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A NONLINEAR CHANNEL EQUALIZATION USING AN ALGEBRAIC APPROACH AND THE AFFINE PROJECTION ALGORITHM

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ABSTRACT

In this paper, the problem of nonlinear equalization is adressed. We use an algebraic approach which allows us to define the existence conditions of a left inverse, for a nonlinear system and therefore the equalization conditions. These existence conditions need the computation of the rank of some jacobian matrices. This approach is applied to a Volterra filter, which represents a nonlinear system. We will show also that these equalizability conditions depend to the coefficients of the nonlinear system and input values therefore we can verify for any channel with assumed known coefficients if this system is ideally equalizable or not. The appropriate algorithm that we have used to test the performance of the equalization is the APA (Affine Projection Algorithm). The choice of APA is justified by the use of a colored excitation signal as an input signal due to the nonlinear channel characteristics.

1. INTRODUCTION

Nonlinear Equalization techniques are becoming increasingly important to improve the performance of telecommunication channels. In fact, many real world communication systems, such a satellite telecommunication channel, high density magnetic, etc, uses a nonlinear devices that generate a nonlinear InterSymbol Interference (ISI). To resolve efficiently this problem we need to verify the existence of a perfect equalizer for those systems.

A few approaches have been proposed in that sense, we can cite the method given in [1] where under certain conditions a linear Finite Impulse Response (FIR) filters can perfectly equalize nonlinear SIMO channels. In fact [1] assume that the so-called channel matrix constructed from the channel coefficients has full column rank. In that case linear FIR equalizer always exists.

Other papers which treat the problem of existence of equalizer for a nonlinear systems are ([2],[3],[4]). The authors present expressions for the exact inverse and the *pth* order inverse for a specific nonlinear model. The *pth* order inverse is used due to that not all nonlinear systems possess an inverse and many nonlinear systems admit an inverse only for a certain subset of input signals.

The main contribution of this paper, is to present a relaxed method and conditions based on the system theory ([5],[6],[7])where using some algebraic tools we will justify the existence of a left inverse for a nonlinear system which coincides with the ideal equalizer. Using these conditions we give an explicit expressions for the subset of input signals for which the nonlinear channel is perfectly equalizable.

Due to the nonlinear channel characteristics i.e the use of colored input, we use the Affine Projection Algorithm (APA). Because it employs several input vectors, then the APA provides faster convergence than the NLMS and the LMS especially when the reference input is highly correlated. The AP Algorithm will be applied to a Volterra equalizer using the Mean Square Error (MSE) criterion versus the number of iterations.

The outline of this paper is the following. In section 2 we present a mathematical background. In section 3 we present the algebraically equalizability conditions that allow us to test the existence of equalizer for a nonlinear system. This approach will be applied on a nonlinear channel given by [1] as we will show in section 4. In section 5 we present the APA that will drive the Volterra equalizer. And finally, we conclude with some simulation results and conclusion.

2. MATHEMATICAL BACKGROUND

To present the algebraic approach, we need to introduce some mathematical tools where for more details you can see([6], [8], [9],[10]).

- Difference fields:
  Let z be the unit delay operator, acting on discrete time signals as:

$$\begin{aligned} z(x(n)) &= x(n-1) \\ z^k(x(n)) &= x(n-k) \\ z(x(n)+y(n)) &= z(x(n))+z(y(n)) \\ z(x(n)y(n)) &= z(x(n))z(y(n)) \end{aligned}$$

  A *constant* is an element c such that $zc = c$.

  Definition 1 *A* `difference field` $\mathscr{K}$ *is a commutative field equipped with the delay operator z.*

- Left module:
  Let $\mathscr{K}$ be a given ground field. We denote by $\mathscr{K}[z]$, where z is the delay operator, the ring of polynomials with coefficients in $\mathscr{K}$ of the form

$$P(z) = \sum_{k=0}^{r} a_k z^k,$$

  $\mathscr{K}[z]$ is in general non-commutative (it is commutative if and only if $\mathscr{K}$ is a field of constants).
  Let M be a left [1] $\mathscr{K}[z]$-module. A finitely generated left $\mathscr{K}[z]$-module spanned by $w(n) = (w_1(n), ..., w_s(n))$ is denoted by $M = [w(n)]$.

[1]because multiplication by scalars (the elements of $\mathscr{K}[z]$) applies on the left

3. ALGEBRAIC EQUALIZABILITY

In this section, we present the algebraic approach that will define the existence conditions for an algebraic equalizer concerning the nonlinear channel $\mathscr{S}$ with a transmitted signal $\mathbf{u}(n) = (u_1(n), \dots, u_m(n))$ and the received signal $\mathbf{y}(n) = (y_1(n), \dots, y_p(n))$. In fact, finding an algebraic equalizer means finding a left inverse for the above system. For that we need to give some results presented in [11].

3.1 Nonlinear left invertibility

This section briefly outlines some notations and results from nonlinear system theory, introduced by Fliess (see [6], [7]). The part of mathematics underlying these results is the theory of `Kähler differentials` (see , [12], [13]). `Kähler differentials` can be seen as the algebraic version of the usual infinitesimal differential calculus. We attach to the difference field $\mathscr{K}(\mathbf{y}(n))$, the left $\mathscr{K}(\mathbf{y}(n))[z]$-module $[d\mathbf{y}(n)]$ spanned by the so-called `Kähler differentials` $dx(n)$, for $x(n) \in \mathscr{K}(\mathbf{y}(n))$. The mapping

$$d : \mathscr{K}(\mathbf{y}(n)) \longrightarrow [d\mathbf{y}(n)]$$

satisfies the following rules

$$z(d\zeta(n)) = d(z\zeta(n)) \quad \forall\, \zeta(n) \in \mathscr{K}(\mathbf{y}(n)) \tag{1}$$
$$d(\alpha(n)\beta(n)) = d(\alpha(n))\beta(n) + \alpha(n)d(\beta(n)) \tag{2}$$
$$d(c) = 0 \quad \forall\, c \in \mathscr{K} \tag{3}$$

In this framework, the rank of a nonlinear system admits a clear-cut definition given by Fliess [6, 7]

Definition 2 *The rank of the input-output system $\mathscr{S}$ with input $\boldsymbol{u}(n)$ and output $\boldsymbol{y}(n)$, denoted as $\mathrm{rk}\{\mathscr{S}\}$, is defined as*

$$\mathrm{rk}\{\mathscr{S}\} \stackrel{\triangle}{=} \mathrm{rk}[d\boldsymbol{y}(n)]. \tag{4}$$

This rank satisfies the following properties:

- $\mathrm{rk}\{\mathscr{S}\} \leq inf(m, p)$
- $\mathrm{rk}\{\mathscr{S}\}$ extends to nonlinear system the usual transfer matrix rank of linear time-invariant system.

And we have the following definition:

Definition 3 *The system $\mathscr{S}$ with the input $\boldsymbol{u}(n)$ and output $\boldsymbol{y}(n)$ is* left invertible *if and only if*

$$\mathrm{rk}\{\mathscr{S}\} = m \tag{5}$$

The left invertibility means that the input variables may be recovered from the output variables by a finite set of difference equations.

3.2 Conditions of Algebraic Equalization

Using the left invertibility condition of the nonlinear system, the rank of $\mathscr{S}$ must be determined. The computation of this rank need the use of notion of filtration.

Definition 4 *A filtration of a system $\mathscr{S}$ with input $\boldsymbol{u}(n)$ and output $\boldsymbol{y}(n)$ is an ascending chain of subspaces:*

$$\mathscr{H}_r = \mathrm{span}_{\mathscr{K}(\boldsymbol{y}(n))}\{d\boldsymbol{y}(n), \dots, d\boldsymbol{y}(n-r)\}$$

Using the above definition, we give the following propositions:

Proposition 1 *:*

- $\dim \mathscr{H}_r = \rho r + \beta$
- $\dim \mathscr{H}_{r+1} - \dim \mathscr{H}_r = \rho = \mathrm{rk}\{\mathscr{S}\}$

for r large enough.

Hence

Proposition 2 *The nonlinear system $\mathscr{S}$ with m-input $\boldsymbol{u}(n)$ and p-output $\boldsymbol{y}(n)$ is algebraically equalizable if and only if,*

$$\rho = \mathrm{rk}\{\mathscr{S}\} = m \tag{6}$$

Matrix formulation: We can also give an equalization test for a nonlinear channel h using a matrix formulation as shown:

Let $\mathbf{y}(n)$be the output of a nonlinear channel $h(\cdot)$ represented by a Volterra model, with input $\mathbf{u}(n)$ as given by the following expression:

$$\mathbf{y}(n) = h(\mathbf{u}(n),\ \mathbf{u}(n-1), \dots, \mathbf{u}(n-N)) \tag{7}$$

Then, the Kähler differential of $\mathbf{y}(n)$ is given by:

$$d\mathbf{y}(n) = \sum_{j=0}^{N} \frac{\partial h}{\partial \mathbf{u}(n-j)} d\mathbf{u}(n-j), \tag{8}$$

so that we have

$$\begin{pmatrix} d\mathbf{y}(n) \\ d\mathbf{y}(n-1) \\ \vdots \\ d\mathbf{y}(n-r) \end{pmatrix} = J_r \begin{pmatrix} d\mathbf{u}(n) \\ d\mathbf{u}(n-1) \\ \vdots \\ d\mathbf{u}(n-N-r) \end{pmatrix} \tag{9}$$

where J_r denotes the Jacobian matrix of $\{\mathbf{y}(n),\ \mathbf{y}(n-1), \dots, \mathbf{y}(n-r)\}$ with respect to $\{\mathbf{u}(n),\ \mathbf{u}(n-1), \dots, \mathbf{u}(n-N-r)\}$. A matrix formulation of the above proposition then reads as:

Proposition 3 *[11][5] The nonlinear channel $h(\cdot)$ is algebraically equalizable if and only if, for $r \geq N$*

- $\mathrm{rk}J_r = mr + \beta$
- $\mathrm{rk}J_{r+1} - \mathrm{rk}J_r = m$

4. APPLICATION EXAMPLE

In this example we choose $\mathscr{K} = \mathbb{R}$. We consider $\mathscr{S}$ the nonlinear channel reported in [1] in example 2 whose input-output expression is given by:

$$\begin{aligned} y(n) \quad &= \quad \sum_{l=0}^{1} h_1(l)u(n-l) + \sum_{l=0}^{1} h_2(l)u(n-l)^2 \\ &+ \quad h_3(0)u(n-1)u(n) + v(n) \end{aligned} \tag{10}$$

where the input $u(n)$ is once chosen as a two-level PAM data ($u(n) = 0, 1$), once as a four-level PAM data ($u(n) = \pm 3, \pm 1$), and $v(n)$ is an additive white gaussian noise. Then, using Kähler differentials we have

$$dy(n) \quad = \alpha_n du(n) + \beta_n du(n-1) \tag{11}$$

where

$$\alpha_n = h_1(0) + 2h_2(0)u(n) + h_3(0)u(n-1) \quad (12)$$
$$\beta_n = h_1(1) + 2h_2(1)u(n-1) + h_3(0)u(n) \quad (13)$$

Note that the system (1) is linear time-varying, and its coefficients belong to the difference field $\mathscr{K}(y(n))$ since they depend on the delayed versions of the input, as for the output $y(n)$.

Recalling the filtration ($\mathscr{H}_r$) associated to the above time-varying system, we have for $r = 0$,

$$\mathscr{H}_0 = \mathrm{span}_{\mathscr{K}(y(n))}\{dy(n)\} \quad (14)$$

Now

$$\dim \mathscr{H}_0 = \mathrm{rk}\,[\alpha_n \quad \beta_n] = 1$$

so long as α_n or $\beta_n \neq 0$ for all n.
Next, for $r = 1$, we have

$$\mathscr{H}_1 = \mathrm{span}_{\mathscr{K}(y(n))}\{dy(n), dy(n-1)\} \quad (15)$$

and $\dim \mathscr{H}_1$ is equal to the rank of the Sylvester matrix

$$\begin{bmatrix} \alpha_n & \beta_n & 0 \\ 0 & \alpha_{n-1} & \beta_{n-1} \end{bmatrix}$$

When $\alpha_n\alpha_{n-1} \neq 0$ or $\beta_n\beta_{n-1} \neq 0$, this rank is equal to 2. We may check that if $\alpha_n\alpha_{n-1} \neq 0$ or $\beta_n\beta_{n-1} \neq 0$ for all $n \in \mathbb{N}$, then

$$\dim \mathscr{H}_r = r + 1, \quad \forall\, r$$

and therefore, $\dim \mathscr{H}_{r+1} - \dim \mathscr{H}_r = 1$ = number of inputs. But if there exists an instant n_0 for which $\alpha_{n_0} = 0$ and $\beta_{n_0} = 0$ i.e

$$\alpha_{n_0} = h_1(0) + 2h_2(0)u(n_0) + h_3(0)u(n_0-1) = 0 \quad (16)$$
$$\beta_{n_0} = h_1(1) + 2h_2(1)u(n_0-1) + h_3(0)u(n_0) = 0 \quad (17)$$

then finding the perfect equalizer which correspond to the left inverse of the nonlinear channel isn't possible. But we may find an estimate of the ideal equalizer using an appropriate algorithm.
Using the values of channel coefficients given in [1] are as follow:

$$[h_1(0), h_1(1), h_2(0), h_2(1), h_3(0)]' = [1, -2.5, 0.01, 0.2, 0.007]'$$

we have verified that for all n and for any input value of $u(n)$ belonging to set {0,1} and the set {-3,-1,1,3}, the conditions of α_n or $\beta_n \neq 0$ and also $\alpha_n\alpha_{n-1} \neq 0$ or $\beta_n\beta_{n-1} \neq 0$ are always verified. This implies that the channel is left invertible and therefore we have an ideal equalization for this channel. Using these equalizability conditions we give a clear definition for the subset of input signal for which we can have a best equalization performances. In fact, the expressions of α_n or $\beta_n \neq 0$ and $\alpha_n\alpha_{n-1} \neq 0$ or $\beta_n\beta_{n-1} \neq 0$ define this subset.
But as shown in [14], we may have a certain input subset for which the nonlinear channel isn't algebraically equalizable.

5. AFFINE PROJECTION ALGORITHM

In this section we present the Affine Projection Algorithm (APA) used to drive a Volterra equalizer for the nonlinear channel represented also by a Volterra model. The APA of order P, in a relaxed and regularized form, is defined as follows:

$$\mathbf{e_n} = \mathbf{u_n} - \mathbf{Y_n}\mathbf{w_n} \quad (18)$$
$$\mathbf{C_n} = [\mathbf{Y'_n}\mathbf{Y_n} + \delta\mathbf{I}] \quad (19)$$
$$\mathbf{w_{n+1}} = \mathbf{w_n} + \mu\mathbf{Y_n}\mathbf{C_n^{-1}}\mathbf{e_n} \quad (20)$$

The excitation signal matrix for the equalizer is,$\mathbf{Y_n}$, and has the structure,

$$\mathbf{Y_n} = [\mathbf{y_n}, \mathbf{y_{n\text{-}1}}, \cdots, \mathbf{y_{n\text{-}P\text{+}1}}] \quad (21)$$

where the $\mathbf{y_n} = [y(n), y(n-1), \ldots, y(n-N_1+1),$ $y(n)y(n), y(n)y(n-1), \ldots, y(n-N_2+1)y(n-N_2+1)]^T$, N_1 and N_2 indicate respectively the linear memory order and the nonlinear memory order of the Volterra equalizer. Also,

$$\mathbf{w_n} = [w_1(1)w_1(2)\ldots w_1(N_1)w_2(1,1)w_2(1,2)\ldots w_2(N_2,N_2)] \quad (22)$$

$$\mathbf{e_n} = [e_n, e_{n-1}, ..., e_{n-P+1}]' \quad (23)$$

The scalar δ is a regularization parameter used to cope with the ill-conditioning in matrix inversion and μ is a step size parameter.

6. SIMULATION RESULTS

In this section, we consider the nonlinear real channel reported in [1] and given by the equation (10) where SNR = 40 dB. Equation (10) suggests the form of the nonlinear Volterra equalizer with input $y(n)$. The output of the equalizer, denoted by $\hat{u}(n)$, consists of a linear combination of all linear terms and all possible combinations of nonlinear terms of $y(n)$. In fact, we have considered a 4-tap 2nd order Volterra equalizer given as follow:

$$\hat{u}(n) = \sum_{l=0}^{3} w(l)y(n-l) + \sum_{l=0}^{3}\sum_{k=l}^{3} w(l,k)y(n-l)y(n-k) \quad (24)$$

As mentioned previously, we will use the Affine Projection Algorithm for adaption the equalizer. This due to the robustness of such algorithm towards the correlated input. We will use the NLMS algorithm (which represent the Affine Projection Algorithm of order 1) and Affine Projection algorithm of order 2 and 3. The step size used to control the convergence speed is equal to 0.1.
Also, as depicted previously, the proposed channel is always algebraically equalizable when using a 2-PAM (i.i.d) data $(u(n) = 0, 1)$ or a 4-PAM (i.i.d) data $(u(n) = \pm 1, \pm 3)$. In fact, when we use at first a 2-PAM (i.i.d) data $(u(n) = 0, 1)$ as input, we can see in Fig.3 that we have a good performance of the equalizer in terms of the Mean Square Error (MSE) criterion obtained over 100 independent trials and also we can increase the convergence speed of the algorithm when we increase the order of the Affine Projection Algorithm. A typical eye diagram of the channel's output is plotted in Fig.1 with its equalized version in Fig.2.

Also, for the 4-PAM (i.i.d) data $(u(n) = \pm 1, \pm 3)$ as input (i.e we have increased the order of constellation), we can see in the Fig.5 that the conditions of algebraic equalizability stay verified. The Fig.6 depicted the MSE performance for a related data.

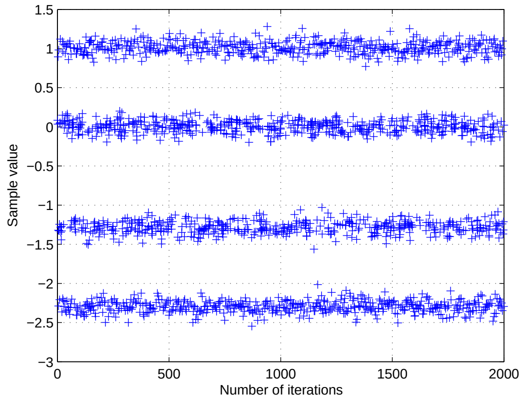

Figure 1: Eye-patterns before equalization

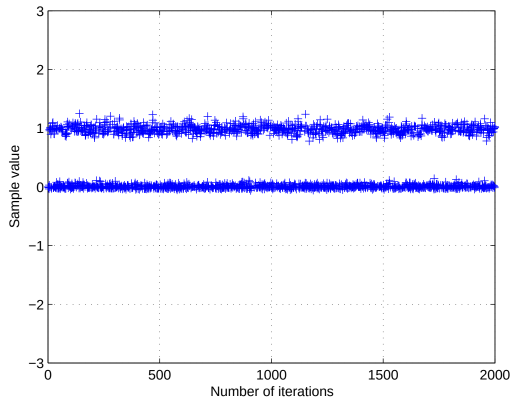

Figure 2: Eye-patterns after equalization

7. CONCLUSIONS

In this paper we have presented an algebraic method that allow us to compute the rank of a nonlinear channel and gives a relaxed conditions using this rank, in order to justify the existence of a nonlinear equalizer. Due to the nonlinear characteristics of the transmission channel in terms of correlated input, we have applied the Affine Projection Algorithm (APA)as an adaptive algorithm, which will drive the equalizer coefficients because such algorithm was robust towards the colored excitation. The simulation results show the coherence between the algebraic equalizability conditions and the performance of the adaptive equalizer in terms of MSE and equalizer output.

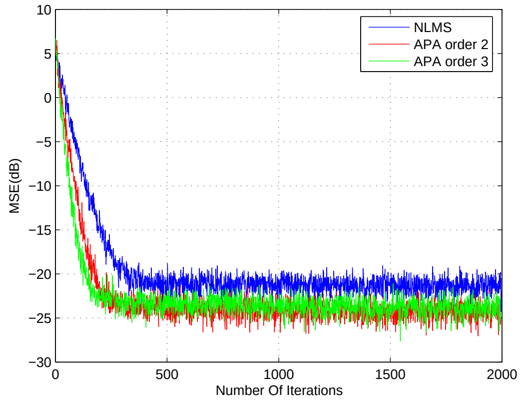

Figure 3: MSE Curves

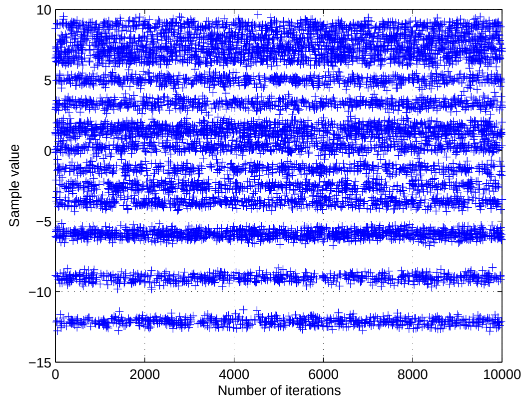

Figure 4: Eye-patterns before equalization

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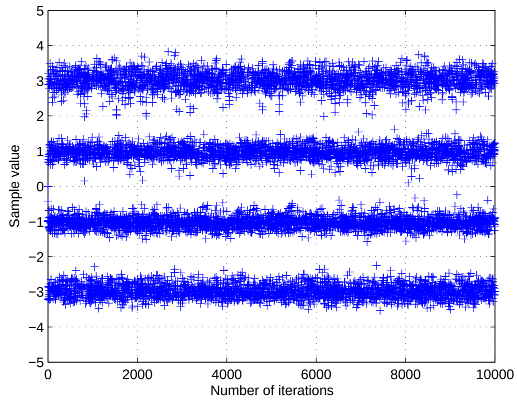

Figure 5: Eye-patterns after equalization

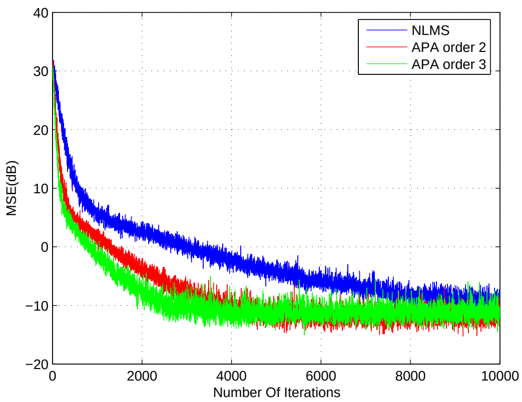

Figure 6: MSE Curves

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