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VARIABLE DIGITAL FILTER DESIGN WITH LEAST SQUARE CRITERION SUBJECT TO PEAK GAIN CONSTRAINTS

Hai Huyen Dam[1,3], *Antonio Cantoni*[2], *Kok Lay Teo*[3] *and Sven Nordholm*[1]

[1] Western Australian Telecommunications Research Institute (WATRI)
Curtin University of Technology
39 Fairway, Crawley WA 6009, Australia

[2]Western Australian Telecommunications Research Institute (WATRI)
The University of Western Australia
39 Fairway, Crawley WA 6009, Australia

[3]Department of Mathematics and Statistics
Curtin University of Technology
Bently, WA 6102, Australia

ABSTRACT

Variable digital filters (VDFs) are useful for various signal processing and communication applications where the frequency characteristics, such as fractional delays and cutoff frequencies, can be varied online. In this paper, we present a formulation that allows the trade-off between the integral squared error and the maximum deviation from the desired response in the passband and stopband. With this formulation, the maximum deviation can be reduced below the least square solution with a slight change in the performance of the integral squared error. Similarly, the total square error can be reduced below the minmax solution with a minor change in the maximum deviation from the minmax solution. Numerical schemes with adaptive grid size are presented for solving the optimization problems.

1. INTRODUCTION

Variable digital filters (VDFs) are digital filters with controllable spectral characteristics such as variable cutoff frequency, adjustable passband width and controllable fractional delay [1], [2]. These spectral characteristics can be varied online. Variable digital filters have many applications in different areas of signal processing and communications. Examples include arbitrary sample rate changers, digital synchronizers and other applications involving online tuning of frequency characteristics and timing adjustment for digital receivers.

The least square design criterion is commonly used to design Farrow-based [3] Finite Impulse Response (FIR) VDFs. This criterion is related to the noise gain of the filter. The design problem is simple and easy to formulate, which results in the filters with low sidelobe energy but having large errors near the discontinuities in the desired response. The least square criterion gives rise to a quadratic optimization problem [4]. Linear programming technique has also been used for the design of variable digital filters with minmax design error criterion. With the minmax criterion, the emphasis is to minimize the maximum amplitude distortion of signals to be passed by a filter without taking into consideration the error energy. Thus, minmax filters typically have high sidelobe energy.

In this paper, we investigate the design of filters with Farrow structure, [3], which allows a trade-off between the minmax and least square criteria. The design poblem can be formulated as a semi-infinite quadratic optimization problem. The design problem with minmax criterion, on the other hand, can be formulated as a semi-infinite linear programming problem. Numerical schemes with adaptive grid size are presented for solving the semi-infinite quadratic and linear optimization problems. Following from general window design [5], we show the trade-off between the integral squared error and the peak error for VDFs. The minmax and least square solutions are at the end points of the trade-off.

The outline of the paper is given as follows. Variable digital filter and the desired frequency response are given in Section 2. The problem formulation is formulated in Section 3. The discretized optimization methods for solving the semi-infinite linear and quadratic programming problems with adaptive scheme are presented in Section 4. Design examples are given in Section 5 and finally, conclusions are given in Section 6.

2. VARIABLE DIGITAL FILTER AND THE DESIRED FREQUENCY RESPONSE

In a VDF design problem, the objective is to achieve a design response $H_d(z,\delta)$ that is a function of a control or tuning parameter δ defined to lie in some range Δ,

$$\Delta = [\delta_{\min}, \delta_{\max}].$$

The desired frequency response for the VDF is specified by

$$H_d(e^{j\omega},\delta) = \begin{cases} e^{-j\omega\tau(\delta)}, & \omega \in \mathscr{P}(\delta) \\ 0, & \omega \in \mathscr{S}(\delta), \end{cases}$$

where $\mathscr{P}(\delta)$, $\mathscr{S}(\delta)$ and $\tau(\delta)$ are the passband, stopband regions, and the desired group delay, respectively.

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The VDF structure consists of L FIR subfilters, depicted as in Figure 1. The z transform of the VDF can be expressed as

$$H(z,\delta) = \sum_{l=0}^{L-1} H_l(z)\delta^l, \tag{1}$$

where $H_l(z)$ is the z transform for the l^{th} subfilter,

$$H_l(z) = \sum_{n=0}^{N-1} h_l(n)z^{-n} = \mathbf{h}_l^T \phi(z). \tag{2}$$

Here, $\mathbf{h}_l$ and $\phi(z)$ are $N \times 1$ vectors,

$$\mathbf{h}_l = [h_l(0), \cdots, h_l(N-1)]^T,$$

and $\phi(z) = [1, \cdots, z^{-(N-1)}]^T$. The notation $[\cdot]^T$ denotes the transposition of a vector $[\cdot]$.

The frequency response of the VDF can be given in the following form

$$H(e^{j\omega}, \delta) = \mathbf{h}^T \mathbf{s}(\omega, \delta),$$

where $\mathbf{h}$ and $\mathbf{s}(\omega,\delta)$ are $NL \times 1$ vectors,

$$\mathbf{h} = [\mathbf{h}_0^T, \cdots, \mathbf{h}_{L-1}^T]^T$$

and

$$\mathbf{s}(\omega,\delta) = [\phi^T(e^{j\omega})\delta^0, \cdots, \phi^T(e^{j\omega})\delta^{L-1}]^T.$$

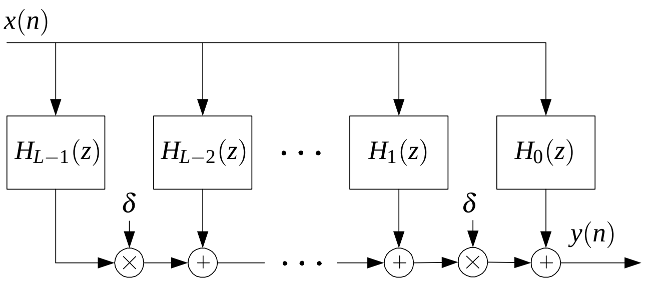

Figure 1: Variable digital filter - Farrow structure.

3. PROBLEM FORMULATION

The integral squared error of the frequency response deviation over all possibilities of δ and ω is given by

$$E(\mathbf{h}) = \int_{\Delta} \int_{\Omega(\delta)} W(\omega,\delta)|H(e^{j\omega},\delta) - H_d(e^{j\omega},\delta)|^2 d\omega d\delta, \tag{3}$$

where $W(\omega,\delta)$ is the weighting function and

$$\Omega(\delta) = \mathscr{P}(\delta) \cup \mathscr{S}(\delta).$$

Since we consider real coefficient vector $\mathbf{h}$, the error in (3) can be reduced to the following quadratic function,

$$E(\mathbf{h}) = \mathbf{h}^T\mathbf{Q}\mathbf{h} + \mathbf{p}^T\mathbf{h} + c, \tag{4}$$

where $\mathbf{Q}$ is an $NL \times NL$ matrix,

$$\mathbf{Q} = \mathscr{R}\left\{\int_{\Delta}\int_{\Omega(\delta)} W(\omega,\delta)\mathbf{s}(\omega,\delta)\mathbf{s}^H(\omega,\delta)d\omega d\delta\right\},$$

$\mathbf{p}$ is an $NL \times 1$ vector,

$$\mathbf{p} = -2\mathscr{R}\left\{\int_{\Delta}\int_{\Omega(\delta)} W(\omega,\delta)\mathbf{s}(\omega,\delta)H_d^*(e^{j\omega},\delta)d\omega d\delta\right\}$$

and c is a constant,

$$c = \int_{\Delta}\int_{\Omega(\delta)} W(\omega,\delta)|H_d(e^{j\omega},\delta)|^2 d\omega d\delta.$$

The notations $(\cdot)^H$ and $(\cdot)^*$ denote the Hermitian transpose and the complex conjugate of $(\cdot)$, respectively. The least square solution is obtained by minimizing the quadratic cost function in (4). The solution to this optimization problem can be expressed as

$$\mathbf{h}_{LS} = -\frac{1}{2}\mathbf{Q}^{-1}\mathbf{p}. \tag{5}$$

Now we consider the minmax criterion. The design problem can be formulated as

$$\min_{\mathbf{h}} \max_{\delta\in\Delta} \max_{\omega\in\Omega(\delta)} W(\omega,\delta)|H(e^{j\omega},\delta) - H_d(e^{j\omega},\delta)|. \tag{6}$$

By introducing a positive parameter γ, the optimization problem (6) can be re-formulated as

$$\begin{cases} \min\limits_{\mathbf{h},\gamma} \gamma \\ \text{subject to} \\ W(\omega,\delta)|H(e^{j\omega},\delta) - H_d(e^{j\omega},\delta)| \le \gamma, \\ \forall\omega \in \Omega(\delta),\ \delta \in \Delta. \end{cases} \tag{7}$$

By using the real rotation theorem [6], the problem (7) can be formulated as

$$\begin{cases} \min\limits_{\mathbf{h},\gamma} \gamma \\ \text{subject to} \\ W(\omega,\delta)\mathscr{R}\{\mathbf{s}(e^{j\omega},\delta)e^{j2\pi\lambda}\}^T\mathbf{h} - \gamma \le W(\omega,\delta)\times \\ \mathscr{R}\left\{H_d(e^{j\omega},\delta)e^{j2\pi\lambda}\right\}, \forall\omega \in \Omega(\delta),\ \delta\in\Delta,\ \lambda\in[0,1]. \end{cases} \tag{8}$$

The problem (8) is a semi-infinite linear optimization problem. Denote by E_{LS}, E_{MM} and γ_{MM}, γ_{LS} the integral squared error and maximum deviation level for the least square problem and the minmax problem (8), respectively. Since the minmax solution has the lowest maximum error deviation while the least square solution has the lowest integral squared error, we have

$$E_{LS} \le E_{MM} \text{ and } \gamma_{MM} \le \gamma_{LS}.$$

In the following, we present a design criterion that allows the trade-off between the minmax and least square criteria. The optimization problem can be formulated as minimizing the integral squared error with the maximum error deviation being restricted to be less than or equal to a positive value α, where

$$\gamma_{MM} \le \alpha \le \gamma_{LS}.$$

This problem can be expressed as

$$\begin{cases} \min_{\mathbf{h}} E(\mathbf{h}) \\ \text{subject to} \\ W(\omega,\delta)|H(e^{j\omega},\delta)-H_d(e^{j\omega},\delta)| \le \alpha, \forall\ \omega \in \Omega(\delta),\ \delta \in \Delta. \end{cases} \tag{9}$$

By using the real rotation theorem, this optimization problem can be reduced to the following semi-infinite quadratic optimization problem

$$\begin{cases} \min_{\mathbf{h}} \mathbf{h}^T\mathbf{Q}\mathbf{h}+\mathbf{p}^T\mathbf{h}+c \\ \text{subject to} \\ W(\omega,\delta)\mathscr{R}\left\{\mathbf{s}(e^{j\omega},\delta)e^{j2\pi\lambda}\right\}^T\mathbf{h} \le \alpha + W(\omega,\delta)\times \\ \mathscr{R}\left\{H_d(e^{j\omega},\delta)e^{j2\pi\lambda}\right\}, \forall\omega \in \Omega(\delta), \delta\in\Delta, \lambda\in[0,1]. \end{cases} \tag{10}$$

4. OPTIMIZATION ALGORITHM

The problem (10) is a semi-infinite quadratic optimization problem with three continuous parameters ω, δ and λ. This problem can be solved by using: (1) semi-infinite quadratic optimization techniques or (2) discretization approach. In this paper, we present a discretization scheme with adaptive grid size for solving the quadratic optimization problem (10). Similar scheme has been developed in [7] for solving the semi-infinite linear programming problem.

If the continuous sets Δ, Ω and $[0,1]$ are approximated with grid sets of sizes K_1, K_2 and K_3, then the problem (10) reduces to a quadratic optimization problem with $K_1K_2K_3$ constraints. It can be seen from [6] that if a unit circle generated by $e^{j2\pi\lambda}$ for $\lambda\in[0,1]$ is approximated by 16 points,

$$\Lambda = [0, 1/16, \cdots, 15/16],$$

then the difference between the absolute complex value and the maximum discretized value is relatively small. Thus, the value of K_3 is set as $K_3 = 16$. For the discretization problem obtained to be a good approximation to the original problem, the values K_1 and K_2 should be sufficiently large. Thus, the discretization problem has a large number of constraints. Consequently, a discretization method with adaptive grid points is employed, where the sequence grid points are refined gradually.

In many cases, the number of constraints turn out to be too large to be handled. Thus, a near active set constraint scheme is used which includes a procedure to eliminate unnecessary constraints [7], [8]. The optimization scheme is presented as follows.

Procedure 4.1 *Quadratic programming approach employing adaptive grid scheme.*

- *Step 1: Initialize the numbers of discretization points K_1, K_2, and two positive numbers ε, ε_1 where $\varepsilon_1 > \varepsilon$. Denote by $\hat{\Delta}$ and $\hat{\Omega}(\delta)$ the uniform discretization sets for Δ and $\Omega(\delta)$, respectively, with K_1 and K_2 grid points for all $\delta\in\hat{\Delta}$.*
- *Step 2: If $\mathbf{h}$ has not been initialized, then set $\hat{\mathscr{X}} = \mathscr{X}$ where $\mathscr{X}$ is the set of all discretization points for (ω,δ) and go to Step 3. Otherwise, let $\hat{\mathscr{X}}$ be the set of (ω,δ) such that*

$$\hat{\mathscr{X}} = \{(\omega,\delta)\in\mathscr{X} : W(\omega,\delta)\times \\ |\mathbf{h}^T\mathbf{s}(e^{j\omega},\delta)-H_d(e^{j\omega},\delta)| \ge \alpha-\varepsilon_1\}. \tag{11}$$

- *Step 3: Solve the problem (10) with constraints*

$$W(\omega,\delta)\mathscr{R}\left\{\mathbf{s}(e^{j\omega},\delta)e^{j2\pi\lambda}\right\}^T\mathbf{h} \le \alpha - \\ \varepsilon + W(\omega,\delta)\mathscr{R}\left\{H_d(e^{j\omega},\delta)e^{j2\pi\lambda}\right\},$$

  for all $(\omega,\delta)\in\hat{\mathscr{X}}$ and $\lambda\in\Lambda$.
- *Step 4: If the numbers of grid points K_1 and K_2 are less than the maximum numbers of discretization points K_1^{max} and K_2^{max}, then increase $K_1 \leftarrow 2K_1$ and $K_2 \leftarrow 2K_2$ and update the discretization set $\mathscr{X}$. Otherwise, $\mathscr{X}$ remains unchanged. If ε is small enough and $K_1 \ge K_1^{max}$, $K_2 \ge K_2^{max}$, then go to Step 6. Otherwise, set $\varepsilon = \varepsilon/10$ and return to Step 2. Here, the value of ε_1 is chosen as $\varepsilon_1 = \max(2\varepsilon, 10^{-2})$.*
- *Step 6: Stop the procedure.* □

The proposed algorithm does not require the discretization of λ in (11). For the problem (8), a scheme similar to that described in Procedure 4.1 can be devised for the discretized linear optimization problem with adaptive grid points. The variable in this case is $\left[\mathbf{h}^T,\gamma\right]^T$ instead of $\mathbf{h}$. Convergence properties for the algorithm can be shown similar to that in [7] and [8].

5. DESIGN EXAMPLES

Case 1: Consider the design of a lowpass VDF with *variable cutoff frequency*. The range of δ is chosen as $\Delta = [0,1]$. The passband and stopband cutoff frequencies $\omega_p(\delta)$ and $\omega_s(\delta)$ change linearly with respect to δ. Consequently, when δ changes from 0 to 1, the normalized passband region increases from

$$[-0.2\pi, 0.2\pi] \ \text{ to } \ [-0.4\pi, 0.4\pi]$$

while the stopband region reduces from

$$[-\pi,-0.4\pi]\cup[0.4\pi,\pi] \ \text{ to } \ [-\pi,-0.6\pi]\cup[0.6\pi,\pi].$$

The length of the filter is $N = 21$ while $L = 5$. The weighting function is one for all the frequencies. The values K_1^{max} and K_2^{max} are chosen as

$$K_1^{max} = 256 \text{ and } K_2^{max} = 128.$$

Figure 2 shows an example of the VDF magnitude responses for minmax criterion and different values of δ. The filters satisfy the specification and have approximately the same levels in the passband and stopband.

Figure 3 shows the trade-off between the integral squared error and the maximum error for cases with the desired delay being reduced from 10 to 6. The min-max and least square solutions are at the two ends of the trade-off curve. It is noted that the maximum and the integral squared errors increase with a reduction in the desired delay. In addition, the maximum error can be reduced from the least square solution with a minor change in the integral squared error. Similarly, the integral squared error can be reduced from the minmax solution with a minor change in the maximum error.

Case 2: Consider the design of a lowpass VDF with *variable delay*. The desired delay changes linearly over one sample delay with

$$\tau(\delta) = \tau_d + \delta,$$

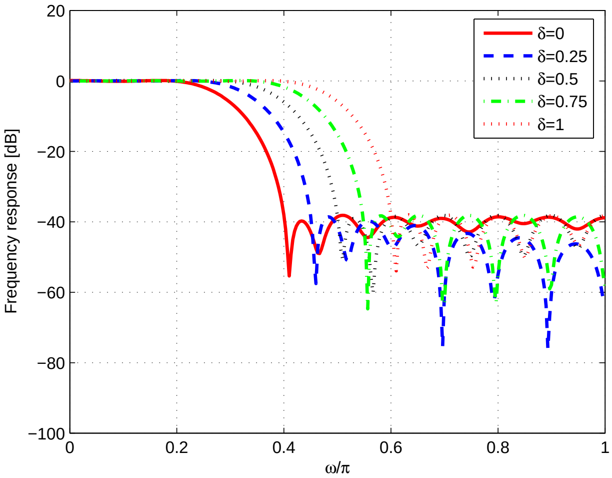

Figure 2: Magnitude response of VDF with variable cutoff frequency and minmax criterion with $\tau(\delta) = \tau_d = 10$.

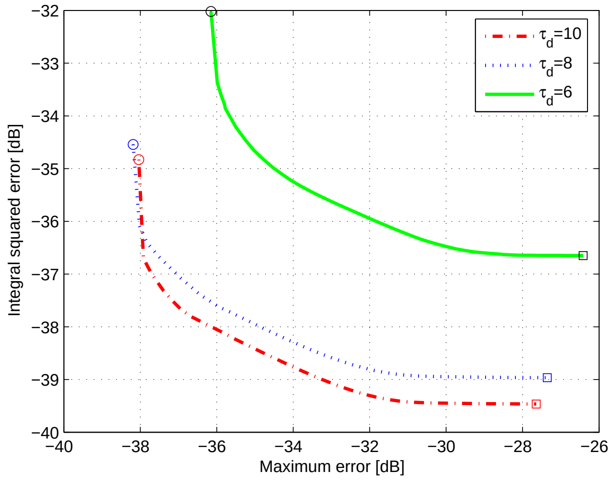

Figure 3: Integral squared error versus maximum error for VDF with variable cutoff frequency and different τ_d.

while the passband and stopband regions are unchanged, $\mathscr{P} = [-0.2\pi, 0.2\pi]$ and $\mathscr{S} = [-\pi, -0.4\pi] \cup [0.4\pi, \pi]$. The parameters are chosen the same as in Case 1.

The trade-off between the integral squared error and peak error is shown in Figure 4 for cases with the desired delay reduced from 10 to 6. The maximum error and integral squared error are increased when the desired delay is decreased. Similar to the first case, the maximum error can be reduced from the least square solution with a minor change on the integral squared error while the integral squared error can be reduced from the minmax solution.

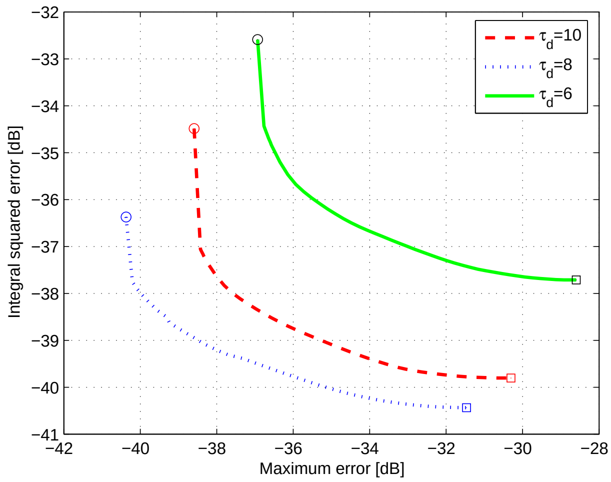

Figure 4: Integral squared error versus maximum error for VDF with variable delay and different τ_d.

6. CONCLUSIONS

In this paper, we have investigated the design of the VDF filters with least square criterion and peak gain constraints. We have shown that a trade-off can be achieved between the maximum error and the integral squared error. The maximum deviation can be reduced below the least square solution with a minor change in the performance of the total squared error. Similarly, the total squared error can be reduced below the minmax solution with a slightly increase in the maximum deviation from the minmax solution.

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