14th European Signal Processing Conference (EUSIPCO 2006), Florence, Italy, September 4-8, 2006, copyright by EURASIP

ANALYSIS OF VECTOR QUANTIZERS USING TRANSFORMED CODEBOOK WITH APPLICATION TO FEEDBACK-BASED MULTIPLE ANTENNA SYSTEMS

Jun Zheng and Bhaskar D. Rao

University of California at San Diego
juzheng@ucsd.edu, brao@ece.ucsd.edu

ABSTRACT

Transformed codebooks are often obtained by a transformation of a codebook, potentially optimum for a particular set of statistical conditions, to best match the statistical environment at hand. The procedure, though suboptimal, has recently been suggested for feedback MISO systems because of their simplicity and effectiveness. We first consider in this paper the analysis of a general vector quantizer with transformed codebook. Bounds on the average distortion of this class of quantizers are provided to characterize the effects of sub-optimality introduced by the transformed codebook on system performance. We then focus our attention on the application of the proposed general framework to providing capacity analysis of a feedback-based MISO system over correlated fading channels using channel quantizers with both optimal and transformed codebooks. In particular, upper and lower bounds on the channel capacity loss of MISO systems with transformed codebooks are provided and compared to that of the optimal quantizers. Numerical and simulation results are presented which confirm the tightness of the theoretical distortion bounds.

1. INTRODUCTION

Communication systems using multiple antennas have recently received much attention due to their promise of providing significant capacity increases. The performance of the multiple antenna systems depends heavily on the availability of the channel state information (CSI) at the transmitter (CSIT) and at the receiver (CSIR). Most of the MIMO system design and analysis adopt one of two extreme CSIT assumptions, *complete CSIT* and *no CSIT*. In this paper, we consider systems with CSI assumptions in between these extremes. We assume perfect CSIR is available at the receiver, and focus our attention on MIMO systems where CSI is conveyed from the receiver to the transmitter through a finite-rate feedback link. Recently, several interesting papers have appeared, proposing design algorithms as well as analytically quantifying the performance of the finite-rate feedback multiple antenna systems.

Most past works on the analysis of finite-rate feedback MIMO systems have adopted one of three approaches. The first is to approximate the channel quantization region corresponding to each code point based on the channel geometric property. Mukkavilli et. al. [1] derived a universal lower bound on the outage probability of quantized MISO beamforming systems with arbitrary number of transmit antennas t over i.i.d. Rayleigh fading channels. Love and et. al. [2] related the problem to that of Grassmannian line packing and provided corresponding performance bounds of multiple antenna systems with finite-rate feedback. The second approach is based on approximating the statistical distribution of the key random variable that characterizes the system performance. This approach was used by Xia et. al. in [3] and Roh et. al. in [4], where the authors analyzed the performance of MISO systems over i.i.d. Rayleigh fading channels, and obtained closed form expressions of the capacity loss (or SNR loss) in terms of feedback rate B and antenna size t. The third approach adopted by Narula et. al. in [5] is based on relating the quantization problem to the rate distortion theory, where the authors obtained an approximation of the expected loss of the received SNR due to finite rate quantization of the beamforming vectors in an MISO system. Moreover, Love and Heath in [6] and Xia et. al. in [3] extended the beamforming codebook design algorithms to correlated MIMO fading channels by using transformed codebooks obtained by a rotation-based transformation on an optimum codebook designed assuming i.i.d Rayleigh fading channels.

Most of the analytical results available to date are case specific and limited to i.i.d. MISO channels, and the approaches are hard to extend to more complicated schemes. In this paper, we consider the analysis of CSI-feedback-based multiple antenna system from a source coding perspective. We do this by using the general framework developed in [7] wherein channel quantization is formulated as a general vector quantization problem with encoder side information, constrained quantization space and non-mean-squared distortion function. The analysis was developed for optimal quantizers with perfect statistical knowledge. This paper first extends the general distortion analysis to sub-optimal quantizers with transformed codebooks. Distortion bounds of this class of quantizers are provided to characterize the effects of sub-optimality introduced by the transformed codebook on system performance. As an utilization of the general framework, this paper further investigates the effects of finite-rate CSI quantization on MISO systems over correlated fading channels. In particular, upper and lower bounds on the system capacity loss due to the finite-rate channel quantization are provided for MISO systems with transformed codebooks. Performance comparisons between MISO CSI quantizers with optimal and transformed codebooks are also provided under different channel correlations. Numerical and simulation results are presented which confirm the tightness of the theoretical distortion bounds.

2. BACKGROUND INFORMATION

The finite-rate feedback-based multiple antenna system can be formulated as a generalized fixed-rate vector quantization problem [7] and analyzed by adapting tools from high resolution quantization theory. In this section, we briefly describe the generalized high rate quantization theory provided in [7]. Extension and application of the distortion analysis to CSI-feedback-based MISO systems in the context of correlated channels are provided in later sections.

2.1 General Vector Quantization Framework

The multiple antenna systems with finite-rate feedback can be modeled as a generalized vector quantization problem with additional attributes such as encoder side information, constrained quantization space and non-mean-squared distortion measures. To be specific, the source variable $\mathbf{x} = (\mathbf{y}, \mathbf{z})$ is a two-vector tuple with vector $\mathbf{y} \in \mathbb{Q}$ representing the actual quantization variable of dimension k_{q} and $\mathbf{z} \in \mathbb{Z}$ being the additional side information of dimension k_{z}. The *encoder side information* $\mathbf{z}$ is available at the encoder but not at the decoder. Based on a particular source realization $\mathbf{x}$, the encoder (or the quantizer) represents vector $\mathbf{y}$ by one of the N vectors $\widehat{\mathbf{y}}_1, \widehat{\mathbf{y}}_2, \cdots, \widehat{\mathbf{y}}_N$, which form the codebook. The encoding or the quantization process is denoted as $\widehat{\mathbf{y}} = \mathcal{Q}(\mathbf{y}, \mathbf{z})$. The distortion of a

This research was supported in part by CoRe grant No.02-10109 sponsored by Ericsson and in part by the U. S. Army Research Office under the Multi-University Research Initiative (MURI) grant # W911NF-04-1-0224.

$$\frac{k_{\mathrm{q}}}{k_{\mathrm{q}}+2}\left(\frac{\left|\mathbf{W}_{\mathbf{z}}(\mathbf{F}(\mathbf{y}))\right|}{\kappa_{k_{\mathrm{q}}}^{2}}\right)^{\frac{1}{k_{\mathrm{q}}}}=\widetilde{I}_{\text{tr-low}}\left(\mathbf{F}(\mathbf{y});\mathbf{z}\right)\leq\widetilde{I}_{\text{tr}}\left(\mathbf{F}(\mathbf{y});\mathbf{z}\right)\leq\widetilde{I}_{\text{tr-upp}}\left(\mathbf{F}(\mathbf{y});\mathbf{z}\right)=\frac{\left|\mathbf{F}_{\mathrm{d}}(\mathbf{y})\right|^{-\frac{2}{k_{\mathrm{q}}}}}{k_{\mathrm{q}}+2}\left(\frac{\left|\mathbf{W}_{0,\mathbf{z}}(\mathbf{y})\right|}{\kappa_{k_{\mathrm{q}}}^{2}}\right)^{\frac{1}{k_{\mathrm{q}}}}\cdot\mathbf{tr}\Big(\mathbf{W}_{0,\mathbf{z}}(\mathbf{y})^{-1}\mathbf{F}_{\mathrm{d}}(\mathbf{y})^{\mathsf{T}}\mathbf{W}_{\mathbf{z}}(\mathbf{F}(\mathbf{y}))\mathbf{F}_{\mathrm{d}}(\mathbf{y})\Big). \tag{7}$$

$$\frac{k_{\mathrm{q}}'}{k_{\mathrm{q}}'+2}\left(\frac{\left|\mathbf{V}_2(\mathbf{F}(\mathbf{y}))^{\mathsf{T}}\cdot\mathbf{W}_{\mathbf{z}}(\mathbf{F}(\mathbf{y}))\cdot\mathbf{V}_2(\mathbf{F}(\mathbf{y}))\right|}{\kappa_{k_{\mathrm{q}'}}^{2}}\right)^{\frac{1}{k_{\mathrm{q}'}}}=\widetilde{I}_{\text{c-tr-low}}\left(\mathbf{F}(\mathbf{y});\mathbf{z}\right)\leq\widetilde{I}_{\text{c-tr}}\left(\mathbf{F}(\mathbf{y});\mathbf{z}\right)\leq\widetilde{I}_{\text{c-tr-upp}}\left(\mathbf{F}(\mathbf{y});\mathbf{z}\right)$$
$$=\frac{\left|\mathbf{V}_2(\mathbf{F}(\mathbf{y}))^{\mathsf{T}}\mathbf{F}_{\mathrm{d}}(\mathbf{y})\mathbf{V}_2(\mathbf{y})\right|^{-\frac{2}{k_{\mathrm{q}}'}}}{k_{\mathrm{q}}'+2}\left(\frac{\left|\mathbf{V}_2(\mathbf{y})^{\mathsf{T}}\mathbf{W}_{0,\mathbf{z}}(\mathbf{y})\mathbf{V}_2(\mathbf{y})\right|}{\kappa_{k_{\mathrm{q}}'}^{2}}\right)^{\frac{1}{k_{\mathrm{q}}'}}\cdot\mathbf{tr}\left(\Big(\mathbf{V}_2(\mathbf{y})^{\mathsf{T}}\mathbf{W}_{0,\mathbf{z}}(\mathbf{y})\mathbf{V}_2(\mathbf{y})\Big)^{-1}\mathbf{V}_2(\mathbf{y})^{\mathsf{T}}\mathbf{F}_{\mathrm{d}}(\mathbf{y})^{\mathsf{T}}\mathbf{W}_{\mathbf{z}}(\mathbf{F}(\mathbf{y}))\,\mathbf{F}_{\mathrm{d}}(\mathbf{y})\mathbf{V}_2(\mathbf{y})\right), \tag{8}$$

finite rate quantizer is defined as

$$D=E_{\mathbf{x}}\left[D_{\mathrm{Q}}\Big(\mathbf{y},\widehat{\mathbf{y}};\mathbf{z}\Big)\right], \tag{1}$$

where $D_{\mathrm{Q}}\Big(\mathbf{y},\widehat{\mathbf{y}};\mathbf{z}\Big)$ is a general *non mean-squared distortion* function between $\mathbf{y}$ and $\widehat{\mathbf{y}}$ that is parameterized by $\mathbf{z}$. It is further assumed that function D_{Q} has a continuous second order derivative (or Hessian matrix w.r.t. to $\mathbf{y}$) $\mathbf{W}_{\mathbf{z}}(\widehat{\mathbf{y}})$ with the $(i,j)^{\text{th}}$ element given by

$$w_{i,j}=\frac{1}{2}\cdot\frac{\partial^2}{\partial y_i\partial y_j}D_{\mathrm{Q}}\big(\mathbf{y},\widehat{\mathbf{y}};\mathbf{z}\big)\bigg|_{\mathbf{y}=\widehat{\mathbf{y}}}. \tag{2}$$

2.2 Distortion Analysis of the General Vector Quantizer

Under high resolution assumptions, large N, the average asymptotic distortion can be represented by the following form, which is similar to the Bennett's integral provided in [8]

$$D=2^{-\frac{2B}{k_{\mathrm{q}}}}\int_{\mathbb{Z}}\int_{\mathbb{Q}}I\big(\mathbf{y};\mathbf{z};\mathbb{E}_{\mathbf{z}}(\mathbf{y})\big)p\big(\mathbf{y},\mathbf{z}\big)\lambda\big(\mathbf{y}\big)^{-\frac{2}{k_{\mathrm{q}}}}\,d\mathbf{y}\,d\mathbf{z}, \tag{3}$$

where $\mathbb{E}_{\mathbf{z}}(\mathbf{y})$ denotes the asymptotic projected Voronoi cell that contains $\mathbf{y}$ with side information $\mathbf{z}$. In equation (3), $\lambda(\mathbf{y})$ is a function representing the relative density of the codepoints, which is called point density, such that $\lambda(\mathbf{y})\,d\mathbf{y}$ is approximately the fraction of quantization points in a small neighborhood of $\mathbf{y}$. Function $I\big(\mathbf{y};\mathbf{z};\mathbb{E}\big)$ is the normalized inertia profile that represents the relative distortion of the quantizer $\mathcal{Q}$ at position $\mathbf{y}$ conditioned on side information $\mathbf{z}$ with Voronoi shape $\mathbb{E}$. Both $\lambda(\mathbf{y})$ and $I\big(\mathbf{y};\mathbf{z};\mathbb{E}\big)$ are the key performance determining characteristics that can be used to analyze the effects of different system parameters, such as source distribution, distortion function, quantization rate etc., on the finite rate quantizer.

Note that if the source variable (vector) $\mathbf{y}$ is further subject to k_{c} *constraints* given by the vector equation $\mathbf{g}(\mathbf{y})=0$, the asymptotic distortion integral given by (3) is still valid under some minor modifications. In these cases, the actual degrees of freedom of the quantization variable reduce from k_{q} to $k_{\mathrm{q}}'=k_{\mathrm{q}}-k_{\mathrm{c}}$, and the average asymptotic distortion decays exponentially with rate $2^{-2B/k_{\mathrm{q}}'}$.

3. QUANTIZERS WITH TRANSFORMED CODEBOOK

In certain situations, the underlying source distribution $p(\mathbf{y},\mathbf{z})$ or the distortion function D_{Q} may vary continuously during the quantization process. However, it is practically infeasible to design separate codebooks optimized for every different source distribution and distortion function. In these cases, quantizers constructed by transforming another codebook based on the current statistical distribution of the source variable is a promising alternative.

3.1 Problem Setup

It is first assumed that all the codebooks are generated from one fixed codebook $\mathcal{C}_0$ which is designed to match source distribution $p_0(\mathbf{y},\mathbf{z})$, and distortion function $D_{0,\mathrm{Q}}$ with sensitivity matrix $\mathbf{W}_{0,\mathbf{z}}(\mathbf{y})$. Codebook $\mathcal{C}_0$ has a point density given by $\lambda_0(\mathbf{y})$, and a normalized inertial profile $I_0\big(\mathbf{y};\mathbf{z};\mathbb{E}_{0,\mathbf{z}}(\mathbf{y})\big)$ that is optimized to matches the distortion function $D_{0,\mathrm{Q}}$, with $\mathbb{E}_{0,\mathbf{z}}(\mathbf{y})$ representing the asymptotic Voronoi cell that contains $\mathbf{y}$ with side information $\mathbf{z}$. If the source distribution changes from $p_0(\mathbf{y},\mathbf{z})$ to $p(\mathbf{y},\mathbf{z})$ and the distortion function becomes D_{Q} instead of $D_{0,\mathrm{Q}}$ with sensitivity matrix $\mathbf{W}_{\mathbf{z}}(\mathbf{y})$ instead of $\mathbf{W}_{0,\mathbf{z}}(\mathbf{y})$, the encoder and decoder will correspondingly adopt a transformed codebook $\mathcal{C}$ from $\mathcal{C}_0$ by a general one-to-one mapping $\mathbf{F}(\cdot)$ with both of its domain and codomain in space $\mathbb{Q}$, i.e.

$$\mathcal{C}=\Big\{\mathbf{F}(\widehat{\mathbf{y}})\;\big|\;\widehat{\mathbf{y}}\in\mathcal{C}_0\Big\}. \tag{4}$$

3.2 Sub-optimal Point Density & Sub-optimal Voronoi Shape

Two types of sub-optimality arise when the transformed codebook is used instead of the optimal one. One comes from the sub-optimal point density $\lambda_{\text{tr}}(\mathbf{y})$ of the transformed codebook, which can be derived from $\lambda_0(\mathbf{y})$ by the following transformation

$$\lambda_{\text{tr}}(\mathbf{y})=\frac{\lambda_0\Big(\mathbf{F}^{-1}(\mathbf{y})\Big)}{\big|\mathbf{F}_{\mathrm{d}}\big(\mathbf{F}^{-1}(\mathbf{y})\big)\big|},\qquad\mathbf{F}_{\mathrm{d}}(\mathbf{y})=\frac{\partial\mathbf{F}(\mathbf{y})}{\partial\mathbf{y}}. \tag{5}$$

If the source variable is subject to k_{c} constraints given by the vector equation $\mathbf{g}(\mathbf{y})=\mathbf{0}$, the transformed point density is given by

$$\lambda_{\text{c-tr}}(\mathbf{y})=\frac{\lambda_0\Big(\mathbf{F}^{-1}(\mathbf{y})\Big)}{\Big|\mathbf{V}_2(\mathbf{y})^{\mathsf{T}}\cdot\mathbf{F}_{\mathrm{d}}\big(\mathbf{F}^{-1}(\mathbf{y})\big)\cdot\mathbf{V}_2\big(\mathbf{F}^{-1}(\mathbf{y})\big)\Big|}, \tag{6}$$

where $\mathbf{V}_2(\mathbf{y})$ is an orthonormal matrix with its columns constituting an orthonormal basis for the orthogonal compliment of the range space $\mathcal{R}\left(\frac{\partial}{\partial\mathbf{y}}\mathbf{g}(\mathbf{y})\right)$. Compared to the optimal point density $\lambda^*(\mathbf{y})$ that matches to $p(\mathbf{y},\mathbf{z})$ and D_{Q}, $\lambda_{\text{tr}}(\mathbf{y})$ given by equation (5) is sub-optimal and will lead to performance degradation. However, given no restrictions on the transformation, there always exists an $\mathbf{F}$ that makes $\lambda_{\text{tr}}(\mathbf{y})$ exactly equal to $\lambda^*(\mathbf{y})$.

The other sub-optimality arises from the fixed location of the code points in the transformed codebook $\mathcal{C}$, in the sense that the Voronoi shape of the transformed codebook does not match the distortion function D_{Q} and hence is not optimized to minimize the inertial profile. However, the Voronoi region $\mathbb{E}_{\text{tr},\mathbf{z}}(\mathbf{y})$ of the transformed codebook is hard to characterize and depends both on the transformation $\mathbf{F}$ as well as the distortion function D_{Q}. Fortunately, the approximated inertial profile $\widetilde{I}_{\text{tr}}\left(\mathbf{F}(\mathbf{y});\mathbf{z}\right)$ of the transformed codebook can be upper and lower bounded by equation (7). Furthermore, if the source variable is subject to k_{c} constraints given by the vector equation $\mathbf{g}(\mathbf{y})=\mathbf{0}$, the constrained inertial profile $\widetilde{I}_{\text{c-tr}}\left(\mathbf{F}(\mathbf{y});\mathbf{z}\right)$ can be similarly bounded by equation (8).

3.3 Distortion Integral of Transformed Codebooks

By substituting the transformed point density (5) and the bounds of the transformed inertial profile given by (7) into the distortion integration (3), we can upper and lower bound the asymptotic system distortion of a transformed quantizer by the following form

$$\widetilde{D}_{\text{tr-Low}}=2^{-\frac{2B}{k_{\mathrm{q}}}}\left(\int_{\mathbb{Z}}\int_{\mathbb{Q}}\widetilde{I}_{\text{tr-low}}\left(\mathbf{y};\mathbf{z}\right)p\big(\mathbf{y},\mathbf{z}\big)\lambda_{\text{tr}}\big(\mathbf{y}\big)^{-\frac{2}{k_{\mathrm{q}}}}\,d\mathbf{y}\,d\mathbf{z}\right)$$
$$\leq\widetilde{D}_{\text{tr}}=2^{-\frac{2B}{k_{\mathrm{q}}}}\left(\int_{\mathbb{Z}}\int_{\mathbb{Q}}\widetilde{I}_{\text{tr}}\left(\mathbf{y};\mathbf{z}\right)p\big(\mathbf{y},\mathbf{z}\big)\lambda_{\text{tr}}\big(\mathbf{y}\big)^{-\frac{2}{k_{\mathrm{q}}}}\,d\mathbf{y}\,d\mathbf{z}\right)$$
$$\leq\widetilde{D}_{\text{tr-Upp}}=2^{-\frac{2B}{k_{\mathrm{q}}}}\left(\int_{\mathbb{Z}}\int_{\mathbb{Q}}\widetilde{I}_{\text{tr-upp}}\left(\mathbf{y};\mathbf{z}\right)p\big(\mathbf{y},\mathbf{z}\big)\lambda_{\text{tr}}\big(\mathbf{y}\big)^{-\frac{2}{k_{\mathrm{q}}}}\,d\mathbf{y}\,d\mathbf{z}\right). \tag{9}$$

Similarly, by substituting (6) and (8) into (3), the asymptotic distortion bounds of a constrained quantizer with transformed codebook can also be obtained.

Similar to the case of the conventional product transformed code [9], there exist trade-offs between the two sub-optimalities: point density loss and Voronoi shape loss. To be specific, it is always possible to find a transformation $\mathbf{F}(\cdot)$ such that the transformed point density $\lambda_{\text{tr}}(\mathbf{y})$ matches exactly to the optimal point density $\lambda^*(\mathbf{y})$. However, by doing so, the transformation may cause severe "oblongitis" of the Voronoi shape in some cases, which will lead to significant increment of the normalized inertial profile. Therefore, a compromised transformation that optimally trades off the two losses should be employed. This tradeoff is directly reflected in the distortion bound $\widetilde{D}_{\text{tr-Upp}}$ that both $\widetilde{I}_{\text{tr-upp}}(\mathbf{y}\,;\mathbf{z})$ and $\lambda_{\text{tr}}(\mathbf{y})$ in integration (9) depend on the transformation $\mathbf{F}(\cdot)$.

4. OPTIMAL MISO CSI QUANTIZER

By utilizing the high-rate distortion analysis provided in Section 2.1, this section provides a detailed investigation of the capacity loss of a finite-rate quantized MISO beamforming system over correlated fading channels.

4.1 System Model

We consider an MISO system with t transmit antennas, one single receive antenna, signaling through a frequency flat block fading channel. The channel impulse response $\mathbf{h}$ is assumed to be perfectly known at the receiver but partially available at the transmitter through CSI feedback. It is assumed that there exists a finite rate feedback link of B ($N = 2^B$) bits per channel update between the transmitter and receiver. To be specific, a codebook $\mathcal{C} = \{\widehat{\mathbf{v}}_1, \cdots, \widehat{\mathbf{v}}_N\}$, which is composed of unit norm transmit beamforming vectors, is assumed known to both the receiver and the transmitter. Based on the channel realization $\mathbf{h}$, the receiver selects the best code point $\widehat{\mathbf{v}}$ from the codebook and sends the corresponding index back to the transmitter. At the transmitter, vector $\widehat{\mathbf{v}}$ is employed as the transmit beamforming vector, and the system channel model can be represented as

$$y = \mathbf{h}^H \cdot (\widehat{\mathbf{v}} \cdot s) + n = \|\mathbf{h}\| \cdot \langle \mathbf{v}, \widehat{\mathbf{v}} \rangle \cdot s + n\,, \tag{10}$$

where y is the received signal (scalar), $n \sim \mathcal{N}_{\text{c}}(0,1)$ is the additive complex Gaussian noise with zero mean and unit variance, $\mathbf{h}^H \in \mathbb{C}^{1\times t}$ is the MISO channel response with distribution given by $\mathbf{h} \sim \mathcal{N}_{\text{c}}(\mathbf{0}, \mathbf{\Sigma}_{\text{h}})$, and vector $\mathbf{v}$ is the channel directional vector given by $\mathbf{v} = \mathbf{h}/\|\mathbf{h}\|$. The transmitted signal s is normalized to have a power constraint given by $E[s^2] = \rho$, with ρ representing the average signal to noise ratio at each receive antenna.

The performance of a finite-rate feedback MISO beamforming system can be characterized by the capacity loss C_{Loss}, which is the expectation of the instantaneous mutual information rate loss $C_{\text{L}}(\mathbf{h}, \widehat{\mathbf{v}})$ due to the finite rate quantization of the transmit beamforming vector. This performance metric was also used in [4] and is defined as

$$C_{\text{L}}(\mathbf{h}, \widehat{\mathbf{v}}) = -\log_2\left(1 - \frac{\rho \cdot \|\mathbf{h}\|^2}{1 + \rho \cdot \|\mathbf{h}\|^2} \cdot \left(1 - |\langle \mathbf{v}, \widehat{\mathbf{v}} \rangle|^2\right)\right), \tag{11}$$

4.2 MISO Systems with Optimal CSI Quantization

By employing the general framework described in Section 2.1, the finite-rate quantized MISO beamforming system can be formulated as a general fixed rate vector quantization problem. Specifically, the source variable to be quantized is the channel directional vector $\mathbf{v}$ of $k_{\text{q}} = 2t$ real dimensions, and the encoder side information is the channel power $\alpha = \|\mathbf{h}\|^2$. Moreover, under the norm and phase constraints, i.e. $\mathbf{v}$ is a unit norm vector and is invariant to arbitrary phase rotation $e^{j\theta}$, the actual free dimensions of vector $\mathbf{v}$ is reduced from k_{q} to $k'_{\text{q}} = 2t - 2$. The instantaneous capacity loss due to effects of finite-rate CSI quantization is taken to be the system distortion function $D_{\text{Q}}(\mathbf{v}, \widehat{\mathbf{v}}\,;\alpha)$, which is given by the following form according to the definition given by (11)

$$D_{\text{Q}}(\mathbf{v}, \widehat{\mathbf{v}}\,;\alpha) = -\log_2\left(1 - \frac{\rho\alpha}{1+\rho\alpha} \cdot \left(1 - |\langle \mathbf{v}, \widehat{\mathbf{v}} \rangle|^2\right)\right). \tag{12}$$

By utilizing the distortion analysis provided in [7], the normalized inertia profile of the MISO system is tightly lower bounded by

$$\widetilde{I}_{\text{c, opt}}(\mathbf{v};\alpha) = \frac{\rho\alpha}{\ln 2 \cdot (1+\rho\alpha)} \cdot \frac{(t-1)\cdot \gamma_t^{-1/(t-1)}}{t}\,, \tag{13}$$

where γ_t is a constant coefficient equal to $\gamma_t = \pi^{t-1}/(t-1)!$. The minimal distortion of the MISO system is hence achieved by using a codebook with an optimal point density given by

$$\lambda^*(\mathbf{v}) = \beta_1(\rho, t, \mathbf{\Sigma}_{\text{h}})^{-1} \cdot \left(\left(\mathbf{v}^H \mathbf{\Sigma}_{\text{h}}^{-1} \mathbf{v}\right)^{-(t+1)} \times {}_2F_0\left(t+1, 1; ; -\frac{\rho}{\mathbf{v}^H \mathbf{\Sigma}_{\text{h}}^{-1} \mathbf{v}}\right)\right)^{(t-1)/t}. \tag{14}$$

where ${}_2F_0$ is the generalized hypergeometric function, and β_1 is a normalization constant that only depends on the antenna size t, channel correlation matrix $\mathbf{\Sigma}_{\text{h}}$ and system SNR ρ. The average system distortion (or capacity loss) of the quantized MISO system is then tightly lower bounded by

$$\widetilde{D}_{\text{c-Low}}(\mathbf{\Sigma}_{\text{h}}) = \frac{\rho(t-1)\beta_1(\rho, t, \mathbf{\Sigma}_{\text{h}})^{t/(t-1)}}{\ln 2 \cdot |\mathbf{\Sigma}_{\text{h}}| \cdot \gamma_t^{t/(t-1)}} \cdot 2^{-B/(t-1)}. \tag{15}$$

5. MISO CSI QUANTIZER WITH TRANSFORMED CODEBOOK

In practically situations, it is impossible to design different codebooks optimized for every instantiation of the channel covariance matrix and use them adaptively. In these situations, it is convenient to use a channel quantizer whose codebook is generated from a fixed pre-designed codebook through a transformation parameterized by the channel covariance matrix.

5.1 Problem Setup

To be specific, suppose $\mathcal{C}_0$ is the optimal codebook designed for the i.i.d. MISO fading channels. When the elements of the fading channel response $\mathbf{h}$ are correlated, i.e. $\mathbf{h} \sim \mathcal{N}_{\text{c}}(\mathbf{\Sigma}_{\text{h}})$, it is evident that codebook $\mathcal{C}_0$ is no longer optimal. In order to compensate the mismatch between $\mathcal{C}_0$ and the current channel statistics, a transformed codebook $\mathcal{C}$ generated by a one-to-one mapping from codebook $\mathcal{C}_0$ given by equation (4) can be used. Optimization of the transformation $\mathbf{F}(\cdot)$ turns out to be difficult, and hence a simple sub-optimal transformation,

$$\mathbf{F}(\widehat{\mathbf{v}}) = \mathbf{G}\,\widehat{\mathbf{v}} / \|\mathbf{G}\,\widehat{\mathbf{v}}\|\,, \tag{16}$$

was proposed in [3] [6] where $\mathbf{G} \in \mathbb{C}^{t\times t}$ is a fixed matrix depends on the channel covariance matrix $\mathbf{\Sigma}_{\text{h}}$.

5.2 Capacity Loss Analysis of Transformed MISO Quantizers

First of all, according to the codebook transformation given by (16), the transformed point density function $\lambda_{\text{c-tr}}(\mathbf{v})$ can be obtained as the following form, from equation (6),

$$\lambda_{\text{c-tr}}(\mathbf{v}) = \gamma_t^{-1} \cdot |\mathbf{\Sigma}|^{-1} \cdot \left(\mathbf{v}^{\mathsf{H}} \mathbf{\Sigma}^{-1} \mathbf{v}\right)^{-t}, \quad \mathbf{\Sigma} = \mathbf{G} \cdot \mathbf{G}^{\mathsf{H}}. \tag{17}$$

which is equivalent to the PDF of a unit-norm complex vector $\mathbf{x}/\|\mathbf{x}\|$ with $\mathbf{x}$ having complex Gaussian distribution $\mathbf{x} \sim \mathcal{N}_{\text{c}}(\mathbf{0}, \mathbf{\Sigma})$. It is evident that the transformed point density given by (17) does not match to the optimal point density function $\lambda^*(\mathbf{v})$ given by (14) in the general case. However, for MISO systems with a large number of antennas and in high-SNR and low-SNR regimes,

$$\widetilde{D}^{\text{H-dim, L-SNR}}_{\text{c-Low}} = \widetilde{D}^{\text{H-dim, L-SNR}}_{\text{c-tr-Low}} \le \widetilde{D}^{\text{H-dim, L-SNR}}_{\text{c-tr}} \le \widetilde{D}^{\text{H-dim, L-SNR}}_{\text{c-tr-Upp}} \le c_1 \cdot \widetilde{D}^{\text{H-dim, L-SNR}}_{\text{c-Low}}, \quad c_1 = (t-1)\sum_{i=1}^{t} \frac{(\ln \lambda_{\text{h},i})/\lambda_{\text{h},i}}{\prod_{k\neq i}(1-\lambda_{\text{h},k}/\lambda_{\text{h},i})}, \tag{24}$$

$$\widetilde{D}^{\text{H-dim, H-SNR}}_{\text{c-Low}} = \widetilde{D}^{\text{H-dim, H-SNR}}_{\text{c-tr-Low}} \le \widetilde{D}^{\text{H-dim, H-SNR}}_{\text{c-tr}} \le \widetilde{D}^{\text{H-dim, H-SNR}}_{\text{c-tr-Upp}} \le c_2 \cdot \widetilde{D}^{\text{H-dim,H-SNR}}_{\text{c-Low}}, \quad c_2 = \left(\frac{\delta(t-2)}{\lambda_{\text{h},1}\cdot\lambda_{\text{h},2}} - (t-1)(t-2)\sum_{i=1}^{t}\frac{(\ln \lambda_{\text{h},i})/\lambda^2_{\text{h},i}}{\prod_{k\neq i}(1-\lambda_{\text{h},k}/\lambda_{\text{h},i})}\right)\Big/ c_1^{\frac{t}{t-1}}. \tag{25}$$

it can be shown that the optimal point density $\lambda^*(\mathbf{v})$ reduces to be the source distribution $p(\mathbf{v})$ of the directional vector $\mathbf{v}$, given by

$$\lim_{t\to\infty} \lambda^*(\mathbf{x}) = p_{\mathbf{v}}(\mathbf{x}) = \gamma_t^{-1}\cdot|\mathbf{\Sigma}_\text{h}|^{-1}\cdot\left(\mathbf{x}^\text{H}\mathbf{\Sigma}_\text{h}^{-1}\mathbf{x}\right)^{-t}. \tag{18}$$

In this case, by choosing matrix $\mathbf{G}$ as a product $\mathbf{G} = \mathbf{U}\mathbf{\Lambda}^{\frac{1}{2}}$ with matrices $\mathbf{U}$ and $\mathbf{\Lambda}$ form the eigen-value decomposition of the channel covariance matrix $\mathbf{\Sigma}_\text{h} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\text{H}$, one can generate a transformed codebook with optimal point density $\lambda_{\text{c-tr}}(\mathbf{v}) \approx \lambda^*(\mathbf{v})$. Hence, there is no distortion loss caused by the point density mismatch, although the system suffers from the oblongitis of the Voronoi shape (or the shape loss).

By substituting the transformation given by (16) into equation (8), the inertial profile of the transformed codebook can be upper and lower bounded by the following form

$$\widetilde{I}_{\text{c-tr-upp}}(\mathbf{v};\alpha) = \frac{\gamma_t^{-\frac{1}{t-1}}\rho\alpha\left(\mathbf{v}^\text{H}\mathbf{\Sigma}^{-1}\mathbf{v}\right)}{t\cdot\ln 2\cdot(1+\rho\alpha)}\cdot\mathbf{tr}\left(\left(I-\mathbf{v}\mathbf{v}^\text{H}\right)\cdot\mathbf{\Sigma}\right)$$
$$\ge \widetilde{I}_{\text{c-tr}}(\mathbf{v};\alpha) \ge \widetilde{I}_{\text{c-tr-low}}(\mathbf{v};\alpha) = \widetilde{I}_{\text{c-opt}}(\mathbf{v};\alpha). \tag{19}$$

where $\widetilde{I}_{\text{c-opt}}(\mathbf{v};\alpha)$ is the optimal inertia profile given by equation (13). It is evident from (19) that except unitary rotations of the i.i.d. codebook, any non-trivial transformation of the codebook will lead to mismatched Voronoi shape and hence causes inertial profile loss. Therefore, a codebook transformation that compromises both the point density loss and the inertial profile loss is favored. However, the optimization of the overall distortion w.r.t. matrix $\mathbf{G}$ is difficult and beyond the scope of this paper. As a special case, when the transformation is chosen to match the point density only, i.e., $\mathbf{G} = \mathbf{U}\mathbf{\Lambda}^{\frac{1}{2}}$, the average distortion bounds of a MISO system with transformed codebook can be obtained

$$\widetilde{D}_{\text{c-tr-Low}}(\mathbf{\Sigma}_\text{h}) = \frac{(t-1)\cdot|\mathbf{\Sigma}_\text{h}|^{\frac{1}{t-1}}\cdot\beta_2}{\ln 2\cdot t}\cdot 2^{-\frac{B}{t-1}}, \tag{20}$$

$$\widetilde{D}_{\text{c-tr-Upp}}(\mathbf{\Sigma}_\text{h}) = \frac{|\mathbf{\Sigma}_\text{h}|^{\frac{1}{t-1}}\cdot\beta_3}{\ln 2\cdot t}\cdot 2^{-\frac{B}{t-1}}. \tag{21}$$

where the coefficients β_2 and β_3 are given by

$$\beta_2 = E\left[\frac{\rho\cdot\left(\mathbf{h}^\text{H}\mathbf{\Sigma}_\text{h}^{-1}\mathbf{h}\right)^{\frac{t}{t-1}}}{(1+\rho\cdot\|\mathbf{h}\|^2)\cdot\|\mathbf{h}\|^{\frac{2}{t-1}}}\right] \tag{22}$$

$$\beta_3 = E\left[\frac{\rho\cdot\left(\mathbf{h}^\text{H}\mathbf{\Sigma}_\text{h}^{-1}\mathbf{h}\right)^{\frac{2t-1}{t-1}}\left(t\cdot\|\mathbf{h}\|^2 - \mathbf{h}^\text{H}\mathbf{\Sigma}_\text{h}\mathbf{h}\right)}{(1+\rho\cdot\|\mathbf{h}\|^2)\cdot\|\mathbf{h}\|^{\frac{4t-2}{t-1}}}\right]. \tag{23}$$

5.3 Comparison with Optimal CSI Quantizers

Interestingly, in high-SNR and low-SNR regimes with a large number transmit antennas t, the average system distortion of CSI quantizers with transformed codebook can be upper and lower bounded by some multiplicative factors of the optimal quantization distortion, which is represented in equations (24) and (25). Note that the constant coefficients c_1 and c_2 can be viewed as the upper bounds of the penalty paid for using a transformed codebook instead of optimal design. As verified by the numerical example shown in Section 6, c_1 and c_2 are slightly greater than 1 for most channels that are not "highly" correlated. This means that the intuitive choice of $\mathbf{F}$ given in [3] [6] is a fairly good solution especially for cases when the channel covariance matrix has a relative small condition number (for channels not so "correlated").

6. NUMERICAL AND SIMULATION RESULTS

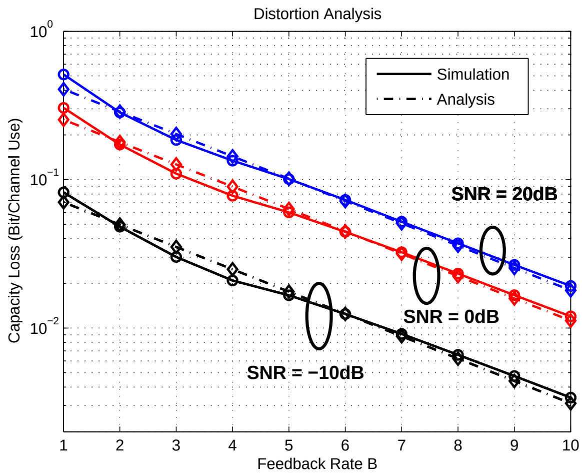

Figure 1: Capacity loss of a 3×1 MISO system with different CSI feedback rate $B = 1,2,\cdots,10$ bits per channel update

We plot in Fig. 1 the system capacity loss due to the finite-rate quantization of the CSI versus the feedback rate B for a 3×1 MISO system over correlated fading channels. The spatially correlated channel is simulated by the correlation model in [10]: A linear antenna array with antenna spacing of half wavelength, uniform angular-spread in $[-30^\circ, 30^\circ]$ and angle of arrival $\phi = 0^\circ$. Both the simulation results with optimum code book generated by the inner product criterion proposed in [4] and the analytical evaluation of distortion lower bound $\widetilde{D}_{\text{c-Low}}$ given by (15) are shown in the plot, demonstrating the accuracy of the proposed asymptotic distortion analysis provided in Section 4.

Fig. 2 demonstrates the system capacity loss by using transformed codebook versus feedback rate B for the same 3×1 MISO system over correlated fading channels with adjacent antenna spacing $D/\lambda = 0.5$ and different system SNRs at $\rho = -10$, and 20 dB, respectively. For comparison purpose, both the distortion lower bound $\widetilde{D}_{\text{c-tr-Low}}$ given by (20) and the distortion upper bound $\widetilde{D}_{\text{c-tr-Upp}}$ given by (21) as well as the system distortion by using the optimal codebooks are also included in the plot. It can be observed from Fig. 2 that the distortion lower bound $\widetilde{D}_{\text{c-tr-Low}}$ is tight and the performance of the CSI quantizer with transformed codebook is close to that of the optimal codebooks.

In order to see the effects of channel correlation on the system performance, we also plot in Fig. 3 the distortion ratio of correlated fading channels to i.i.d. fading channels (normalized capacity loss) of a 3×1 MISO system versus antenna spacing D/λ with both optimal and transformed codebooks, and with system SNR $\rho = -10$

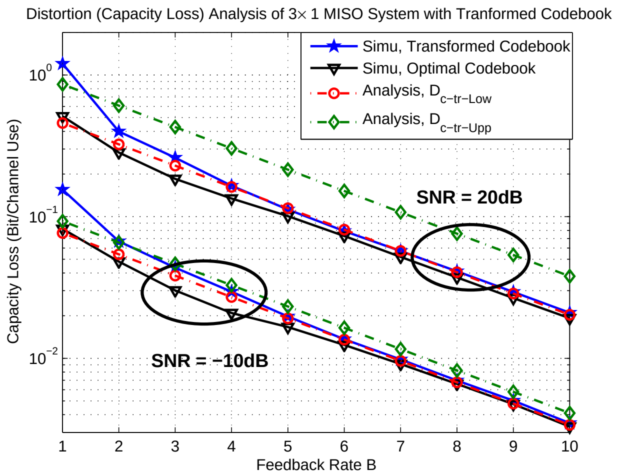

Figure 2: Capacity loss of a 3×1 correlated MISO system using CSI quantizers with transformed codebook.

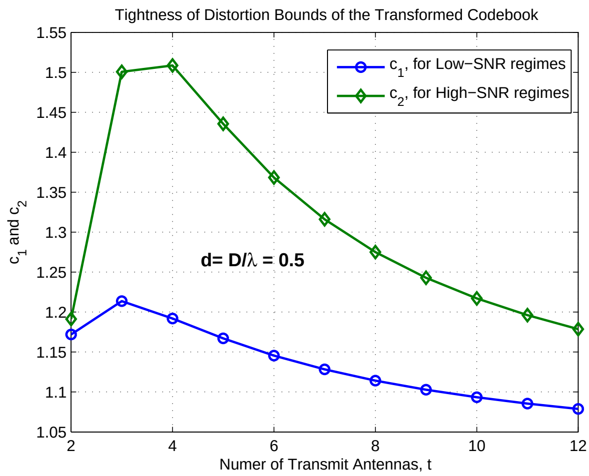

Figure 4: Demonstration of the tightness of the distortion bounds.

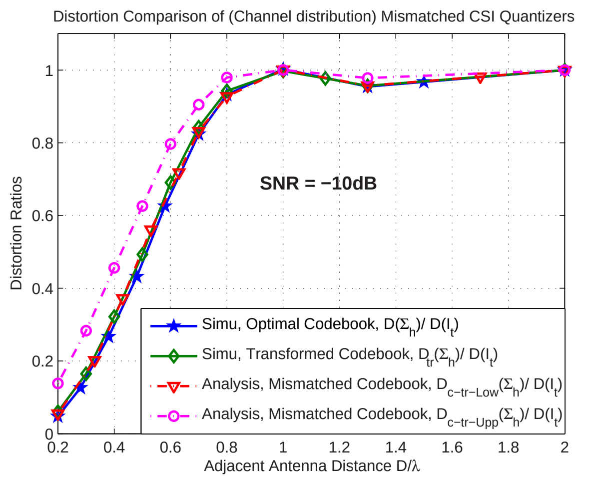

Figure 3: Comparison of normalized capacity loss of a 3×1 MISO system with optimal and transformed codebooks.

dB and quantization rate $B = 10$ bits. For comparison purpose, the ratio of the distortion bounds, i.e. $\widetilde{D}_{\text{c-tr-Low}}(\mathbf{\Sigma}_{\text{h}})/\widetilde{D}_{\text{c-tr-Low}}(I_t)$ and $\widetilde{D}_{\text{c-tr-Upp}}(\mathbf{\Sigma}_{\text{h}})/\widetilde{D}_{\text{c-tr-Upp}}(I_t)$, are also included in the plot. It can be observed from Fig. 3 that the analytical bounds agree well with the obtained simulation results. In order to see the tightness of the distortion bounds $\widetilde{D}_{\text{c-tr-Upp}}$ and $\widetilde{D}_{\text{c-tr-Low}}$ in high-SNR and low-SNR regimes, Fig. 4 plots the constant coefficient c_1 and c_2 versus the number of transmit antennas t for correlated MISO channels with adjacent antenna spacing $D/\lambda = 0.5$. It can be observed from the plot that the performance degradation caused by the transformed codebook is less than 10% in low-SNR regimes and 22% in high-SNR regimes for MISO systems with more than 10 transmit antennas.

7. CONCLUSION

We first investigated in this paper a general vector quantizer with transformed codebook. Bounds on the average distortion of this class of quantizers were provided to characterize the effects of sub-optimality introduced by the transformed codebook on system performance. As an application of the proposed general framework, we provided in this paper a capacity analysis of the feedback-based MISO systems over correlated fading channels using channel quantizers with both optimal and transformed codebooks. To be specific, upper and lower bounds on the channel capacity loss of MISO systems with transformed codebooks were provided and compared to that of the optimal quantizers. Numerical and simulation results were presented and further confirmed the tightness of the theoretical distortion bounds.

REFERENCES

[1] K. K. Mukkavilli, A. Sabharwal, E.Erkip, and B.Aazhang, “On beamforming with finite rate feedback in multiple-antenna systems,” *IEEE Trans. on Information Theory*, vol. 49, no. 10, pp. 2562–2579, Oct. 2003.

[2] D. J. Love, R. W. Heath, Jr., and T. Strohmer, “Grassmannian beamforming for multiple-input multiple-output wireless systems,” *IEEE Trans. on Information Theory*, vol. 49, pp. 2735–2747, Oct. 2003.

[3] P. Xia and G. B. Giannakis, “Design and analysis of transmit-beamforming based on limited-rate feedback,” *IEEE Trans. on Signal Processing*, 2006 (to appear).

[4] J. Roh, *Multiple-Antenna Communication with Finite Rate Feedback*, Ph.D. thesis, Univ. of California, San Diego, 2005.

[5] A. Narula, M. J. Lopez, M. D. Trott, and G. W. Wornell, “Efficient use of side information in multiple-antenna data transmission over fading channels,” *IEEE Journal on Selected Areas in Communications*, vol. 16, pp. 1423–1436, Oct. 1998.

[6] D. J. Love and R. W. Heath Jr., “Grassmannian beamforming on correlated MIMO channels,” in *IEEE Globecom 2004*, Dallas, TX, Dec. 2004, vol. 1, pp. 106–110.

[7] J. Zheng, Ethan Duni, and B. D. Rao, “Analysis of multiple antenna systems with finite-rate feedback using high resolution quantization theory,” in *accepted by Proc. IEEE Data Compression Conference*, Snowbird, UT, Mar. 2006.

[8] W. R. Bennett, “Spectra of quantized signals,” *Bell System Technical Journal*, vol. 27, pp. 446–472, July 1948.

[9] S. Na and D. Neuhoff, “Bennett’s integral for vector quantizers,” *IEEE Trans. on Information Theory*, vol. 41, no. 4, pp. 886–900, 1995.

[10] J. Salz and J. H. Winters, “Effect of fading correlation on adaptive arrays in digital mobile radio,” *IEEE Trans. on Vehicular Technology*, vol. 43, pp. 1049–1057, Nov. 1994.