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NP-HARDNESS OF BIT ALLOCATION IN MULTIUSER MULTICARRIER COMMUNICATIONS

Manish Vemulapalli, Soura Dasgupta

Department of Electrical Computer Engineering
The University of Iowa
Iowa City, IA-52242, USA.
{mvemulap, dasgupta}@engineering.uiowa.edu

ABSTRACT

In this paper, we consider the problem of optimal bit allocation for a multiuser multicarrier communications. Some of the existing papers comment, without a proof, on the intractability of the problem and provide algorithms resulting in suboptimal bit allocation to reduce the computation complexity. A formal proof for classifying this problem as being NP-hard is presented in this article.

1. INTRODUCTION

In recent years multicarrier communications, with Orthogonal frequency division multiplexing (OFDM) representing a flagship example, has been assuming increased importance, [1]. Direct applications include, ADSL, VDSL and the IEEE 802.11a standard. Bit loading has been used as an effective tool for improving performance and enhancing throughput, [10]-[12]. Specifically, bit loading seeks to distribute the bit rate across the various subchannels characterizing a multicarrier system, to achieve optimality. For example, given channel conditions, one may distribute the bit rates across the carriers to minimize the transmission power needed to meet appropriate QoS requirements.

This paper examines bit loading within the context of multiuser communications, i.e. when multiple services with different data rate and QoS requirements must share the same multicarrier communications system. In this case the goal will be not just to distribute the bit rates across the carriers, but also to assign specific carriers to different services. For example, in [4], one has K users supported on an N-subchannel system. The i-th user must receive a total bit rate of β_i, and achieve symbol error rate (SER) of η_i. For an N-subchannel system, suppose user k is assigned the subchannels indexed by the set $\mathcal{I}_k \subset \{1, 2, \cdots, N\}$. Suppose x_i is the number of bits allocated to the i-th subchannel. Then it has been shown in, [4] that the net transmission power equals

$$J = \sum_{i=1}^{K} \sum_{k \in \mathcal{I}_i} \phi_k(x_k),$$

where typically

$$\phi_k(x_k) = \alpha_k 2^{x_k} \text{ with } \alpha_k > 0. \tag{1}$$

α_k's reflect target SER performance, and channel and interference conditions experienced in the k-th subchannel [4]. A high α_k value reflects adverse channel conditions and/or stringent performance goals; x_k is the (positive integer) number of bits assigned to each symbol in the cognizant subchannel. The K-user bit loading problem then becomes: under (1) find nonnegative integers x_i, and index sets $\mathcal{I}_k$, to minimize J above, subject to the condition that for all $k \in \{1, \cdots, K\}$

$$\sum_{k \in \mathcal{I}_i} x_k = \beta_i.$$

Approximation algorithms achieving a suboptimal multiuser bit loading exist in the literature. These include, [3], [5], [6], and [7] the latter considering a variation. The motivation given for studying suboptimal bit allocation is underscored by implicit and explicit claims made in some of these papers that the underlying optimization problem is NP-hard, i.e. an algorithm whose run time grows polynomially with N is unlikely to be found. We note for the sake of completeness that in the theory of computational complexity there exist a class of problems called NP-complete which are known not to have, as yet, algorithms that solve them with a runtime that is a polynomial in the input size. Thus classifying a problem as NP-hard would mean that the problem is just as "hard" to solve as an NP-complete problem and therefore cannot be solved in polynomial time.

However, the justification of the NP-hardness of the multiuser bit loading problem, used in turn as a justification of seeking suboptimal solutions, has thus far been made without any proof or reference. The paper [3], does cite a result from [13] to justify the claim of NP-hardness. However, [13] demonstrates the NP-hardness of the following problem. Given arbitrary convex f_i, positive real a_{ij}, find non-negative integers x_{ij} to:

$$\text{Minimize : } Z(X) = \sum_{i=1}^{n} f_i(\sum_{j=1}^{m} a_{ij} x_{ij}), \tag{2}$$

$$\text{Subject to : } \sum_{i=1}^{n} x_{ij} = N_j, j = 1, 2, \ldots, m. \tag{3}$$

This problem is far more general than the K-user bit loading problem, and its NP-hardness does not prove the NP-hardness of the latter. Consequently, if for no other reason than the sake of completeness, we present a proof for NP-hardness of the K-user bit loading problem, thereby formally justifying the search for suboptimal solutions.

To this end we focus on showing that the 2-user bit loading problem is it self NP-hard. Then NP-hardness of the K-user problem follows for $K \geq 2$. The two user problem on its part can be recast as below.

$\mathcal{Q}$: Under (1), given positive integers β_1 and β_2, positive numbers α_i, find nonnegative integers x_k and $\delta_k \in \{0,1\}$ such that (4) is minimized subject to (5).

$$\text{Minimize} : P(x_1, .., x_N) = \sum_{k=1}^{N} \phi_k(x_k), \tag{4}$$

$$\text{Constraint} : \begin{cases} \sum_{k=1}^{N} \delta_k x_k = \beta_1, \text{where} \quad \delta_k \in \{0,1\}, \\ \sum_{k=1}^{N} x_k = \beta_1 + \beta_2, \\ x_k \in \{0, 1, ..., \max\{\beta_1, \beta_2\}\}. \end{cases} \tag{5}$$

Note $\delta_k = 1$ indicates that k-th subchannel is assigned to user 1. The target bit rates for users 1 and 2 are β_1, and β_2 respectively.

Our proof approach recognizes that a problem A is said to be NP-hard if a problem B known to be NP-complete can be transformed to a problem instance of A (in polynomial time) in the sense that any problem instance of B has a solution if and only if the transformed instance A has a solution. (Thus A is not easier than B since any instance of B can be solved by solving the transformed instance of A). Note that the whole set of problem instances of B may be transformed to only a subset of problem instances of A.

The organization of this paper is as follows. In section 2 we introduce an NP-complete problem which is used to prove that $\mathcal{Q}$ is NP-hard and present a polynomial time transformation of the NP-complete problem to a problem instance of $\mathcal{Q}$. The proof for NP-hardness is provided in Section 3.

2. EQUIVALENCE OF $\mathcal{Q}$ AND THE SUBSET COVER PROBLEM

We now present a problem that qualifies as a simpler instance of $\mathcal{Q}$. Thus if this new problem is NP-Hard so is $\mathcal{Q}$.

Problem $\mathcal{A}$: Under (1), given a set of positive real numbers α_i, nonnegative integers b_i, $b_i \geq b_{i+1}$, and nonnegative integers β_1 and β_2, find nonnegative integers x_k, δ_k satisfying the following set of equations.

$$P(x_1, .., x_N) \leq P(b_1, .., b_N), \tag{6}$$

$$\text{Constraint} : \begin{cases} \sum_{k=1}^{N} \delta_k x_k = \beta_1, \text{where} \quad \delta_k \in \{0,1\}, \\ \sum_{k=1}^{N} x_k = \beta_1 + \beta_2, \\ x_k \in \{0, 1, ..., \max\{\beta_1, \beta_2\}\} \end{cases} \tag{7}$$

Problem $\mathcal{A}$ is no harder than the corresponding minimization problem $\mathcal{Q}$ because the minimum value $P(x_1^*, x_2^*, \ldots, x_N^*)$ of the minimization problem immediately shows whether $P(x_1, x_2, \ldots, x_N) \leq \mathcal{K}$ for a certain constant $\mathcal{K}$ is possible or not. Note that in Problem $\mathcal{A}$, $\mathcal{K}$ is chosen to be $P(b_1, \ldots, b_N)$, for certain $\{b_1, \ldots, b_N\}$. The reason for this choice of b_k's will be apparent by the end of this section.

In order to show that $\mathcal{Q}$ is NP-hard we then simply need to prove that a Problem-$\mathcal{B}$ which is known to be NP-complete could be transformed to an instance of $\mathcal{A}$ in polynomial time. Problem-$\mathcal{B}$ turns out to be the *Subset Cover* problem, which is known to be NP-complete, and has the following formulation,

Problem $\mathcal{B}$: Given a set of nonnegative integers S = $\{b_1, \ldots, b_N\}$, $b_i \geq b_{i+1}$ and a positive integer β_1, determine if there exists a subset $S_1 \subseteq S$ such that elements of S_1 add up to β_1. This is equivalent to finding δ_k such that

$$\begin{aligned} &\sum_{k=1}^{N} \delta_k . b_k = \beta_1, \; \delta_k \in \{0,1\}, \forall k. \\ &\sum_{k=1}^{N} b_k = \beta_1 + \beta_2 \end{aligned} \tag{8}$$

We show in the next section that problem $\mathcal{B}$ can be transformed to an instance of problem $\mathcal{A}$ in polynomial time, and that problem $\mathcal{B}$ has a solution if and only if that instance of problem $\mathcal{A}$ has a solution. More specifically, for every b_i and β_i for which problem $\mathcal{B}$ has a solution, there exist $\alpha_i > 0$, obtained in a polynomial time from the b_i, for which the resulting Problem $\mathcal{A}$ has a solution. What is more the x_i solving Problem $\mathcal{A}$ equal b_i, and the δ_i solving the two problems are identical.

The α_i's are chosen as follows: They must obey,

$$2^{b_1 - b_i - 1} < \alpha_i < 2^{b_1 - b_i}, \; \forall i, \tag{9}$$

and

$$\begin{aligned} &\alpha_i < \alpha_j, \; \forall i < j, \\ &\alpha_i \neq 2^C \alpha_j, \; \forall i < j, \; \text{C an integer} \end{aligned} \tag{10}$$

Since α_i's are real numbers, there is always a choice of α_i's satisfying (9) and (10). What is more the above transformation can be completed in O(N).

3. PROOF FOR NP-HARDNESS OF $\mathcal{Q}$

With the choice of α_i's as explained in Section 2 we now show that $\mathcal{Q}$ is NP-hard. In particular we prove the following theorem.

Theorem 1 *Given a set of nonnegative integers S = $\{b_1, \ldots, b_N\}$, $b_i \geq b_{i+1}$, positive integers β_1 and β_2, and α_i as in (9, 10), the only $\{x_1, x_2, \ldots, x_N\}$ for which P($x_1, \ldots, x_N$) ≤ P($b_1, \ldots, b_N$) under the constraint $\sum_{k=1}^{N} x_k = \beta_1 + \beta_2$, is $x_1 = b_1$, $x_2 = b_2$, …, $x_N = b_N$.*

Consequently, if there exist $\delta_1, \delta_2 \in \{0,1\}$, that solve problem $\mathcal{B}$ for this choice of b_i, then with α_i defined in (9, 10), problem $\mathcal{A}$ has a solution with $x_i = b_i$ and δ_i that solve problem $\mathcal{B}$. Further since $x_i = b_i$ are the only solutions to $P(x_1, .., x_N) \leq P(b_1, .., b_N)$, under

$$\sum_{i=1}^{N} x_i = \beta_1 + \beta_2,$$

problem $\mathcal{A}$ will not have a solution if problem $\mathcal{B}$ does not.

Thus indeed Theorem 1 proves that given any choice of b_i, β_i, there exist α_i obtained in a linear time from the b_i, for which the solution of problem $\mathcal{A}$ exists iff problem $\mathcal{B}$ also has a solution, and that indeed the solution to problem $\mathcal{A}$ is obtained entirely from that of problem $\mathcal{B}$. Thus the NP-hardness of problem $\mathcal{A}$ and hence also of problem $\mathcal{Q}$ follows.

To prove Theorem 1, we provide four lemmas. The first lemma proves that $\{x_1, \ldots, x_N\} = \{b_1, \ldots, b_N\}$ is an optimal solution for the allocation of $\beta_1+\beta_2$ bits. The last shows that $\{x_1, \ldots, x_N\} = \{b_1, \ldots, b_N\}$ is the only optimal solution under the constraint $\sum_{k=1}^{N} x_k = \beta_1 + \beta_2$.

Lemma 1 uses a result form [2] that solves the following single user bit allocation problem. With $\phi_k(.)$ as in (1) and a positive integer B, and find nonnegative integers x_k $\forall k$ such that (11) is minimized subject to (12).

$$\text{Minimize} : P(x_1, .., x_N) = \sum_{k=1}^{N} \phi_k(x_k), \quad (11)$$

$$\text{Constraint} : \sum_{k=1}^{N} x_k = B. \quad (12)$$

The solution in [2] defines $\psi_k(x) = \phi_k(x) - \phi_k(x-1)$, $l_i = \left\lceil log_2(\frac{\alpha_i}{\alpha_1}) \right\rceil$ and proceeds using:

Step-1: Find the smallest k such that

$$R_k = \sum_{i=1}^{k-1} (l_k - l_i) \geq B$$

Step-2: Define

$$\Delta = B - R_{k-1}$$
$$r = \Delta mod(k-1)$$
$$q = \Delta div(k-1)$$

suppose $\forall\ j_i \in \{1, ..., k-1\}$

$$\psi_{j_i}(l_{k-1} - l_{j_i} + q + 1) \leq \psi_{j_{i+1}}(l_{k-1} - l_{j_{i+1}} + q + 1)$$

Now $\forall\ j_i \in \{1, 2, .., k-1\}$

$$b_{j_i} = \begin{cases} l_{k-1} - l_{j_i} + q + 1 & \text{if} \quad 1 \leq i \leq r, \\ l_{k-1} - l_{j_i} + q & \text{else.} \end{cases}$$

We can now prove Lemma 1.

Lemma 1 *A solution to the optimization problem*

$$\begin{aligned} &\textit{Minimize} : P(x_1, .., x_N) = \sum_{k=1}^{N} \phi_k(x_k), \textit{where} \quad \phi_k(x_k) = \alpha_k 2^{x_k}, \\ &\textit{Constraint} : \sum_{k=1}^{N} x_k = \beta_1 + \beta_2, \end{aligned} \quad (13)$$

is given by $\{x_1, x_2, \ldots, x_N\} = \{b_1, b_2, \ldots, b_N\}$.

Proof: Using the single user discrete bit-loading algorithm presented in [2] for the choice of α_i's described earlier we have,

$$l_i = \left\lceil log_2(\frac{\alpha_i}{\alpha_1}) \right\rceil = b_1 - b_i.$$

We need to determine the smallest k for which the following holds.

$$R_k = \sum_{i=1}^{k-1} (l_k - l_i) \geq \beta_1 + \beta_2 \quad (14)$$

It can be seen that

$$\begin{aligned} R_N &= \sum_{i=1}^{N-1} (l_N - l_i) \\ &= \sum_{i=1}^{N-1} (b_i - b_N) \\ &= \beta_1 + \beta_2 - (N \times b_N) \\ &\leq \beta_1 + \beta_2. \end{aligned}$$

Therefore the smallest k for which (14) holds is $k = N + 1$ (since $l_{N+1} = \infty$).

$$\Delta = \beta_1 + \beta_2 - R_N = N \times b_N$$
$$r = \Delta \,\mathrm{mod}(N) = 0$$
$$q = \Delta \,\mathrm{div}(N) = b_N$$

Since r = 0,

$$\begin{aligned} x_i &= l_N - l_i + q \\ &= b_i - b_N + b_N \\ &= b_i, \forall i. \end{aligned}$$

■

Lemmas 2 and 3 are preparatory to proving Lemma 4. In particularly Lemma 2 exposes an ordering among the α_i.

Lemma 2 *If* $\{x_1, x_2, \ldots, x_N\}$ *is an optimal solution to the constrained optimization problem in Lemma(1) then,* $x_i \geq x_{i+1}$, $\forall i$

Proof: Supposing that $\exists\, i, j$ for which $x_i < x_j$ for some $i < j$.

$$\alpha_i(2^{x_j} - 2^{x_i}) < \alpha_j(2^{x_j} - 2^{x_i}), \text{ because } \alpha_j > \alpha_i, \text{ for } j > i$$
$$\alpha_i.2^{x_j} + \alpha_j.2^{x_i} < \alpha_i.2^{x_i} + \alpha_j.2^{x_j}$$

This means that by swapping the number bits assigned to channels i and j (while retaining the allocation for the remaining channels) we could minimize the cost. This is a contradiction since the initial assignment was know to be optimal. ■

Two further inequalities are provided by Lemma 3.

Lemma 3 *With the choice of* α_i*'s explained by (9) and (10) the following inequalities hold,*

$$\begin{aligned} \alpha_1.2^{b_1} &< \alpha_i.2^{b_i+1}, \forall i \\ \alpha_i.2^{b_i} &< \alpha_1.2^{b_1}, \forall i \end{aligned} \quad (15)$$

Proof: From equations (9) and (10) we have the following.

$$\alpha_1.2^{b_1-b_i-1} < \alpha_i < \alpha_1.2^{b_1-b_i}$$

Using the first half of the above inequality we have,

$$\alpha_1.2^{b_1-b_i-1} < \alpha_i$$
$$\alpha_1.2^{b_1-1} < \alpha_i.2^{b_i}$$
$$\alpha_1.2^{b_1} < \alpha_i.2^{b_i+1}$$

Now using the second half of the inequality,

$$\alpha_i < \alpha_1.2^{b_1-b_i}$$
$$\alpha_i.2^{b_i} < \alpha_1.2^{b_1}$$

We now prove the last Lemma that together with Lemma 1 proves Theorem 1.

Lemma 4 *If we have* $\sum_{k=1}^{N} \alpha_k.2^{x_k} = \sum_{k=1}^{N} \alpha_k.2^{b_k}$ *with* $\sum_{k=1}^{N} x_k = \sum_{k=1}^{N} b_k = \beta_1 + \beta_2$*, with the* α_i*'s chosen as described earlier we have* $\{x_1, \dots, x_N\} = \{b_1, \dots, b_N\}$

Proof: Let us assume that there exists some k such that $x_k \neq b_k$ otherwise we are done.

CASE I: $x_1 > b_1$

There exists some k such that $x_k = b_k - C_2$ for some positive constant C_2 while $x_1 = b_1 + C_1$ (for some positive constant C_1). Using Lemma(3) we have.

$$\begin{aligned} &\alpha_1.2^{b_1+C_1-1} > \alpha_k.2^{b_k-C_2}, \text{ since } C_1, C_2 \geq 1 \\ &\alpha_1.2^{b_1+C_1} + \alpha_k.2^{b_k-C_2} > \alpha_1.2^{b_1+C_1-1} + \alpha_k.2^{b_k-C_2+1} \\ &\alpha_1.2^{x_1} + \alpha_k.2^{x_k} > \alpha_1.2^{x_1-1} + \alpha_k.2^{x_k+1} \end{aligned}$$

The last equation implies that an allocation of $\{x_1-1, x_2, \dots, x_k+1, \dots, x_N\}$ yields a smaller cost. This is a contradiction.

CASE II: $x_1 < b_1$

There exists some k such that $x_k = b_k + C_2$ for some positive constant C_2 while $x_1 = b_1 - C_1$ (for some positive constant C_1). Using Lemma(3) we have.

$$\begin{aligned} &\alpha_k.2^{b_k+C_2-1} > \alpha_1.2^{b_1-C_1}, \text{ since } C_1, C_2 \geq 1 \\ &\alpha_1.2^{b_1-C_1} + \alpha_k.2^{b_k+C_2} > \alpha_1.2^{b_1-C_1+1} + \alpha_k.2^{b_k+C_2-1} \\ &\alpha_1.2^{x_1} + \alpha_k.2^{x_k} > \alpha_1.2^{x_1+1} + \alpha_k.2^{x_k-1} \end{aligned}$$

The last equation implies that an allocation of $\{x_1+1, x_2, \dots, x_k-1, \dots, x_N\}$ yields a smaller cost. This, again, is a contradiction and therefore $x_1 = b_1$. We now consider this scenario.

CASE III: $x_1 = b_1$.

Now $\exists\, k$ such that $x_k = b_k + C$ for some positive constant C. Using Lemma(3) we have,

$$\begin{aligned} &\alpha_k.2^{b_k+C} > \alpha_1.2^{b_1}, \text{ since } C \geq 1 \\ &\alpha_1.2^{b_1} + \alpha_k.2^{b_k+C} > \alpha_1.2^{b_1+1} + \alpha_k.2^{b_k+C-1} \\ &\alpha_1.2^{x_1} + \alpha_k.2^{x_k} > \alpha_1.2^{x_1+1} + \alpha_k.2^{x_k-1} \end{aligned}$$

The last equation clearly implies that an allocation of $\{x_1+1, x_2, \dots, x_k-1, \dots, x_N\}$ yields a smaller cost. This is a contradiction. Therefore from Cases I, II and III we can conclude that $x_k = b_k, \ \forall k$

■

4. CONCLUSIONS

In this paper we have presented a formal proof for classifying the multiuser bitloading for multicarrier systems as NP-hard. This was accomplished by proving the equivalence of the problem to the *Subset Cover* problem which is known to be NP-complete. This thus formally provides a hitherto unjustified rationale for seeking suboptimal solutions to the multiuser bit loading problem.

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