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THE DESIGN OF LOW-DELAY NONUNIFORM PSEUDO QMF BANKS

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ABSTRACT

This paper presents a method for designing low-delay nonuniform pseudo QMF banks. The method is motivated by the work of Li, Nguyen and Tantaratana, in which the nonuniform filter bank is realized by combining an appropriate number of adjacent subbands of a uniform pseudo QMF filter bank. In prior work, the prototype filter of the uniform pseudo QMF is constrained to have linear phase and the overall delay associated with the filter bank was often unacceptably large for filter banks with a large number of subbands. By relaxing the linear phase constraints, this paper proposes a pseudo QMF filter bank design technique that significantly reduces the delay. An example that experimentally verifies the capabilities of the design technique is presented.

1. INTRODUCTION

Nonuniform filter banks have been widely used in applications such as speech, audio and image processing. There are mainly two ways of achieving a nonuniform division of the signal spectrum. One is the wavelet/wavelet packet transform. Wavelet/wavelet packet transform can be conveniently performed using tree-structured filter banks [1]. However, nonuniform filter banks created by cascading wavelet bases often result in subband filters with poor frequency localization [8]. In most applications, subband signals are usually processed differently within different subbands. When decompositions with poor frequency localization are employed, aliasing distortion will result in the reconstructed output signal.

A second approach that involves the direct design of nonuniform filter banks have been proposed recently in an attempt to resolve the poor frequency localization problem [3, 2, 6]. In [3], the authors proposed a two-stage least squares method that employed a frequency domain criteria for the design of the analysis and synthesis filter banks. The design criteria were set to minimize the aliasing in each subband. In [2], the authors proposed a method of designing nonuniform filter banks by joining sections of different uniform filter banks using transition filters. The prototype filters of the uniform filter banks are designed to have sharp transition bands and high stopband attenuations. Both design approaches are somewhat complicated. Li *et al.* [6] proposed a simple and efficient method for designing nonuniform filter banks by combining an appropriate number of subbands of a uniform pseudo QMF bank [7].

Besides frequency localization, the delay introduced into the signal by the decomposition method is also a crucial component for many applications such as speech communications. Because of the linear phase constraint, the delay of the nonuniform filter bank proposed by Li *et al.* [6] is $N-1$ samples of the filter bank input, where N is the length of the prototype filter. For good frequency localization in filter banks with a large number of subbands, the filter length N must be large, and therefore the delay associated with the filter bank will also be large. Heller *et al.* [4] and Schuller *et al.* [10] proposed two different approaches for designing perfect reconstruction cosine-modulated filter banks with arbitrary delay. However, perfect reconstruction is overly restrictive in many practical applications. More importantly, perfect reconstruction filter banks usually cannot achieve as high stopband attenuation as nearly perfect reconstruction filter banks can [7]. High stopband attenuation is important to minimize the distortion caused by the combining of uniform bands to achieve nonuniform bands in [6].

In this paper, we present a method for designing low-delay nonuniform pseudo QMF banks that achieves sharp transition band and high stopband attenuation. The low-delay nonuniform pseudo QMF bank is constructed by combining an appropriate number of bands of a low-delay uniform pseudo QMF bank. The low-delay uniform pseudo QMF bank design is achieved by relaxing the linear phase condition of the prototype filter in [7]. New design constraints on the prototype filter for the low-delay uniform pseudo QMF bank is derived. We prove that by imposing these constraints on the design, near perfect reconstruction can be guaranteed. The combining process in achieving the nonuniform filter bank is similar to that in [6]. Our design also assumes an oversampled filter bank.

The rest of this paper is organized as follows. Section 2 briefly reviews the theoretic background of pseudo QMF banks [1] and the work by Nguyen [7] and Li *et al.* [6]. In Section 3, we present a new scheme for designing low-delay, oversampled nonuniform pseudo QMF banks. Section 4 provides a design example. Finally, we make our concluding remarks in Section 5.

2. REVIEW OF PRIOR WORK

For an M-channel maximally decimated uniform filter bank, the reconstructed signal $\hat{X}(z)$ can be expressed as [1]

$$\hat{X}(z) = \sum_{l=0}^{M-1} X(zW_M^l)T_l(z), \tag{1}$$

where $W_M = e^{-j\frac{2\pi}{M}}$ and

$$T_l(z) = \frac{1}{M}\sum_{k=0}^{M-1} H_k(zW_M^l)F_k(z). \tag{2}$$

For perfect reconstruction of $X(z)$, we need to have $T_0(z) = z^{-\Delta}$ and $T_l(z) = 0$, $1 \le l \le M-1$, where Δ is a positive integer corresponding to the delay of the filter bank. The system function $T_l(z)$ contributes to the aliasing associated with $X(zW_M^l)$. The system developed in this paper is a pseudo QMF bank which is a nearly perfect reconstruction cosine modulated filter bank. Nearly perfect reconstruction means that only "adjacent channel aliasing" (significant aliasing) is canceled [1]. When the stopband attenuation of the prototype filter is high, the "non-adjacent channel aliasing" will be small.

2.1 Uniform Pseudo QMF bank

For uniform pseudo QMF banks, the real valued coefficients of the analysis and synthesis filters are generated using [1, 7]

$$H_k(z) = a_k U_k(z) + a_k^* V_k(z), \quad (3)$$
$$F_k(z) = b_k U_k(z) + b_k^* V_k(z), \quad (4)$$

where $U_k(z)$ and $V_k(z)$ are defined as

$$U_k(z) = c_k H(zW_{2M}^{(k+0.5)}) \quad (5)$$

and

$$V_k(z) = c_k^* H(zW_{2M}^{-(k+0.5)}), \quad (6)$$

respectively. Here, $W_{2M} = e^{-j\frac{\pi}{M}}$, $H(z)$ is the prototype lowpass filter, and the coefficients a_k, b_k and c_k are constants with unit magnitude. The prototype filter $H(z)$ was restricted to be a linear phase filter with symmetry in [1, 7]. It was shown in [1] that if we choose $a_k = e^{j(-1)^k\frac{\pi}{4}}$, $b_k = a_k^*$ and $c_k = W_{2M}^{\frac{(N-1)(k+0.5)}{2}}$, the significant aliasing components are canceled and the distortion function reduces to

$$T_0(e^{j\omega}) = e^{-j\omega(N-1)} \frac{1}{M} \sum_{k=0}^{M-1} |H_k(e^{j\omega})|^2, \quad (7)$$

where N is the length of the prototype filter. It was proved in [7] that if we further constrain the prototype filter $H(z)$ to be a linear phase spectral factor of a $2M$th band filter, the overall distortion function $T_0(z)$ is a pure delay.

2.2 Nonuniform Pseudo QMF bank

Li, *et al.* [6] proposed a feasible partition nonuniform pseudo QMF bank design approach by simply combining the neighboring bands of a uniform pseudo QMF bank. For detailed information about feasible partition filter banks, refer to [5]. Consider a feasible partition nonuniform filter bank in Figure 1. In [6], a uniform M-channel pseudo QMF bank is first

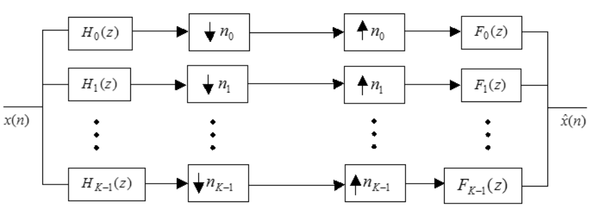


Figure 1: A K-channel nonuniform filter bank.

designed using the technique in [7], where M is equal to the least common multiple of $n_0, n_1, \cdots, n_{K-1}$. The pth subband of the nonuniform filter bank is then formed by combining $\frac{M}{n_p}$ uniform subbands starting at $k_p = \sum_{i=0}^{p-1} \frac{M}{n_i} + 1$ so that

$$H_p^{NU}(z) = \frac{1}{\sqrt{M/n_p}} \sum_{j=0}^{\frac{M}{n_p}-1} H_{k_p+j}(z) \quad (8)$$

and

$$F_p^{NU}(z) = \frac{1}{\sqrt{M/n_p}} \sum_{j=0}^{\frac{M}{n_p}-1} F_{k_p+j}(z). \quad (9)$$

The distortion resulting from the combining operation may be expressed as $T_0(z) = \frac{1}{M}(z^{-(N-1)} + D(z))$, where $D(z)$ is small when the stopband attenuation of the prototype filter is sufficiently high [6].

3. DESIGN OF LOW-DELAY NONUNIFORM PSEUDO QMF BANKS

In this section, we discuss the design of low-delay, maximally decimated and oversampled uniform pseudo QMF banks. The nonuniform filter bank is then constructed similar to that in [6].

3.1 Theoretic Statements

We start with the following two lemmas for uniform pseudo QMF banks. Lemma 1 extends the theory of pseudo QMF bank, which in the past has been limited to linear phase prototype filters, and shows that the same theory is applicable to nonlinear phase prototype filters as well. A proof of this lemma is given in the Appendix.

Lemma 1: Consider the prototype filter $H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}$, and let $G(z) = H^2(z) = \sum_{n=0}^{2(N-1)} g(n)z^{-n}$ be a $2M$th band filter, *i.e.*, for some delay Δ and integer values p

$$g(\Delta + p2M) = \begin{cases} 1/2; & p = 0 \\ 0; & \text{otherwise.} \end{cases} \quad (10)$$

By choosing $a_k = e^{j(-1)^k\frac{\pi}{4}}$, $b_k = a_k^*$ and $c_k = W_{2M}^{\frac{\Delta(k+0.5)}{2}}$, the pseudo QMF bank in which the analysis and synthesis filters are given by

$$H_k(z) = a_k c_k H(zW_{2M}^{(k+0.5)}) + a_k^* c_k^* H(zW_{2M}^{-(k+0.5)}),$$
$$F_k(z) = b_k c_k H(zW_{2M}^{(k+0.5)}) + b_k^* c_k^* H(zW_{2M}^{-(k+0.5)}),$$

or equivalently in the time domain by

$$h_k(n) = 2h(n)\cos\left(\frac{\pi}{M}(k+0.5)\left(n - \frac{\Delta}{2}\right) + (-1)^k \frac{\pi}{4}\right),$$
$$f_k(n) = 2h(n)\cos\left(\frac{\pi}{M}(k+0.5)\left(n - \frac{\Delta}{2}\right) - (-1)^k \frac{\pi}{4}\right),$$

exhibits no amplitude and phase distortion with $T_0(z) = z^{-\Delta}$. Furthermore, the significant aliasing components, *i.e.*, the "adjacent channel aliasing" terms are completely canceled by the above choice of a_k and b_k.

From this Lemma, it is clear that we can design a low-delay maximally decimated uniform pseudo QMF bank by relaxing the linear phase constraint on the prototype filter.

We note that as formulated here, Δ is a free design parameter that we can select.

All the pseudo QMF banks mentioned before assume maximal decimation. In general, the pseudo QMF banks lose their aliasing cancelation property for the oversampled case. However, for a special subclass of oversampling factors, we can design oversampled filter banks by simply changing the sampling factor associated with a maximally-decimated filter bank. This is formally stated in the next lemma.

Lemma 2: Let R be an integer such that $K = M/R$ is an integer. Then an oversampled pseudo QMF bank with oversampling factor R can be designed from a maximally decimated pseudo QMF bank such that there is no amplitude and phase distortion and the significant aliasing terms are completely canceled.

Proof of this lemma is simple. If M/R is an integer, the set of nearly perfect reconstruction constraints of an oversampled filter bank will be a subset of the constraints of the maximally decimated filter bank. This indicates that if we design a maximally decimated nearly perfect reconstruction filter bank, an oversampled nearly perfect reconstruction filter bank can be obtained by simply increasing the sampling rate of the maximally decimated filter bank R times.

Using the above lemmas, one may argue that the method of [6] can be applied on a low-delay uniform oversampled filter bank from which a low-delay nonuniform oversampled filter bank can be constructed.

3.2 Design Procedure

From Lemma 2, we know it is simple to construct an oversampled pseudo QMF bank from a maximally decimated pseudo QMF bank if the oversampling factor is an integer factor of the maximal decimation rate. We thus only discuss the design procedure of a maximally decimated pseudo QMF bank.

Let $\mathbf{h} = [h(0) \quad h(1) \quad \cdots \quad h(N-1)]^T$ and $\mathbf{e}(z) = [1 \quad z^{-1} \quad \cdots \quad z^{-(N-1)}]^T$, where the superscript T denotes transposition. The transfer function of the prototype filter is $H(z) = \mathbf{h}^T\mathbf{e}(z)$. Then,

$$\begin{aligned} G(z) &= H^2(z) = \sum_{n=0}^{2(N-1)} g(n) z^{-n} \\ &= \sum_{n=0}^{2(N-1)} (\mathbf{h}^T\mathbf{S}_n\mathbf{h}) z^{-n}. \end{aligned} \tag{11}$$

In the above equation, $\mathbf{S}_n$ are constant matrices whose elements take values from 0 or 1

$$[\mathbf{S}_n]_{k,l} = \begin{cases} 1; & k+l=n \\ 0; & \text{otherwise}. \end{cases} \tag{12}$$

From Lemma 1, we know that we can design a low-delay pseudo QMF bank by designing a prototype filter $H(z)$ with high stopband attenuation and satisfying (10) with $p \in [-\lfloor \frac{\Delta}{2M} \rfloor, \lfloor \frac{2(N-1)-\Delta}{2M} \rfloor]$. Comparing (10) and (11), we can now constrain the filter coefficients vector $\mathbf{h}$ so that whenever $n = \Delta + p2M$,

$$\mathbf{h}^T\mathbf{S}_n\mathbf{h} = \begin{cases} 1/2; & p=0 \\ 0; & \text{otherwise}. \end{cases} \tag{13}$$

In addition to the above constraints, $\mathbf{h}$ should also yield a prototype filter with good stopband attenuation. That is, we need to minimize the stopband energy

$$\begin{aligned} E_s &= \frac{1}{2\pi}\int_{\omega_s}^{2\pi-\omega_s} |H(e^{j\omega})|^2 d\omega \\ &= \mathbf{h}^T\mathbf{\Phi_s}\mathbf{h}, \end{aligned} \tag{14}$$

where, the (i,j)th element of $\mathbf{\Phi}_s$ can be expressed as

$$\mathbf{\Phi}_s(i,j) = \begin{cases} 1-\dfrac{\omega_s}{\pi}; & i=j \\ -\dfrac{\sin[\omega_s(i-j)]}{\pi(i-j)}; & i \neq j. \end{cases} \tag{15}$$

The optimization problem can be summarized as minimizing (14) subject to the constraints in (13). Here, we adopt the iterative least squares design technique developed by Rossi *et al* [9] to solve the nonlinear optimization problem. First, apply Cholesky factorization to obtain $\mathbf{\Phi}_s = \mathbf{C}^T\mathbf{C}$ and (14) can be rewritten as $E_s = (\mathbf{Ch})^T\mathbf{Ch} = \|\mathbf{Ch}\|^2$. Minimization of $\mathbf{\Phi}_s$ can be accomplished by minimizing the length of the vector $\mathbf{Ch}$. The constraints (13) can be combined together and written as

$$\begin{pmatrix} \mathbf{h}^T\mathbf{S}_{\Delta+p(1)2M} \\ \vdots \\ \mathbf{h}^T\mathbf{S}_{\Delta} \\ \vdots \end{pmatrix}\mathbf{h} - \begin{pmatrix} 0 \\ \vdots \\ 1/2 \\ \vdots \end{pmatrix} = \mathbf{0}. \tag{16}$$

Assume that $\mathbf{h}_i$ is a vector that is close to its optimum value. An iterative process for finding the optimum solution proceeds as follows:

1. Evaluate the matrix

$$\mathbf{B}_i = \begin{pmatrix} \mathbf{h}_i^T\mathbf{S}_{\Delta+p(1)2M} \\ \vdots \\ \mathbf{h}_i^T\mathbf{S}_{\Delta} \\ \vdots \end{pmatrix} \tag{17}$$

   and form the error vector

$$\mathbf{v}_i = \mathbf{D}_i\mathbf{h}_i - \mathbf{u}, \tag{18}$$

   where

$$\mathbf{D_i} = \begin{pmatrix} \mathbf{B}_i \\ \gamma\mathbf{C} \end{pmatrix} \quad \text{and} \quad \mathbf{u} = \begin{pmatrix} 0 \\ \vdots \\ 1/2 \\ 0 \\ \vdots \end{pmatrix}. \tag{19}$$

   The parameter γ is a constant that is chosen to strike a balance between the stopband attenuation and the accuracy of the constraints (13).

2. Find a vector $\tilde{\mathbf{h}}_i$ that minimizes the cost function $\|\mathbf{v}_i\|^2$. This is a least squares problem and has the solution $\tilde{\mathbf{h}}_i = (\mathbf{D}_i^T\mathbf{D}_i)^{-1}\mathbf{D}_i^T\mathbf{u}$.

3. Average the current solution $\tilde{\mathbf{h}}_i$ with the initial vector $\mathbf{h}_i$ to obtain a new value for the next iteration, *i.e.*, let $\mathbf{h}_{i+1}$ equal to $\frac{\tilde{\mathbf{h}}_i+\mathbf{h}_i}{2}$.
4. Evaluate $\|\mathbf{v}_{i+1}\|^2$ for $\mathbf{h}_{i+1}$. Stop the iterations if it is smaller than a pre-determined error or if a certain number of iterations has been executed. Otherwise, go to 1) and continue with the next iteration.

For convergence of the iterative least squares approach, a proper initial $\mathbf{h}_i$ has to be designed. Here, we adopted the design approach in [7] to design an initial filter $\mathbf{h}_i$. The design approach in [7] was for linear phase filter. The major difference is that we do not impose symmetrical structure in the low-delay design. For details of the design, refer to [11].

4. DESIGN EXAMPLE

We present the design result of a 9-channel nonuniform filter bank that was obtained from combining subbands of a 16-channel pseudo QMF bank. The first 6 channels of the nonuniform filter bank were the same as those in the uniform filter bank. The 7th channel of the nonuniform filter bank was created by combining the 7th and 8th channels, the 8th by combining the 9th - 12th channels and the 9th channel by combining the 13th - 16th channels of the uniform filter bank.

Figure 2 shows the design of a filter bank with a delay of 192 samples (solid line) and a linear phase design with a delay of 383 samples (dashed line). The plots presented in Figure 2(a)-(d) are the magnitude responses of the prototype filter $H(z)$, the analysis filters $H_k(z)$, $k = 1,7$ and 9, the overall distortion function $T_0(z)$, and the aliasing transfer functions $T_l(z)$, $l = 1$, respectively. The parameters of the prototype filter were $N = 384$, $\omega_s = 0.059\pi$. In order for a fair comparison, we chose γ that both the low-delay and the linear phase design had an amplitude distortion in the same level. In other words, we chose γ that satisfied the constraints (13) in the low-delay case and the constraints (29) in [7] to the same degree. In this example, γ is chosen to be 0.015 and 0.65 for the low-delay case and the linear phase case, respectively. This results in an amplitude distortion at the level of 5×10^{-5} dB in both cases as can be seen from Figure 2(c) (where no combining is done). 100 iterations were run. Comparing the magnitude responses in Figure 2(a), we can see that the lower delay was obtained at the cost of lower stopband attenuation. Lower stopband attenuation caused higher amplitude distortion and aliasing in the combining process to achieve nonuniform filter bank as can be seen from Figure 2(c) (where combining is done) and Figure 2(d), respectively. The maximum amplitude distortion of the low-delay design is less than 0.0015 dB and the aliasing distortion is below -100 dB for all cases. Such distortions are negligible and thus acceptable in a variety of applications. When we processed speech signals with the analysis and synthesis filter banks of this example, there were no audible differences between input and output signals.

To further illustrate the effectiveness of the low-delay design, we compare the above low-delay design with a linear phase design that has the same delay of 192 samples. The parameters for this linear phase design were $N = 193$, $\omega_s = 0.059\pi$. γ was chosen to be 0.001 to achieve an amplitude distortion at the level of 5×10^{-5} dB. Figure 3 compares the low-delay design (solid line) with the linear phase design

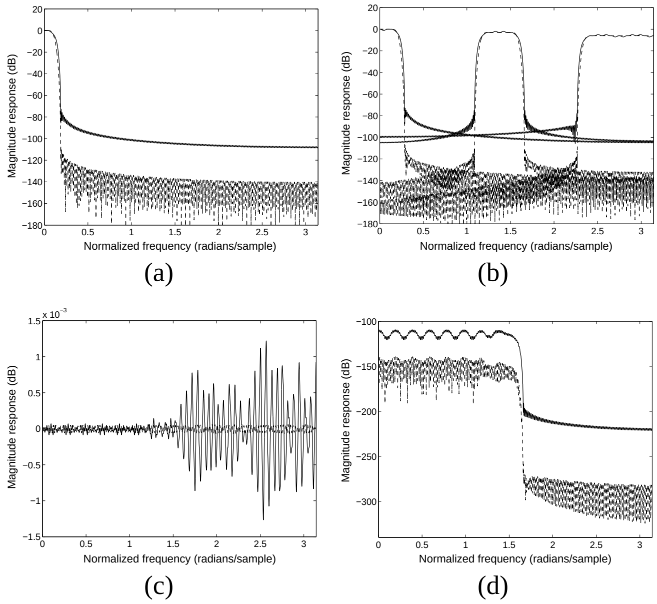


Figure 2: Comparison of the two filter bank designs in the example: Solid: low-delay design with $\Delta = 192$, Dashed: linear phase design with $\Delta = 383$ (a) Magnitude response of the prototype filter $H(z)$; (b) magnitude response of the nonuniform analysis filters $H_k(z)$, $k = 1,7,9$; (c) magnitude response of the overall distortion $T_0(z)$; (d) magnitude response plots for the aliasing transfer functions $T_l(z)$, $l = 1$.

(dashed line). From the plots in Figure 3, we can see for the same delay, the linear phase design has lower stopband attenuation, but also higher amplitude and aliasing distortion. The maximum amplitude distortion of the linear phase design is more than 0.01 dB and the aliasing distortion is above -100 dB for all cases. The linear phase design then has higher risk of producing audible distortions than the low-delay design, especially when the delay is further reduced.

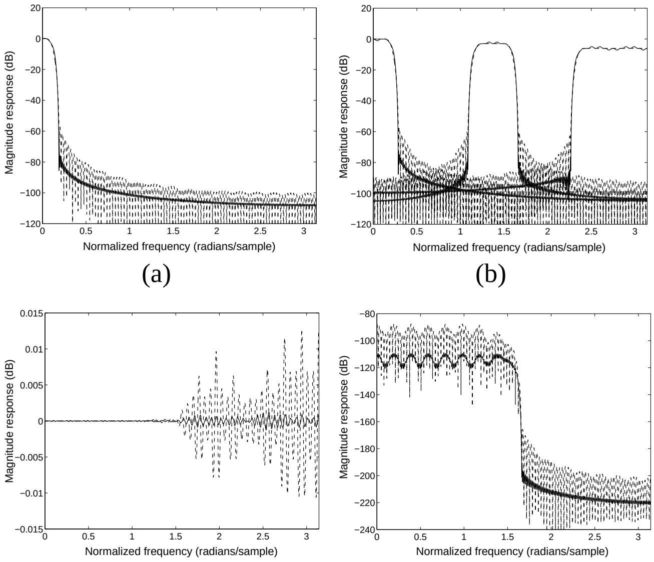
5. CONCLUSIONS

This paper presented an approach for designing low-delay nonuniform filter banks. The low delay is achieved by relaxing the linear phase constraints of traditional pseudo QMF banks. A design example was provided to demonstrate the effectiveness of the method. An application of this method in speech enhancement is described in [11]. The authors believe that the reduced delay design will facilitate the application of nonuniform filter banks in a variety of situations where tree-structured filter banks cannot be employed because of the unacceptably large delays associated with them.

6. APPENDIX

With the choice of $b_k = a_k^*$, $a_k = e^{j(-1)^k\frac{\pi}{4}}$, the proof of the significant aliasing cancelation is the same as that given in [1]. We now show that the distortion function is a pure delay, *i.e.*, $T_0(z) = z^{-\Delta}$.

Using the same definitions of $U_k(z)$ and $V_k(z)$ as in (5) and (6) and using the facts that $a_k b_k^* + a_k^* b_k = 0$ and $a_k b_k = a_k^* b_k^* = 1$ for our choice of the parameters, it is can be shown



(a) (b)

(c) (d)

Figure 3: Comparison of the two filter bank designs with $\Delta = 192$ in the example: Solid: low-delay design, Dashed: linear phase design (a) Magnitude response of the prototype filter $H(z)$; (b) magnitude response of the nonuniform analysis filters $H_k(z)$, $k = 1,7,9$; (c) magnitude response of the overall distortion $T_0(z)$; (d) magnitude response plots for the aliasing transfer functions $T_l(z)$, $l = 1$.

that

$$\begin{aligned} T_0(z) &= \frac{1}{M}\sum_{k=0}^{M-1} H_k(z)F_k(z) \\ &= \frac{1}{M}\sum_{k=0}^{M-1}[U_k^2(z)+V_k^2(z)] \\ &= \frac{1}{M}\sum_{k=0}^{M-1}[c_k^2H^2(zW_{2M}^{(k+0.5)})+(c_k^2)^*H^2(zW_{2M}^{-(k+0.5)})]. \end{aligned}$$

Since $c_k = W_{2M}^{\frac{\Delta(k+0.5)}{2}}$, we can write

$$\begin{aligned} T_0(z) &= \frac{1}{M}\sum_{k=0}^{M-1}[W_{2M}^{\Delta(k+0.5)}H^2(zW_{2M}^{(k+0.5)}) \\ &\quad +W_{2M}^{-\Delta(k+0.5)}H^2(zW_{2M}^{-(k+0.5)})] \\ &= \frac{1}{M}\sum_{k=0}^{M-1}[W_{2M}^{\Delta(k+0.5)}H^2(zW_{2M}^{(k+0.5)}) \\ &\quad +W_{2M}^{\Delta(2M-1-k+0.5)}H^2(zW_{2M}^{(2M-1-k+0.5)})] \\ &= \frac{1}{M}\sum_{k=0}^{2M-1}W_{2M}^{\Delta(k+0.5)}H^2(zW_{2M}^{(k+0.5)}). \end{aligned} \tag{20}$$

Substituting $G(z) = H^2(z) = \sum_{n=0}^{2(N-1)} g(n)z^{-n}$ in (20), we get

$$T_0(z) = \frac{1}{M}\sum_{n=0}^{2(N-1)} g(n)z^{-n}\sum_{k=0}^{2M-1}W_{2M}^{(\Delta-n)(k+0.5)}. \tag{21}$$

Since

$$\sum_{k=0}^{2M-1}W_{2M}^{(\Delta-n)(k+0.5)} = \begin{cases} (-1)^p 2M; & n = \Delta + p2M \\ 0; & \text{otherwise}, \end{cases} \tag{22}$$

and

$$g(\Delta + p2M) = \begin{cases} 1/2; & p = 0 \\ 0; & \text{otherwise}, \end{cases} \tag{23}$$

(21) reduces to

$$T_0(z) = z^{-\Delta}. \tag{24}$$

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