCATTERING CELLS

Augusto Sarti and Giovanni De Sanctis

Dipartimento di Elettronica e Informazione - Politecnico di Milano
Piazza Leonardo Da Vinci 32, 20133 Milano, Italy
desancti/sarti@elet.polimi.it

ABSTRACT

In this article we prove that a computable tree-like interconnection of parallel/series Wave Digital adaptors with memory (which are characterized by reflection filters instead of reflection coefficients) is equivalent to a standard (instantaneous) multi-port adaptor whose ports are connected to mutators (2-port adaptors with memory). We prove this by providing a methodology for extracting the memory from a macro-adaptor, which simplifies the implementation of WD structures.

1. INTRODUCTION

A Wave Digital (WD) structure [1] can be generally seen as a set of WD elements connected with each other through a tree-like network of elementary (series or parallel) Dynamic Scattering Junctions (DSJ). In order for this multiport DSJ to be computable, non-adapted ports can only be connected to adapted ports, therefore $M-1$ of the M elementary DSJs need, in fact, to be dynamic adaptors [1]. When all M DSJs are dynamic adaptors, the multiport DSJ turns out to be a multi-port adaptor, and the adapted port can be used for accommodating a nonlinear element in the structure. This is very useful for modeling systems that embed a lumped nonlinearity.

Although we can assume with no loss of generality that the DSJs are 2-port or 3-port elements, the potential variety of building blocks for such WD structures is, in fact, formidable. This is a disadvantage with respect to traditional Wave Digital Filters [2], which are based on a limited collection of junctions. In order to overcome this problem, we would like to define some structural transformation that can be used for significantly reducing the number of potential building blocks. One way to do so is to construct a WD structure that is functionally equivalent to a multi-port DSJ but is made of an interconnection of standard (memoryless) 2-port and/or 3-port adaptors [2] whose peripheral ports can be connected to WD mutators [1] (2-port scattering cells with memory). In this paper we will see that this *memory extraction* operation is, in fact rather simple to perform (from a procedural standpoint). This result turns out to play a key role in the automatic implementation of WD structures in a wide range of applications, from nonlinear circuit simulation to musical acoustics.

2. MEMORY EXTRACTION FROM ADAPTORS

In this section we show that any dynamic 3-port series (parallel) adaptor can always be implemented as a standard series (parallel) WDF adaptor, whose adapted ports is connected to a DSC, as shown in Fig. 1, where $\widetilde{\Gamma}_3(z)$ is obtained from the reflection filter of the adapted port by removing the instantaneous I/O connection.

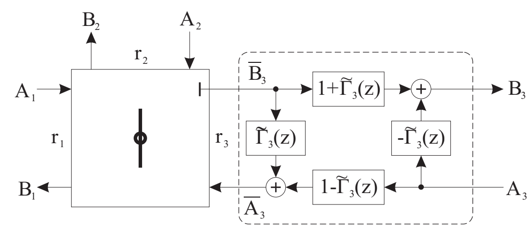

Figure 1: Any 3-port parallel (series) adaptor can always be implemented as a standard parallel (series) WDF adaptor, whose adapted port is connected to 2-port scattering cell.

2.1 The dynamic series adaptor

Let us consider a 3-port dynamic series adaptor with the following rational, causal and stable Reference Transfer Functions (RTF)

$$R_k(z) = \frac{r_k + \sum_{i=1}^{N_k} c_{ik} z^{-i}}{1 + \sum_{i=1}^{M_k} d_{ik} z^{-i}} = r_k + \frac{\sum_{i=1}^{N_k} r_{ik} z^{-i}}{1 + \sum_{i=1}^{M_k} d_{ik} z^{-i}},$$

$k = 1, 2, 3$ being the port index. The right-hand side of this equation is obtained by computing one step of the long division, where r_k is the result of the division and $\sum_{i=1}^{N_k} r_{ik} z^{-i}$ is the reminder. The reflection transfer functions are

$$\Gamma_k(z) = \frac{2R_k(z)}{\sum R_i(z)} = \gamma_k + \widetilde{\Gamma}_k(z), \tag{1}$$

from which we can extract a constant γ_k, while the rest can be written as $\widetilde{\Gamma}_k(z) = z^{-1}\hat{\Gamma}_k(z)$, where $\hat{\Gamma}_k(z)$ is assumed as causal and stable. From (1) we can thus derive

$$\Gamma_1(z) + \Gamma_2(z) + \Gamma_3(z) = 2 \Rightarrow \begin{cases} \gamma_1 + \gamma_2 + \gamma_3 = 2 \\ \widetilde{\Gamma}_1(z) + \widetilde{\Gamma}_2(z) + \widetilde{\Gamma}_3(z) = 0 \end{cases}. \tag{2}$$

Also the scattering matrix of this adaptor can be decomposed into the sum of a constant and a filtered term[1]

$$\mathbf{B} = \begin{bmatrix} 1-\Gamma_1 & -\Gamma_1 & -\Gamma_1 \\ -\Gamma_2 & 1-\Gamma_2 & -\Gamma_2 \\ -\Gamma_3 & -\Gamma_3 & 1-\Gamma_3 \end{bmatrix} \mathbf{A} =$$

[1] All the variables denoted with a capital letter depend on z. As always, vectors and matrices are denoted with bold letters, capital for matrices and lowercase for vectors. The only exceptions are the waves A e B, which are also vectors that depend on z, and yet they are denoted in capital bold letters for reasons of compliance with the literature.

$$= \left(\begin{bmatrix} 1-\gamma_1 & -\gamma_1 & -\gamma_1 \\ -\gamma_2 & 1-\gamma_2 & -\gamma_2 \\ -\gamma_3 & -\gamma_3 & 1-\gamma_3 \end{bmatrix} + \begin{bmatrix} -\widetilde{\Gamma}_1 & -\widetilde{\Gamma}_1 & -\widetilde{\Gamma}_1 \\ -\widetilde{\Gamma}_2 & -\widetilde{\Gamma}_2 & -\widetilde{\Gamma}_2 \\ -\widetilde{\Gamma}_3 & -\widetilde{\Gamma}_3 & -\widetilde{\Gamma}_3 \end{bmatrix} \right) \mathbf{A}$$

The first term represents the classical series junction, whose reflected waves are to be added the contribution of the second term, as we can see in Fig. 2.

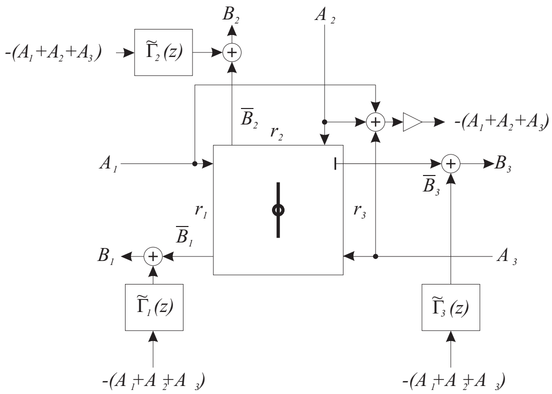

Figure 2: Direct implementation of eq. (2.1).

This structure, however, cannot be used for constructing wave digital structures, as it is not made of cells that are connected to each other through ports. If we could express $-(A_1+A_2+A_3)$ (the input of the three filters $\widetilde{\Gamma}_k(z)$) as a linear function of just the incident and reflected waves at each one of the adaptor ports, $A_1+A_2+A_3 = \alpha A_k + \beta B_k$, we would obtain an implementation of the $\widetilde{\Gamma}_k(z)$ as two-port cells. We notice that each reflected wave ends up depending on just one of the three reflection coefficients and for the instantaneous term we have

$$\begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \\ \overline{B}_3 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} - \begin{bmatrix} \gamma_1(A_1+A_2+A_3) \\ \gamma_2(A_1+A_2+A_3) \\ \gamma_3(A_1+A_2+A_3) \end{bmatrix}. \qquad (3)$$

For each row we can thus write

$$-(A_1+A_2+A_3) = \frac{1}{\gamma_k}(\overline{B}_k - A_k), \quad k = 1,2,3, \qquad (4)$$

which yields

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1-\gamma_1 & -\gamma_1 & -\gamma_1 \\ -\gamma_2 & 1-\gamma_2 & -\gamma_2 \\ -\gamma_3 & -\gamma_3 & 1-\gamma_3 \end{bmatrix} \mathbf{A}$$
$$+ \begin{bmatrix} -\frac{\widetilde{\Gamma}_1}{\gamma_1}(\overline{B}_1 - A_1) & -\frac{\widetilde{\Gamma}_2}{\gamma_2}(\overline{B}_2 - A_2) & -\frac{\widetilde{\Gamma}_3}{\gamma_3}(\overline{B}_3 - A_3) \end{bmatrix}^T.$$

If we rewrite this relationship while keeping the adaptation into account, $\gamma_3 = 1, \gamma_1 + \gamma_2 = 1$, we obtain

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1-\gamma_1 & -\gamma_1 & -\gamma_1 \\ \gamma_1 - 1 & \gamma_1 & \gamma_1 - 1 \\ -1 & -1 & 0 \end{bmatrix} \mathbf{A}$$
$$+ \begin{bmatrix} -\frac{\widetilde{\Gamma}_1}{\gamma_1}(\overline{B}_1 - A_1) & -\frac{\widetilde{\Gamma}_2}{1-\gamma_1}(\overline{B}_2 - A_2) & -\widetilde{\Gamma}_3(\overline{B}_3 - A_3) \end{bmatrix}^T$$

whose implementation is shown in Fig. 3.

We have thus obtained a structure made of an instantaneous series adaptor and three junctions that account for the whole dynamics of the initial adaptor. Yet, such junctions are not scattering cells as we might have expected, instead they form only half of a scattering cell (see Fig.3). By redrawing the three filter as shown in Fig. 2, where they all have the same input $x = -(A_1+A_2+A_3)$, it can be easily shown that the filter $\widetilde{\Gamma}_3(z)$ (whose output is summed to the wave $\overline{B}_3$) can now be moved to the inputs of the other two ports, as long as we multiply it by the corresponding γ_k, after a sign change.

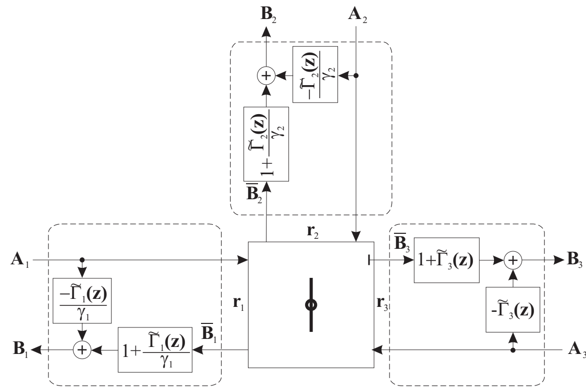

Figure 3: An intermediate step of the memory extraction process from an adaptor. Notice that each filter accounts for just half of the scattering junction.

At each one of the two non-adapted ports we now have two filters having the same input signal x. The output of the first one, $\widetilde{\Gamma}_m(z)$, $m = 1,2$, is summed to the reflected wave $\overline{B}_m$, while the output of the second, $\gamma_m \widetilde{\Gamma}_3(z)$, obtained by moving the filter that was at the adapted port, is summed to the incident wave A_m. In order for such filters to become a scattering junction, they need to be equal to each other, i.e.

$$\widetilde{\Gamma}_1(z) = -\gamma_1 \widetilde{\Gamma}_3(z), \widetilde{\Gamma}_2(z) = -\gamma_2 \widetilde{\Gamma}_3(z). \qquad (5)$$

Eqs. (2) and the conditions of instantaneous adaptation

$$\widetilde{\Gamma}_1(z) + \widetilde{\Gamma}_2(z) + \widetilde{\Gamma}_3(z) = 0,$$

$$\begin{cases} \gamma_1 + \gamma_2 = 1 \\ \gamma_3 = 1 \end{cases},$$

are satisfied by (5)

$$\begin{aligned} \widetilde{\Gamma}_1(z) + \widetilde{\Gamma}_2(z) + \widetilde{\Gamma}_3(z) &= -\gamma_1 \widetilde{\Gamma}_3(z) - \gamma_2 \widetilde{\Gamma}_3(z) + \widetilde{\Gamma}_3(z) \\ &= (-\gamma_1 - \gamma_2 + 1)\widetilde{\Gamma}_3(z) = 0. \end{aligned}$$

In conclusion, by moving the filter that was connected to the adapted port, we obtained a structure made of a static adaptor and two dynamic scattering junctions placed at the non-adapted ports. As we can see from Fig. **??**, if that filter had been moved at the input of port 3, then we would have obtained an equivalent structure with two filters at the outputs of the adapted ports, whose transfer functions are once again multiplied by the respective value of γ after a sign change.

Notice that if we had two filters of the form $-\gamma_1 K(z)$ and $-\gamma_2 K(z)$ at the outputs $\overline{B}_1$ and $\overline{B}_2$, respectively, then we could move such filters at the input a_3 to form a single filter $K(z)$. As a matter of fact, this is exactly our situation, as we have $-\gamma_m K(z) = -\gamma_m \widetilde{\Gamma}_3(z)$ at the non adapted ports. This can be

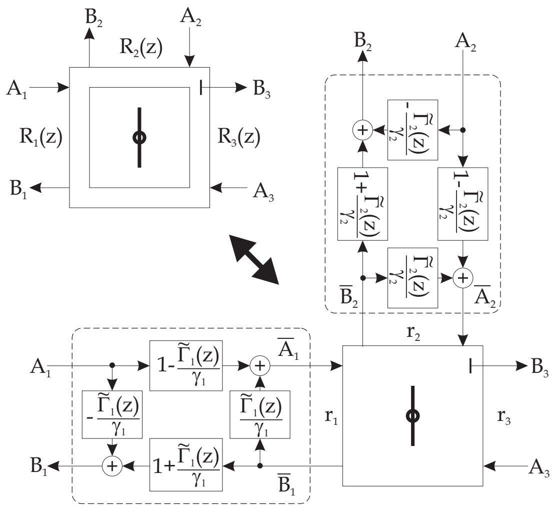

Figure 4: Equivalence between a dynamic series adaptor and a static series adaptor with two dynamic scattering junctions.

easily verified by replacing $\widetilde{\Gamma}_m$ with $-\gamma_m\widetilde{\Gamma}_3$. If we move such filters at the input port 3 then, together with the filter $\widetilde{\Gamma}_3$ that we extracted before, we obtain a single scattering cell that accounts for the whole dynamics of the adaptor.

A dynamic series adaptor is equivalent to its instantaneous counterpart whose adapted port is connected to a DSC. The reflection filter of the DSC is that of the adapted port of the initial adaptor, up to the constant term

It is important to notice that the scattering cells that we obtain with this procedure turn out to be always computable, as from each filter $\widetilde{\Gamma}_k(z)$ we can always extract a delay, as defined in (1):

$$\widetilde{\Gamma}_k(z) = z^{-1}\widehat{\Gamma}_k(z)\,.$$

A first consequence of this result is that, in the case of total adaptation on a dynamic series adaptor, we end up with a memoryless cell. In fact, $\Gamma_3(z) = 1$ implies $\widetilde{\Gamma}_3(z) = 0$, therefore the scattering cell becomes a direct input/output connection.

2.2 The dynamic parallel adaptor

As far as the parallel adaptor is concerned, reaching the same conclusions as before is not as immediate. This is due to the fact that the rows of the scattering matrix do not contain just one reflection coefficient like in the series case, but all three of them, which makes it impossible to write the sum of incident waves $(A_1 + A_2 + A_3)$ as a function of just the incident and reflected waves at each port

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} \Delta_1 - 1 & \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_2 - 1 & \Delta_3 \\ \Delta_1 & \Delta_2 & \Delta_3 - 1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}.$$

The above difficulty can be overcome by exploiting the property of the gyrator to transform a parallel adaptor into its dual, i.e. a series adaptor. A parallel dynamic adaptor whose RTF (port admittances) are

$$G_k(z) = \frac{g_k + \sum_{i=1}^{N_k} e_{ik} z^{-i}}{1 + \sum_{i=1}^{M_k} d_{ik} z^{-i}} = g_k + \frac{\sum_{i=1}^{N_k} g_{ik} z^{-i}}{1 + \sum_{i=1}^{M_k} d_{ik} z^{-i}}\,,$$

is equivalent to a series dynamic adaptor with the same RTFs as before (which are here to be interpreted as impedances), whose ports are connected to dynamic gyrators with unit gyration resistance. In the wave domain, a unit-resistance gyrator connected to a port with RTF $R(z)$ is implemented as a block that individually filters the incident wave with $R(z)$, and the reflected wave with $-1/R(z)$

$$\mathbf{B} = \mathbf{M_p}\mathbf{A} = -\mathbf{R}^{-1}\mathbf{M_s}\mathbf{R}\mathbf{A} \qquad (6)$$

where $\mathbf{R}$ is the matrix of gyration resistances[2]. By solving for the constant part of the scattering matrix and by applying again to that the equivalence principle, using as gyration coefficients the constant part of the RTFs, we obtain the instantaneous part $\overline{\mathbf{M}}_p$ of the parallel adaptor. We thus obtain a structure that is similar to that of a series adaptor, therefore we can extract the dynamics from it. Unlike the series case, however, in this case we have a *dynamic transformer* at each port of the adaptor.

3. MEMORY EXTRACTION FROM A MACRO-ADAPTOR

Using the tools introduced in Section 2, it is now possible to solve the more general problem of extracting the dynamics from a Macro-Adaptor (MA) (which is an arbitrary interconnection of elementary dynamic adaptors). Given a MA, we want to find an equivalent structure that is made of an instantaneous MA and a number of DSCs connected to some of (or all) the ports.

3.1 Structural equivalences

A MA made of the interconnection of a number of memoryless 3-port adaptors and DSCs can always be transformed into a new structure made of a memoryless MA surrounded by DSCs as shown in Fig. 5. This can be achieved by having all dynamic elements “slide through” the inner adaptors according to specific rules, until they reach the periphery of the interaction WD structure.

Our problem is to characterize the “sliding rules” that enable the “extraction of the memory” from inside the MA, which means that we need to find the equivalence that exists between a 3-port memoryless adaptor that has one port connected to a DSC, and another 3-port memoryless adaptor of the same type that has two DSCs connected to the other two ports. We will first consider the simpler case of a series adaptor, and we will show later how to adapt the same results to the parallel adaptor.

Let us consider a memoryless series adaptor whose adapted port is connected to a DSC whose reflection filter is $K(z)$.

The first step is to decompose the DSC into two filters with the same transfer function, driven by $X(z) = -(A_1(z) +$

[2] We recall that R_m are the RTFs, therefore in the case of a series adaptor, they represent the port impedances

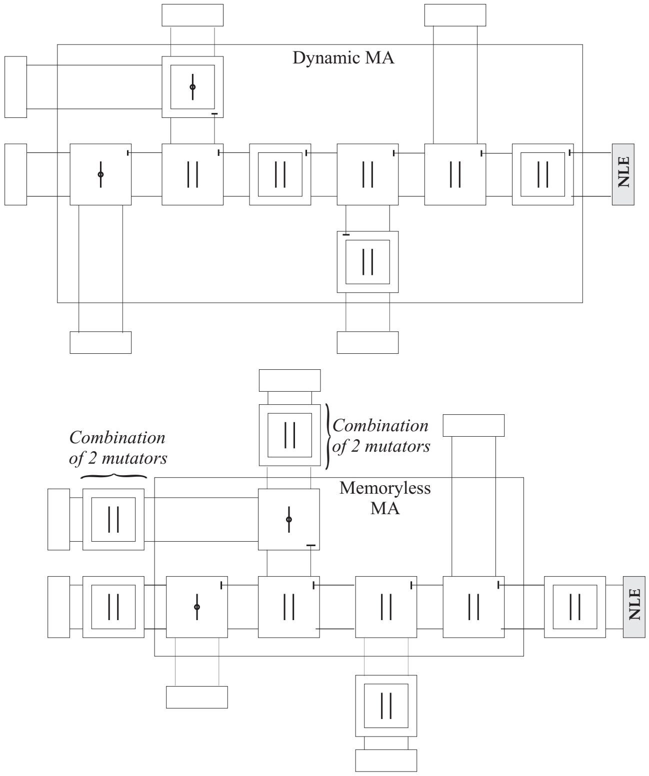

Figure 5: Extracting the dynamics from a macro-adaptor.

$A_2(z)+A_3(z)) = \frac{1}{\gamma_3}(B_3 - A_3) = B_3 - A_3$. Such filter can be moved onto the other two ports (see previous Section) provided that we multiply them by $-\gamma$ and that we change the sign of the port's reflection coefficient. We can now express the input of these filters as a function of the port waves to obtain

$$\begin{aligned} X(z)K(z) &= -\gamma_m(-(A_1(z)+A_2(z)+A_3(z)))K(z) = \\ &= -\gamma_m K(z)\left(\frac{1}{\gamma_m}\left(B_m(z)-A_m(z)\right)\right) = \\ &= K(z)\left(A_m(z)-B_m(z)\right), \quad m=1,2. \end{aligned}$$

This corresponds to the same scattering cell that we started with, in which we swapped the ports[3].

In conclusion: *a structure made of a memoryless series adaptor and a DSC connected to the junction's adapted port is equivalent to the same adaptor whose ports that are non-adapted are connected to similar DSCs.*

Under appropriate conditions, the above equivalency rule holds true in the opposite direction: two identical DSCs connected to the two non-adapted ports of an instantaneous series adaptor make a structure that is equivalent to the same adaptor connected through its adapted port to the same DSC. If only one of the two non-adapted ports is connected to a DSC, we can always connect the other non-adapted port to the cascade of the same DSC with another having an RTF of opposite sign. In fact we can show that two DSCs with opposite RTFs (and same initial conditions) cancel each other out. The pair of DSCs that are connected to the non-adapted ports of the junction can now be moved to the adapted port. As a result, we end up with a similar result as before: *a structure made of a memoryless series adaptor and a DSC connected to any of its ports is equivalent to the same adaptor whose other two ports are connected to similar DSC.* This result is reasonable as the adaptor's task is to implement continuity laws and, at the same time, guarantee the structure's computability.

Finally, it can be shown that the same results hold true for parallel 3-port junction as well, exploiting once again the properties of the gyrators.

[3] We recall that changing the sign to the RTF of a scattering cell corresponds to swapping the cell's ports

3.2 Initial conditions

When we replace a structure that includes a DSC with an equivalent one having two DSCs, such elements with memory cannot be independent on one another as this would correspond to increasing the number of state variables (and of initial conditions). Consequently, the initialization of both DSCs will depend on that of the original one. In order to derive this dependency we can write the reflected wave relative to any of the ports for both structures and then we can equate them. If we express the scattering filter in the form $X(z) = -(A_1(z)+A_2(z)+A_3(z))$ and take the adaptation condition $K(z) = z^{-1}\widehat{K}(z)$ into account, at the port 3 of the one-DSC structure we will have

$$B_3' = -A_1 - A_2 + K(z)X(z),$$

while for the two-DSC structure we have

$$\begin{aligned} B_3'' &= -\{A_1 + [-\gamma_1 K(z)]X(z)\} - \{A_2 + [-\gamma_2 K(z)]X(z)\} \\ &= -\{A_1 + [-K(z)]\gamma_1 X(z)\} - \{A_2 + [-K(z)]\gamma_2 X(z)\}. \end{aligned}$$

B_3' and B_3'' are the same, as the sign change of the filter $K(z)$ in the expression of B_3'' depends on the orientation of the DSC and $\gamma_1 + \gamma_2 = 1$.

The coefficients γ_1 and γ_2 can thus be interpreted as the weights that decide how to partition the initial conditions between the two DSCs in the equivalent structure.

4. CONCLUSIONS

In this paper we showed that a tree-like network of adaptors with memory in WD structures can always be replaced with a like network of memoryless WD adaptors of the same type (parallel/series) whose peripheral ports can be connected to two-port junctions with memory (mutators, transformers, etc.). This process of memory extraction plays a key role in the automatic implementation of WD structures in a wide range of applications. The proof that we proposed for this result is operative, in that it provides a procedure for automatic the memory extraction process.

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