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REPRESENTING LAPLACIAN PYRAMIDS WITH VARYING AMOUNT OF REDUNDANCY

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ABSTRACT

The Laplacian pyramid (LP) is a useful tool for obtaining spatially scalable representations of visual signals such as image and video. However, the LP is overcomplete or redundant and has lower compression efficiency compared to critical representations such as wavelets and subband coding. In this paper, we propose to improve the rate-distortion (R-D) performance of the LP by varying its redundancy through decimation of the detail signals. We present two reconstruction algorithms based on the frame theory and the coding theory, and then show them to be equivalent. Simulation results with various standard test images suggest that, using suitable quantization parameters, it is possible to have better R-D performance over the usual or the dual frame based reconstruction.

1. INTRODUCTION

The Laplacian pyramid (LP) [1] is a useful tool for obtaining multiresolution representations of visual signals such as image and video. In the context of present day multimedia communications over heterogeneous media with varying capacities, its impact can be hardly underestimated. The on-going scalable video coding standard (SVC), for instance, incorporates the LP structure as a principal component for achieving the spatial scalability in the form of standard definition (SD), CIF, and QCIF resolutions [2]. In addition to the multiscale representation, the LP also provides a means to compactly represent visual signals. In comparison to other compact representation techniques such as wavelets and subband coding, the LP has the advantage that it provides greater freedom in designing the decimation and interpolation filters.

An LP achieves the multiscale representation of a signal as a coarse signal at lower resolution together with several detail signals at successive higher resolutions. Since the number of coefficients of the LP is larger than the number of samples of the original signal, an LP representation is overcomplete or redundant. Therefore it can be studied using the frame theory, which provides a mathematical framework for overcomplete systems. In [3], Do and Vetterli consider the LP as a frame expansion and propose a dual frame based structure for the reconstruction. Given an LP representation, the usual reconstruction procedure is to iteratively interpolate the coarse signal and to add the detail signals successively up to the desired resolution. In [3], Do and Vetterli show that the dual frame based reconstruction has lesser error than the usual reconstruction method when the LP coefficients are corrupted with noise. Since the proposed structure in [3] requires biorthogonal filters, the authors in [4] modify the LP by including an update step so that the reconstruction structure is valid for any pair of decimation and interpolation filters.

In the context of scalable compression, LP is a natural choice for obtaining lower resolution signals from a higher resolution signal. However, the redundancy of the LP is still an undesirable feature from the compression point of view. In [3], Do and Vetterli improve the rate-distortion (R-D) performance by utilizing the dual frame based reconstruction structure, but the original LP, and consequently its redundancy, remain untouched. The lifted pyramid proposed in [4] modifies the coarse signal (which is undesirable in the context of scalable video compression), but the new pyramid is

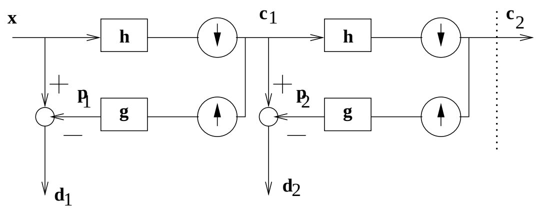

Figure 1: Laplacian pyramid decomposition.

still overcomplete. In this paper, we propose to improve the R-D performance by varying the redundancy of the LP through downsampling of the detail signals. A closer look at LP reveals that it is the detail signals which contain the redundancy of the LP. Therefore the redundancy of the LP can be varied by varying the decimation factor of the detail signals. We first show that, in the absence of noise, perfect reconstruction is possible even if the LP is modified with downsampling of detail signals up to the critical representation. We present two reconstruction algorithms, one based on the frame theory and the other based on the coding theory, and then show them to be equivalent. In the presence of quantization noise, the reconstruction algorithms provide the best estimates in the sense of minimum mean square error (MMSE).

2. LAPLACIAN PYRAMID

The LP structure proposed by Burt and Adelson [1] is shown in Fig. 1. For convenience of notation, we will consider here only 1-D signals; the extension to the 2-D case is straightforward. The input signal $\mathbf{x}$ is first lowpass filtered using the decimation filter $\mathbf{h}$ and then downsampled producing the coarse signal $\mathbf{c}_1$. This coarse signal is upsampled and then filtered using the interpolating filter $\mathbf{g}$ producing the prediction signal $\mathbf{p}_1$. The prediction error $\mathbf{d}_1$ is the first level of detail signal. The process is repeated on the coarse signal $\mathbf{c}_1$ until the final resolution is reached. Note that the subscript in Fig. 1 denotes the index of the pyramid level. Here we have used vector notations in order to facilitate matrix operations. By convention 1-D signals are assumed to be column vectors.

For convenience of explanation, let us consider an LP with only one level of decomposition. Considering an input signal of N samples, the coarse and the detail signals can be derived as

$$\mathbf{c} = H\mathbf{x} \quad \text{and} \quad \mathbf{d} = \mathbf{x} - G\mathbf{c} = (I_N - GH)\mathbf{x}, \tag{1}$$

where I_N denotes the identity matrix of order N, and H and G denote the decimation and the interpolation matrices which have the following structures:

$$H = \begin{bmatrix} \ddots & & & & & & \\ \dots & h(2) & h(1) & h(0) & \dots\dots\dots & & \\ \dots\dots\dots & & & h(2) & h(1) & h(0) & \dots \\ & & & & & & \ddots \end{bmatrix}, \tag{2}$$

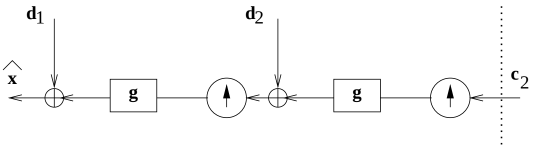

Figure 2: Standard reconstruction structure for LP.

and

$$G = \begin{bmatrix} \ddots & & & & & & \\ \cdots & g(0) & g(1) & g(2) & \cdots\cdots\cdots & & \\ \cdots\cdots\cdots & & & g(0) & g(1) & g(2) & \cdots \\ & & & & & & \ddots \end{bmatrix}^t. \tag{3}$$

Here the superscript t denotes the matrix transpose operation. If the coarse signal has resolution K, then the matrices H and G are of dimension $K \times N$ and $N \times K$ respectively.

Given an LP representation, the standard reconstruction method builds the original signal simply by iteratively interpolating the coarse signal and adding the detail signals at each level up to the final resolution. The standard reconstruction method is shown in Fig. 2. Considering an LP with only one level of decomposition, we can reconstruct the original signal as

$$\hat{\mathbf{x}}_s = G\mathbf{c} + \mathbf{d}. \tag{4}$$

3. LAPLACIAN PYRAMID AS A FRAME EXPANSION

Consider the N-dimensional Euclidean complex space, i.e., $\mathbb{C}^N$. A set of N-dimensional vectors $\Phi_F \equiv \{\varphi_k\}_{k=1}^M$, $M \geq N$, is called a frame if there exist $B_1 > 0$ and $B_2 < \infty$ such that

$$B_1\|\mathbf{z}\|^2 \leq \sum_{k=1}^{M} |\langle \mathbf{z}, \varphi_k \rangle|^2 \leq B_2\|\mathbf{z}\|^2, \quad \text{for all } \mathbf{z} \in \mathbb{C}^N, \tag{5}$$

where $\langle \mathbf{z}, \varphi_k \rangle$ denotes the inner product of $\mathbf{z}$ and φ_k, and $\|\mathbf{z}\|$ denotes the Euclidean norm of $\mathbf{z}$. B_1 and B_2 are called the frame bounds. The inner product $\langle \mathbf{z}, \varphi_k \rangle$ gives the kth frame expansion coefficient of $\mathbf{z}$. Any finite set of vectors that spans $\mathbb{C}^N$ is a frame. A *subframe* is defined as a subset of a frame which is itself a frame, that is, the subset of frame vectors spans $\mathbb{C}^N$. A frame is called *tight* if its bounds are equal, i.e., $B_1 = B_2$.

The frame Φ_F is associated with a frame operator F which is defined as follows:

$$(F\mathbf{z})_k \equiv \langle \mathbf{z}, \varphi_k \rangle, \quad \text{for } k = 1, 2, \ldots, M. \tag{6}$$

Therefore the frame expansion coefficients of $\mathbf{z}$ are given by $F\mathbf{z}$. Φ_F is tight if and only if $F^h F = BI_N$, where F^h denotes the conjugate transpose of F, and $B = B_1 = B_2$. This implies that the columns of F are orthogonal.

Associated with the frame Φ_F, there exists a dual frame whose frame operator is given as $\tilde{F} = F(F^h F)^{-1}$. Given the frame expansion coefficients of any vector $\mathbf{z}$, the vector can be reconstructed using the dual frame operator as $\mathbf{z} = \tilde{F}^h(F\mathbf{z})$. The conjugate transpose of the dual frame operator is the pseudo-inverse of the frame operator, and it minimizes the reconstruction error when the expansion coefficients are quantized [5]. For an in-depth treatment of the frame theory, the reader is referred to [6].

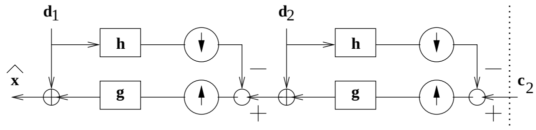

Figure 3: Dual frame based reconstruction structure of Do and Vetterli [3].

Considering now the LP, the coarse and the detail signals in Eqn. 1 can be jointly expressed as

$$\begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} H \\ I_N - GH \end{bmatrix} \mathbf{x} \equiv F\mathbf{x}, \tag{7}$$

where F denotes the matrix on the right hand side. Since the information in $\mathbf{x}$ is preserved in the coarse and the detail signals, the rank of F is N. The rows of F constitute a frame and F can be called the frame operator associated with the LP.

The usual reconstruction shown in Eqn. 4 can be equivalently expressed using the reconstruction operator $[G \;\; I_N]$ as

$$\hat{\mathbf{x}}_s = [G \;\; I_N] \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix}. \tag{8}$$

It is trivial to prove that $[G \, I_N]F = I_N$. However, this reconstruction operator is not equal to the dual frame operator of F, which is $(F^t F)^{-1} F^t$. The reconstruction proposed by Do and Vetterli [3] aims to reconstruct the original signal using the dual frame operator. It can be shown that, if the decimation and the interpolation filters are orthogonal, i.e., $G^t G = HH^t = I_K$, $G = H^t$, the dual frame operator of F is $[G \;\; I_N - GH]$. In this case, the frame associated with the LP is tight [3]. If the filters are biorthogonal, i.e., $HG = I_K$, the above operator is still an inverse operator (i.e., it is a left-inverse of F) even though it is not the dual frame operator [3]. Therefore, with either orthogonal or biorthogonal filters, the original signal can be reconstructed as

$$\hat{\mathbf{x}}_f = [G \;\; I_N - GH] \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix} = G(\mathbf{c} - H\mathbf{d}) + \mathbf{d}. \tag{9}$$

Their proposed reconstruction structure is shown in Fig. 3. Under either orthogonality or biorthogonality,

$$H\mathbf{d} = H(I_N - GH)\mathbf{x} = (H - HGH)\mathbf{x} = \mathbf{0}_{K\times 1}, \tag{10}$$

where $\mathbf{0}_{K\times 1}$ denotes a null vector of length K. Therefore the usual reconstruction and the dual-frame based reconstruction methods lead to identical results when there is no noise in the LP coefficients.

4. LP WITH VARIABLE REDUNDANCY

Since an LP is overcomplete, it is certainly not required to transmit all of its coefficients in order to be able to reconstruct the original signal. The latter can be perfectly reconstructed even if some of the coefficients are intentionally not transmitted and the remaining coefficients are received without any noise at the receiver. This partial transmission of the LP coefficients is analogous to the case of erasures of the frame expansion coefficients during transmission over erasure channels. Therefore one could apply the signal reconstruction algorithms applicable to erasure channels [5, 8] to reconstruct the original signal from the partially received LP coefficients.

For the reconstruction to be feasible, the frame vectors (i.e., the rows of F) corresponding to the transmitted coefficients must make a subframe [5, 8]. Consider the case when the decimation and the interpolation filters are either biorthogonal or orthogonal. In this case, since $HG = I_K$, the rank of the matrix $I_N - GH$ is $N - K$. Therefore, only $N-K$ row vectors of $I_N - GH$ are linearly independent, and they span a $(N-K)$-dimensional subspace. In order to span the original N-dimensional space, which contains the signal vector $\mathbf{x}$, we need all the row vectors of H. This means that we cannot discard coefficients from the coarse signal $\mathbf{c}$ for the reconstruction to be feasible. At the same time, we can dispose of at best K coefficients of $\mathbf{d}$, in which case the transmitted LP coefficients have a critical representation.

On the other hand, if the decimation and the interpolation filters are neither biorthogonal nor orthogonal, this condition may not hold. For instance, if the filter coefficients are such that $I_N - GH$ is a full-rank matrix, we could discard the coarse signal $\mathbf{c}$ altogether and reconstruct $\mathbf{x}$ from the detail signal $\mathbf{d}$ simply by applying $(I_N - GH)^{-1}$ on $\mathbf{d}$. In view of the practical application of scalable compression, here we assume that the coarse signal is fully transmitted and consider discarding coefficients only from the detail signals.

4.1 Frame-theoretic reconstruction

First, let us assume that the LP coefficients are not quantized. Let the number of detail coefficients transmitted be R where $R \geq N - K$. Let the detail signal $\mathbf{d}$ be partitioned as $\mathbf{d} \equiv \begin{bmatrix} \mathbf{d}_R \\ \mathbf{d}_E \end{bmatrix}$, where $\mathbf{d}_R$ and $\mathbf{d}_E$ denote the vector of transmitted coefficients and the vector of discarded coefficients, respectively. Considering only the transmitted coefficients, we could express Eqn.7 as

$$\begin{bmatrix} \mathbf{c} \\ \mathbf{d}_R \end{bmatrix} = \begin{bmatrix} H \\ I_{R\times N} - G_R H \end{bmatrix} \mathbf{x} \equiv F_R \mathbf{x}, \quad (11)$$

where $I_{R\times N}$ and G_R denote the R rows of I_N and G corresponding to the transmitted coefficients indices. If the rows of F_R make a subframe, we can reconstruct $\mathbf{x}$ as follows:

$$\hat{\mathbf{x}}_p = (F_R^t F_R)^{-1} F_R^t \begin{bmatrix} \mathbf{c} \\ \mathbf{d}_R \end{bmatrix}. \quad (12)$$

In the critical representation case, i.e., when $R = N - K$, F_R is a square matrix and we obtain $\hat{\mathbf{x}} = F_R^{-1} \begin{bmatrix} \mathbf{c} \\ \mathbf{d}_R \end{bmatrix}$. Observe that, for the reconstruction to be feasible, the inverse matrix operations in Eqn.12 and above should be valid. Therefore the coefficients to be discarded have to be chosen accordingly.

Let us partition the vector $\mathbf{x}$ as $\mathbf{x} \equiv \begin{bmatrix} \mathbf{x}_R \\ \mathbf{x}_E \end{bmatrix}$, where $\mathbf{x}_R$ and $\mathbf{x}_E$ have the same indices as the transmitted coefficients and discarded coefficients respectively (recall that $\mathbf{d}$ and $\mathbf{x}$ have the same resolution.). By rearranging the columns of F_R in the according manner and using simple matrix algebra, Eqn. 12 can be expressed as

$$\hat{\mathbf{x}}_p = \begin{bmatrix} \hat{\mathbf{x}}_R \\ \hat{\mathbf{x}}_E \end{bmatrix} = \begin{bmatrix} G_R & I_R \\ H_E^{+}(I_K - H_R G_R) & -H_E^{+} H_R \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{d}_R \end{bmatrix}, \quad (13)$$

where H_R and H_E denote the columns of H having the same indices as the transmitted coefficients and the discarded coefficients respectively, and $H_E^{+} \equiv (H_E^t H_E)^{-1} H_E^t$. Note that, for the above equation to be valid, H_E must have full-column rank. In the critical representation case, H_E is a $K \times K$ square matrix and therefore the above expression can be rewritten as

$$\hat{\mathbf{x}}_p = \begin{bmatrix} \hat{\mathbf{x}}_R \\ \hat{\mathbf{x}}_E \end{bmatrix} = \begin{bmatrix} G_R & I_R \\ H_E^{-1}(I_K - H_R G_R) & -H_E^{-1} H_R \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{d}_R \end{bmatrix}. \quad (14)$$

Since there is no quantization error, the reconstruction in Eqn. 13 is perfect and it is identical to $\hat{\mathbf{x}}_s$ in Eqn. 4 and $\hat{\mathbf{x}}_f$ in Eqn. 9.

From the above equation, we obtain

$$\begin{aligned} \hat{\mathbf{x}}_R &= G_R \mathbf{c} + \mathbf{d}_R; && (15) \\ \hat{\mathbf{x}}_E &= H_E^{-1}\mathbf{c} - H_E^{-1} H_R (G_R \mathbf{c} + \mathbf{d}_R) \\ &= H_E^{-1}\mathbf{c} - H_E^{-1} H_R \hat{\mathbf{x}}_R. && (16) \end{aligned}$$

Observe that reconstruction of $\mathbf{x}_R$ is done using the usual method whereas the missing samples are calculated by solving the equation $H_E \mathbf{x}_E + H_R \mathbf{x}_R = \mathbf{c}$, or equivalently, $H\mathbf{x} = \mathbf{c}$.

4.2 Coding-theoretic reconstruction

It is known that frames in finite dimensional spaces are associated with codes in the complex or the real fields [8]. The frame operator F an be looked upon as the generator matrix of the associated code, and the vector of frame expansion coefficients $\begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix}$ can be seen as the codevector corresponding to the input vector $\mathbf{x}$. A parity check matrix for this code can be given as

$$P = \begin{bmatrix} -(I_K - HG) & H \end{bmatrix},$$

where I_K denotes the identity matrix of order K. It is easy to prove that $PF = \mathbf{0}_{K\times K}$. The missing detail coefficients can be recovered using syndrome decoding [7] as follows. Using the property that every codevector lies in the codespace, we get

$$P \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \mathbf{0}_{K\times 1}. \quad (17)$$

Substituting the expression for P, we get

$$\begin{aligned} -(I_K - HG)\mathbf{c} + H\mathbf{d} &= \mathbf{0}_{K\times 1} && (18) \\ \Rightarrow H\mathbf{d} &= (I_K - HG)\mathbf{c}. && (19) \end{aligned}$$

Now, partitioning H and $\mathbf{d}$ into the transmitted and discarded parts, we get

$$\begin{aligned} & H_E \mathbf{d}_E + H_R \mathbf{d}_R = (I_K - HG)\mathbf{c} && (20) \\ \Rightarrow \quad & \mathbf{d}_E = H_E^{+}\left(-H_R \mathbf{d}_R + (I_K - HG)\mathbf{c}\right). && (21) \end{aligned}$$

Observe that, in order that the above equation be valid, H_E must have full-column rank. In the case of critical representation, the above expression can be rewritten as

$$\mathbf{d}_E = H_E^{-1}\left(-H_R \mathbf{d}_R + (I_K - HG)\mathbf{c}\right). \quad (22)$$

When the filters are either biorthogonal or orthogonal, $HG = I_K$, and the above expressions simplify to

$$\begin{aligned} \mathbf{d}_E &= -H_E^{+} H_R \mathbf{d}_R && (23) \\ &= -H_E^{-1} H_R \mathbf{d}_R, \quad \text{for critical representation.} && (24) \end{aligned}$$

Once the missing detail signal coefficients are recovered, the original signal can be reconstructed by adding the detail signal to the interpolated coarse signal, as in the standard reconstruction:

$$\begin{bmatrix} \hat{\mathbf{x}}_R \\ \hat{\mathbf{x}}_E \end{bmatrix} = \begin{bmatrix} G_R & I_R \\ G_E + H_E^{+}(I_K - HG) & -H_E^{+} H_R \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{d}_R \end{bmatrix}, \quad (25)$$

where G_E denotes the rows of G having the same indices as the discarded detail coefficients. Like the previous method, since there is no quantization noise, the reconstruction of the original signal is perfect. Observe that, when the filters are either biorthogonal or orthogonal, the parity check matrix P given earlier is $[\mathbf{0}_{K\times K} \;\; H]$. In this case, Eqn. 19 is the same as the Eqn. 10 mentioned in section 3.

Now consider the realistic case when the LP coefficients are quantized. In this case, following the frame-theoretic reconstruction, the original signal can be estimated by substituting the quantized values of the coarse signal and the transmitted detail signal coefficients in Eqn.13 or in Eqn.14 (for critical representation). Or, following the coding-theoretic reconstruction, first the missing detail signal coefficients can be estimated by substituting the quantized values of the coarse signal and the transmitted detail signal coefficients in Eqn.21 or in Eqn.22 (for critical representation), and then adding the estimated detail signal to the interpolated quantized coarse signal.

To prove that the above two reconstructions are equivalent, we see that

$$G_E + H_E^{+}(I_K - HG) = G_E + H_E^{+}(I_K - H_E G_E - H_R G_R) = H_E^{+}(I_K - H_R G_R).$$

Therefore, Eqn. 13 and Eqn. 25 produce identical results. In the following, we use coding-theoretic reconstruction to analyze the reconstruction error.

5. RECONSTRUCTION ERROR ANALYSIS

In a practical application setup, the LP coefficients will be quantized before being encoded. Here we will consider only the case where the quantization of the coarse signal is outside the prediction loop. This structure is called the "open-loop prediction" in the literature [3].

Let $\mathbf{c}_q$ and $\mathbf{d}_q$ denote the quantized coarse signal and the quantized detail signal with the standard reconstruction. Let $\mathbf{d}_{Rq}$ denote the transmitted detail signal coefficients. With the usual method and the dual frame based method of Do and Vetterli, the decoder receives all the quantized LP coefficients and reconstructs the original signal using Eqn.4 and Eqn.9. The resulting reconstruction errors can be expressed as

$$\mathbf{e}_s = G\mathbf{q}_c + \mathbf{q}_d, \quad \text{and} \qquad (26)$$

$$\mathbf{e}_f = G\mathbf{q}_c + (I_N - GH)\mathbf{q}_d, \qquad (27)$$

where $\mathbf{q}_c$, $\mathbf{q}_d$ denote the quantization noise vectors for the coarse signal and the detail signal respectively. Here, for the sake of simplicity of analysis, we will assume that the coarse and the detail signals are scalar quantized. The quantization step sizes are smal- l enough so that the corresponding quantization noises can be assumed to be white and uncorrelated. Furthermore, because of the open-loop structure, the quantization noises of the coarse and the detail signals can be assumed to be uncorrelated as well. The respective mean square errors can be computed as follows:

$$MSE_s = \frac{1}{N}\mathbb{E}\|\mathbf{e}_s\|^2 = \frac{1}{N}\mathbb{E}(G\mathbf{q}_c + \mathbf{q}_d)^t(G\mathbf{q}_c + \mathbf{q}_d)) = \frac{1}{N}\sigma_c^2 tr(G^tG) + \sigma_d^2; \qquad (28)$$

$$MSE_f = \frac{1}{N}\mathbb{E}\|\mathbf{e}_f\|^2 = \frac{1}{N}\sigma_c^2 tr(G^tG) + \frac{1}{N}\sigma_d^2 tr((I_N - GH)^t(I_N - GH)), \qquad (29)$$

where σ_c^2 and σ_d^2 denote the quantization noise variances for the coarse signal and the detail signal respectively, $\mathbb{E}$ denotes the mathematical expectation, and $tr(.)$ denotes the trace of a matrix. In the special case when the filters are orthogonal, the above expressions can be simplified as

$$MSE_s = \frac{K}{N}\sigma_c^2 + \sigma_d^2, \quad \text{and} \qquad (30)$$

$$MSE_f = \frac{K}{N}\sigma_c^2 + (1 - \frac{K}{N})\sigma_d^2. \qquad (31)$$

With the proposed method, the decoder receives the quantized coarse signal $\mathbf{c}_q$ and the decimated detail signal $\mathbf{d}_{Rq}$. For the worst case scenario, it receives the critically decimated detail signal. Following the coding theoretic approach, it reconstructs the original signal as shown in Eqn. 25. The resulting reconstruction error for the critical case can be expressed as

$$\mathbf{e}_p = \begin{bmatrix} G_R & I_R \\ G_E + H_E^{-1}(I_K - HG) & -H_E^{-1}H_R \end{bmatrix} \begin{bmatrix} \mathbf{q}_c \\ \mathbf{q}_{dR} \end{bmatrix}, \qquad (32)$$

where $\mathbf{q}_{dR}$ denotes the quantization noise vector for the decimated detail signal. Therefore the mean square error can be derived as

$$\begin{aligned} MSE_p &= \frac{1}{N}\mathbb{E}\|\mathbf{e}_p\|^2 \\ &= \frac{1}{N}\sigma_c^2 tr(G^tG) + \frac{2}{N}\sigma_c^2 tr(G_E^t H_E^{-1}(I_K - HG)) \\ &+ \frac{1}{N}\sigma_c^2 tr((I_K - HG)^t(H_E H_E^t)^{-1}(I_K - HG)) \\ &+ (1 - \frac{K}{N})\sigma_d^2 + \frac{1}{N}\sigma_d^2 tr(H_R^t(H_E H_E^t)^{-1}H_R). \end{aligned} \qquad (33)$$

In the special case when the filters are orthogonal, the above expression can be simplified as

$$MSE_p = \frac{K}{N}\sigma_c^2 + (1 - \frac{K}{N})\sigma_d^2 + \frac{1}{N}\sigma_d^2 tr(H_R^t(H_E H_E^t)^{-1}H_R). \qquad (34)$$

Comparing the above expressions, we observe that the mean square error of the reconstructed signal is larger than that obtained with the dual-frame based reconstruction. This is expected since we intend to trade MSE for the bit rate. We also observe that the mean square error is a function of the filter coefficients. Therefore, the reconstruction error can be kept low by choosing the filters properly.

6. SIMULATION RESULTS

In order to test the proposed algorithm, we performed simulations over various standard images. To keep the computational complexity of the matrix operations low, we built LPs over blocks of size 16 and performed two levels of decomposition with downsampling factor 2. We used the Daubechies 9/7 wavelet filters for the lowpass filtering and interpolation even if the use of wavelet filters is not a necessity here. For all the simulations, the encoding of the coarse signal was performed with a JPEG-like algorithm with quality factor 50 whereas the detail images were scalar quantized and entropy coded. For the proposed method, we critically decimated the two detail signal levels by discarding the top-left coefficient in every 2×2 block. Fig. 4 shows the peak signal-to-noise ratio (PSNR) vs bits per pixel (bpp) for the "Barbara" image for all the three methods. The plots were obtained by varying the quantization step-sizes from 1 to 16 and finding the convex-hull of the resulting PSNR-rate pairs. The better performance of Do and Vetterli's reconstruction over the standard reconstruction is because of the use of biorthogonal filters and is already known [3]. We observe that the proposed approach can lead to higher compression performance at the same PSNR, or can lead to higher PSNR at the same bit rate by choosing proper quantization step sizes.

Fig. 5 shows the reconstructed images when the two detail layers are quantized with step sizes 16 and 7 respectively. We observe that the proposed method (bottom-left) requires lesser bpp but decreases the PSNR slightly compared to the other two reconstruction schemes. At the bottom-right, the proposed reconstruction at the same bpp (obtained with quantization step sizes 11 and 5) has better quality than the other two.

We also performed simulations over various other standard images. We observed that the PSNR vs rate plots for these images have

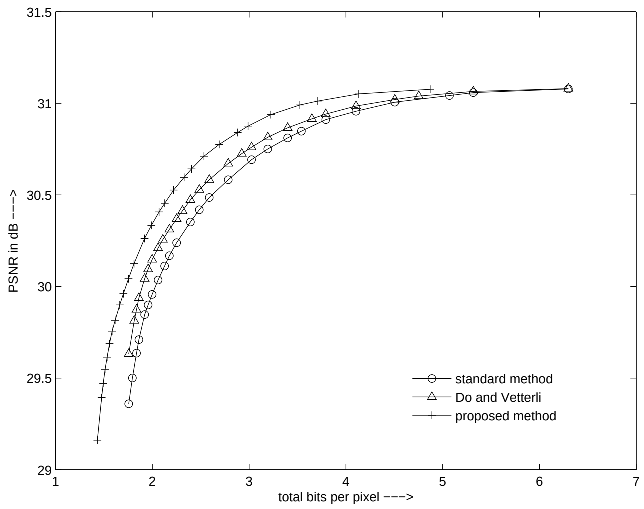

Figure 4: PSNR vs rate for "Barbara" represented using 2 levels of LP with 9/7 biorthogonal filters.

image source	bpp	PSNR		
		standard	Do and Vetterli	proposed
Barbara	1.99	29.96	30.15	30.33
Lena	2.3	30.01	30.06	30.12
Boat	1.35	28.25	28.44	28.54
Baboon	2.65	26.46	26.64	26.91
Peppers	2.82	30.36	30.41	30.48
Sailboat	2.33	28.27	28.32	28.39
Goldhill	2.17	29.90	30.00	30.10

Table 1: Bit rate and PSNR results for various standard images represented using 2 levels of Laplacian pyramid with 9/7 biorthogonal filters.

similar characteristics as of the ones for "Barbara" image. In general, the gain in compression efficiency is higher when the image contains significant detail components. Table 6 shows the PSNR of the reconstructed images for the three reconstruction methods at the same bits per pixel. We observe that the proposed algorithm results in higher PSNR values than the other two algorithms.

7. CONCLUSION

In this paper, we have reexamined the Laplacian pyramid from a frame representation point of view. This representation had been studied earlier by Do and Vetterli [3], who had proposed an improved reconstruction structure based on the dual frame. Here, on the other hand, we have proposed varying the redundancy of the LP through decimation of the detail signals. The decimation factor could be increased up to the critical representation.

For the decimated LP, we have presented two reconstruction algorithms. These algorithms were borrowed from the frame theory and the coding theory literature and were adapted to the LP representation. The reconstruction algorithm based on the frame theory aimed at estimating the original signal directly from the received coarse signal and the decimated detail signals through a dual subframe operator. The reconstruction algorithm based on syndrome decoding, however, aimed at recovering the decimated detail signals completely and then estimating the original signal by the usual reconstruction procedure. The two reconstruction methods were shown to produce identical output results.

Using a simple scalar quantization noise model, we have analyzed the mean square reconstruction error with the proposed

Figure 5: "Barbara" reconstructed from 2 levels of LP with 9/7 biorthogonal filters. top: (left) standard reconstruction (1.99 bpp, 29.96 dB), (right) frame-based reconstruction (1.99 bpp, 30.15 dB); bottom: (left) proposed method (1.62 bpp, 29.82 dB), (right) proposed method (1.99 bpp, 30.33 dB).

method and compared it to the errors obtained with the usual reconstruction and the dual frame based reconstruction. The analytical mean square reconstruction error was observed to depend on the decimation and the interpolation filters, and on the decimation pattern of the detail signals. The simulation results suggest that, using proper quantization parameters, it is possible to have better R-D performance over the standard reconstruction and the dual frame based reconstruction, where all the detail signal coefficients are transmitted.

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