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TIME-DELAY ESTIMATION USING FARROW-BASED FRACTIONAL-DELAY FIR FILTERS: FILTER APPROXIMATION VS. ESTIMATION ERRORS

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ABSTRACT

This paper provides error analysis regarding filter approximation errors versus estimation errors when utilizing Farrow-based fractional-delay filters for time-delay estimation. Further, a new technique is introduced which works on batches of samples and utilizes the Newton-Raphson technique for finding the minimum of the corresponding cost function.

1. INTRODUCTION

The need for estimating time-delays between two signals arises in many different fields, including biomedicine, communications, geophysics, radar, and ultrasonics. In [1], a technique utilizing Farrow-based digital fractional-delay (FD) filters [2] was introduced for this purpose. The use of Farrow-based FD filters has two major advantages over other delay estimation techniques working in the digital domain. First, it is eminently suitable to handle delays that are fractions of the sampling interval. This is in contrast to cross correlation-based methods that require additional interpolation [3]. Second, it can handle general bandlimited signals. This is in contrast to techniques that assume a known input signal, like a sinusoidal signal [4].

In [1], the idea of using Farrow-based FD filters for delay-estimation was proposed. However, no analysis was provided as to filter approximation error versus estimation error. Such an analysis will be provided in this paper. Furthermore, a new technique is introduced that works on batches of samples and utilizes the Newton-Raphson technique for finding the minimum of the corresponding cost function. Since the fractional delay of Farrow-based FD filters is governed by only one parameter, analytical derivatives can be derived for this purpose. Thereby, the problems associated with the use of numerical derivatives are avoided.

Following this introduction, Section 2 will provide a short introduction to time-delay estimation using FD filters, followed by a presentation of the delay-estimation technique in Section 3 and an error analysis in Section 4. In Section 5 we write FD filter design and in Section 6 we verify the error analysis. Finally some conclusions are drawn.

2. TIME-DELAY ESTIMATION USING FD FILTERS

Two (or more) discrete-time signals, originally coming from one source, might experience different delays. We model this as

$$\begin{aligned} x(n) &= x_a(nT) + e_1 && (1) \\ v(n) &= x_a(nT - d_0 T) + e_2 && (2) \end{aligned}$$

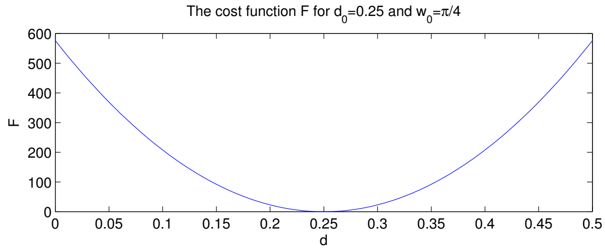

Figure 1: *Example of the cost function F for a sinus input with $d_0 = 0.25$.*

where d_0 is the unknown difference in delay between the signals, T is the sampling period and e_1 and e_2 are uncorrelated additive noise[1]. We assume that the delay d_0 is a fraction of T and that any integer sample delay has already been taken care of in a proper manner.

Now, let $v(n)$ act as a reference signal and let the other signal $x(n)$ pass through a FD filter generating the output $y(n,d)$ [see (7)]. The average squared difference function (ASDF) $F(d)$ for a certain delay d over a batch of N samples can then be written as

$$F(d) = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} \left(y(n,d) - v(n)\right)^2. \quad (3)$$

An estimate $\widehat{d}$ of the unknown fractional delay d_0 in the reference signal can then be computed by minimizing F as

$$\widehat{d} = \arg\min_d F(d). \quad (4)$$

An example of this cost function F for a sinusoidal input can be seen in Fig. 1. Ideally, the function would be equal to $2\sin^2(\frac{\omega(d-d_0)}{2})$, which is approximately square for small d, however, if noise, magnitude errors, delay errors, etc, are present, it will deviate from the square shape.

[1]Due to lack of space, the effect of noise is not included in this paper. It will not affect the filter approximation error vs. estimation error which is in focus in this paper. The noise can be reduced to a level that is negliable compared to the filter approximation error by increasing the batch length N.

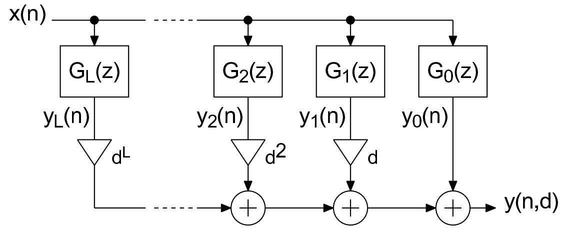

Figure 2: *The modified Farrow-based FD Filter where $G_k(z)$ are linear-phase FIR filters. In an implementation delays must added to make sure the delay is equal for all the subfilters.*

3. PROPOSED TIME-DELAY ESTIMATOR

3.1 Farrow-Based FD Filters

The desired frequency response of an FD filter is

$$H_{\text{des}}(e^{j\omega T}, d) = e^{-j(D+d)\omega T}, \; |\omega T| < \omega_c T < \pi \tag{5}$$

where D is an integer delay and d is a subsample (fractional) delay. In practice an approximation of (5), designed for frequencies up to $\omega_c T$, is used.

The FD filters in this paper make use of the modified Farrow structure shown in Fig. 2 [5], [6] which is a modification of the original Farrow structure [2] in that the subfilters $G_k(z)$ are linear-phase FIR filters. The advantage of using linear-phase filters is that their impulse responses are symmetrical and can be implemented with fewer multiplications. The overall transfer function can thus be written as

$$H(z) = \sum_{k=0}^{L} d^k G_k(z) \tag{6}$$

where $G_k(z)$ are linear-phase FIR filters of either odd or even order, say M_k, and with symmetric (anti-symmetric) impulse responses $g_k(n)$ for k even (k odd), i.e., $g_k(n) = g_k(M_k - n)$ $[g_k(n) = -g_k(M_k - n)]$ for k even (k odd) [7]. When d is fixed, the overall filter approximates an allpass filter with the fractional delay d, provided that the subfilters have been designed in a proper manner [5], [6].

The output $y(n,d)$ from the FD filter in Fig. 2 can be written as

$$y(n,d) = \sum_{k=0}^{L} d^k y_k(n). \tag{7}$$

where

$$y_k(n) = g_k(n) * x(n). \tag{8}$$

In this paper, it is assumed that the order $M = \max\{M_k\}$ is even. The reason is that the delay of $H(z)$ is then an integer, $D = M/2$, when $d = 0$. This is suitable for the time-delay estimation problem. The filter is normally optimized for $|d| \leq 0.5$ to cover one sampling interval.

3.2 The Estimator

To find the minimum of F with respect to d, the well-known Newton Raphson (NR) algorithm is used. The algorithm is iterative and tends towards the closest zero of a one-dimensional function, in this case the derivative of F. The update equation is here

$$\widehat{d}_{n+1} = \widehat{d}_n + \frac{F'(\widehat{d}_n)}{F''(\widehat{d}_n)} \tag{9}$$

where $F'(d)$ and $F''(d)$ are the derivatives of $F(d)$. The principle of the estimator is depicted in Fig. 3. Compared to the common Least-Mean-Square (LMS) algorithm we do not need to specify a step length since it is computed explicitly as $1/F''(\widehat{d}_n)$. For a perfectly quadratic function this step length is optimal and only one iteration is needed. However, in a real situation, a few more iterations are needed. Each new iteration can use a new batch of samples N or the same batch, depending on the amount of memory at hand and if the estimation is run on-line or off-line.

To be able to calculate the next iterative estimate in (9), the first and second derivatives of $F(n,d)$ with respect to d are needed, which can be calculated as

$$F'(d) = 2\frac{1}{N}\sum_{n=n_0}^{n_0+N-1} (y(n,d) - v(n))\, y'(n,d) \tag{10}$$

and

$$F''(d) = 2\frac{1}{N}\sum_{n=n_0}^{n_0+N-1} (y(n,d) - v(n))y''(n,d) + (y'(n,d))^2. \tag{11}$$

When a Farrow-based FD-filter is used, the derivatives of (7) can be calculated analytically as

$$y'(n,d) = \sum_{k=1}^{L} k d^{k-1} y_k(n) \tag{12}$$

and

$$y''(n,d) = \sum_{k=2}^{L} k(k-1) d^{k-2} y_k(n), \tag{13}$$

respectively. In Fig. 4, a straightforward implementation of the derivatives (10) and (11) can be seen. Note that the subfilters $G_k(z)$ only have to be used once.

Since a Farrow-based FD filer is an approximation of the ideal response in (5), the approximation errors will affect the estimator performance. In the next section we will investigate the performance of the algorithm when the FD filter is not ideal.

4. APPROXIMATION ERROR VS. ESTIMATION ERROR

We model the non-ideal FD filter as

$$H(e^{j\omega}, d) = (1 + \delta(\omega, d))e^{-j\omega(D+d+\tilde{d}(\omega,d))} \tag{14}$$

where $\delta(d,\omega)$ denotes the magnitude error and $\tilde{d}(d,\omega)$ denotes the delay error of the FD filter for a certain fractional delay d and a certain angular frequency ω. D is the integer delay of the FD-filter. This is a formulation which is general for all types of FD filters, not just the Farrow-based FD filter used in this paper, and the resulting expressions in this section is hence independent of the FD filter type. In Section 5, expressions specific for Farrow-based FD filters are derived.

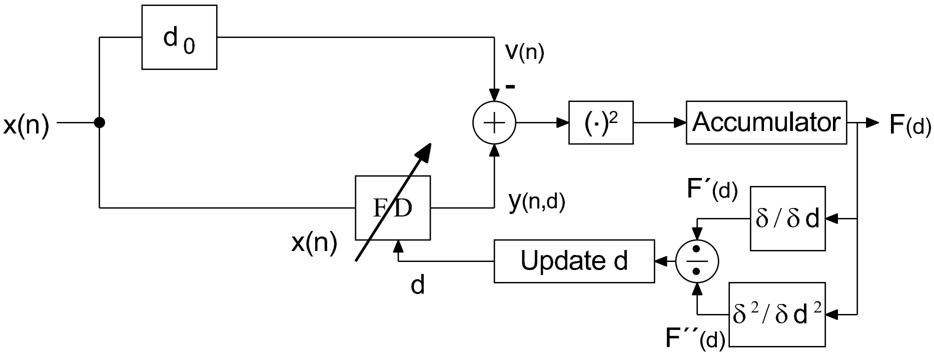

Figure 3: *The principle of the proposed estimator.*

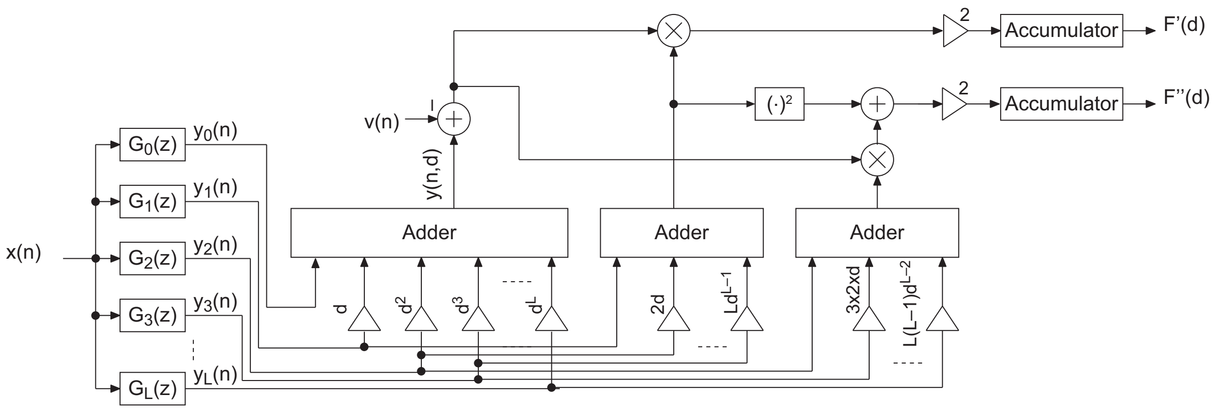

Figure 4: *A straightforward implementation of the derivatives.*

Now, if we assume that $x(n)$ is sinusoidal (for the choice of signal see the conclusions), the output from the FD filter at a frequency ω_0 is

$$y(n,d) = (1+\delta)\sin(\omega_0(n-d-\tilde{d})) \tag{15}$$

and the reference signal $v(n)$ is

$$v(n) = \sin(\omega_0(n-d_0)). \tag{16}$$

Henceforth, the dependence of δ and δ' on d and ω_0 will be omitted in the notation for the sake of clarity.

After inserting (15) and (16) into (3) and some simplifications we arrive at

$$\begin{aligned}
F'(\omega,d) = {} & \delta'[(1+\delta)-\cos(\omega(d-d_0+\tilde{d}))]+ \\
& +(1+\delta)\omega(1+\tilde{d}')\sin(\omega(d-d_0+\tilde{d}))+ \\
& \frac{1}{N}\big[(1+\delta)\omega_0(1+\tilde{d}')\sin(\omega_0(-(d_0+d+\tilde{d})+2\varphi)- \\
& -(1+\delta)^2\omega_0(1+\tilde{d}')\sin(\omega_0(-(d+\tilde{d})+2\varphi)+ \\
& +\delta'\cos(\omega_0(-(d_0+d+\tilde{d})+2\varphi)- \\
& -(1+\delta)\delta'\cos(\omega_0(-(d0+\tilde{d})+2\varphi)\big]\frac{\sin(N\omega_0)}{\sin(\omega_0)} \\
& = F_0'(\omega,d)+F_N'(\omega,d)
\end{aligned} \tag{17}$$

where F_0' does not depend on the batch length N and F_N' contains the terms that do. At the minimum of F, as we will see later, $d+\tilde{d}$ will be close to d_0. Using this fact we can rewrite F_N' as

$$\begin{aligned}
F_N'(\omega,d)|_{d+\tilde{d}=d_0} = {} & -\frac{\delta\sin(N\omega_0)}{N\sin(\omega_0)}\big[(1+\delta)\omega_0(1+\tilde{d}') \\
& \sin(\omega_0(2d_0+N-1))+\delta'\cos(2\omega_0(2d_0+N-1))\big].
\end{aligned} \tag{18}$$

Since δ and δ' usually are very small, (18) will be small even for a small N. However, when ω_0 tends towards 0, (18) will tend towards $-\delta'\delta$, independently of N.

The iterative Newton-Raphson algorithm in (9) will, if the function is behaving well, converge towards the zero of F_0'. If we assume that N is so large that F_N' can be approximated as zero for all d and ω_0, it can be seen from (17) that if δ' and $\tilde{d}$ are zero, F' will be zero when $d = d_0$, irrespective of δ. On the other hand, if δ' is small, but not zero, the effect of a constant magnitude error δ will affect the estimator. The effect of a delay error derivative $\tilde{d}'$ is normally neglible since it tends to be small compared to 1.

Unfortunately, it is impossible to directly from (17) analytically find the d that makes F_0' zero. It can be done numerically and to find an approximation of the estimation error we do a first-order Taylor expansion of the sine and cosine in F_0' and write it as

$$F_0' \approx \delta'(\delta+\frac{\omega_0^2}{2}(d-d_0+\tilde{d})^2)-(1+\delta)(1+\tilde{d}')\omega_0^2(d-d_0+\tilde{d}) \tag{19}$$

If $1+\tilde{d}$ and $1+\delta$ are approximated by 1, we can solve for $d_{\text{err}} \approx d - d_0$ and get

$$d_{\text{err}} \approx -\frac{1}{\delta'} + \frac{1}{\omega_0 \delta'}\sqrt{\omega_0^2 - 2(\delta')^2\delta} - \tilde{d}. \tag{20}$$

This expression can be used to predict the final estimation error. The first part becomes small when ω_0^2 is large compared to $2(\delta')^2\delta$ and the error will then be dominated by $\tilde{d}$. For small ω_0, if δ or δ' are not zero, the error deviates more and more from $\tilde{d}$. If $d' = 0$ the error becomes $d_{\text{err}} = \tilde{d}$.

5. FILTER DESIGN

As was seen in (17) the main error source is the delay error $\tilde{d}$ which directly affects the estimate. However, when the derivative of the magnitude error, δ', is nonzero the magnitude error δ and the frequency ω_0 will come into effect. To find an FD filter which is optimized for small estimation errors, the expression in (20) can be used as cost function in an optimization problem. The optimization problem can for example be formulated as a minimax problem

$$\text{minimize } \varepsilon \text{ subject to } |d_{\text{err}}(\omega, d)| < \varepsilon \tag{21}$$

which is then optimized over a range of angular frequencies ω_k and delays d_k. The `fminimax`-routine in MATLAB efficiently implements a sequential quadratic programming method that is capable of solving the minimax constraint problem in (21). However, the solution is not guaranteed to be the global minimum as the problem is nonlinear.

The derivative of δ, which is needed to calculate the expected estimation error, can be found analytically by noting that the magnitude of the Farrow-based FD filter can be written as

$$\begin{aligned} |H(d, e^{j\omega T})| &= \left| \sum_{k=0}^{\lfloor L/2 \rfloor} d^{2k} G_{(2k)R}(\omega T) + \right. \\ &\quad \left. + j \sum_{k=1}^{\lfloor (L+1)/2 \rfloor} d^{2k-1} G_{(2k-1)R}(\omega T) \right| \\ &= |A(d) + jB(d)| = 1 + \delta \end{aligned} \tag{22}$$

where $G_{(2k)R}$ and $G_{(2k-1)R}$ are the zero phase frequency responses for the even-order linear-phase FIR filters with impulse responses $g_k(n)$.

Since the derivative of the magnitude is equal to the derivative of the magnitude error we can calculate δ' as

$$\delta' = 2\frac{A(d)A'(d) + B(d)B'(d)}{\sqrt{A(d)^2 + B(d)^2}} \tag{23}$$

where $A'(d)$ and $B'(d)$ can be calculated as

$$A'(d) = \sum_{k=1}^{\lfloor L/2 \rfloor} 2k d^{2k-1} G_{(2k)R}(\omega T) \tag{24}$$

and

$$B'(d) = \sum_{k=1}^{\lfloor (L+1)/2 \rfloor} (2k-1) d^{2k-2} G_{(2k-1)R}(\omega T). \tag{25}$$

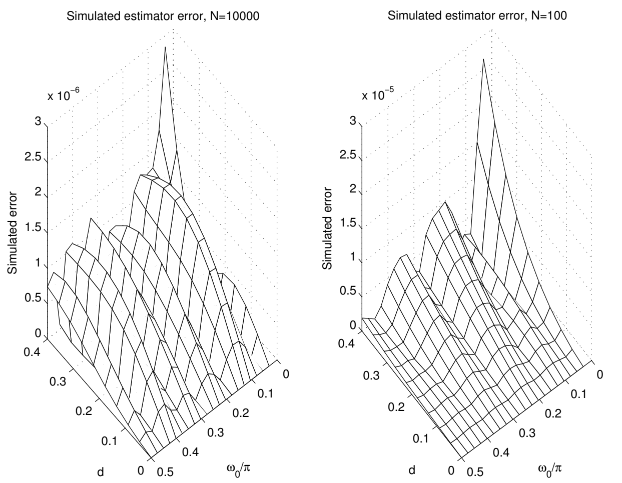

Figure 5: *The simulated estimation error for* $N = 10000$ *and* $N = 100$ *samples.* $0.1 < \omega_0 < \pi/2$.

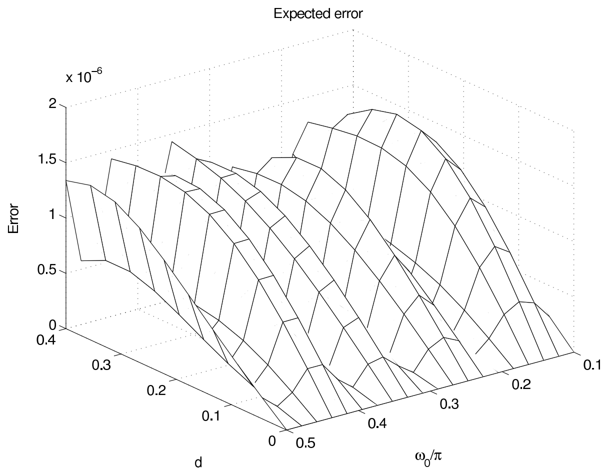

Figure 6: *The expected error.* $0.1\pi < \omega_0 < \pi/2$.

6. PERFORMANCE OF THE ESTIMATION ERROR PREDICTION

To verify the error analysis, a number of simulations were performed. An FD filter with $L = 7$ subfilters was optimized for minimal error up to the frequency $\omega_0 = \pi/2$. The resulting even-order subfilters had orders M_k ranging from 8 to 18. The maximum magnitude error was $\delta_{\max} = 1.83 \cdot 10^{-4}$, the maximum delay error was $\tilde{d}_{\max} = 1.168 \cdot 10^{-5}$ and the maximum derivative of the delay error was $\tilde{d}'_{\max} = 1.01 \cdot 10^{-4}$.

In Fig. 5 the simulated estimation error for $N = 10000$ and $N = 100$ samples can be seen. As predicted by (20) the estimation error increases for decreasing frequencies when there is a magnitude error. The reason for the lower performance for $N = 100$ samples is that N must be larger for the variance of F'_N to become small, especially when δ and δ' are relatively large, which they tend to be for small ω_0 and large d. In Fig. 6 the expected error d_{err} can be seen. Compared to the simulated error in Fig. 5 a smaller frequency range has been used to enhance the details.

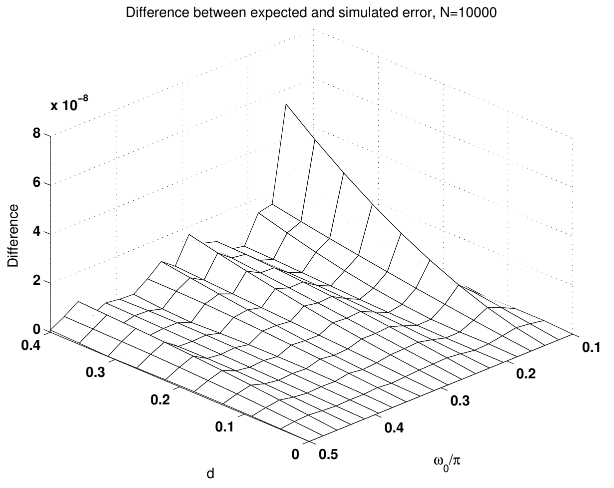

Figure 7: *The difference between the expected estimation error and the simulated error,* $N = 10000$. $0.1\pi < \omega_0 < \pi/2$.

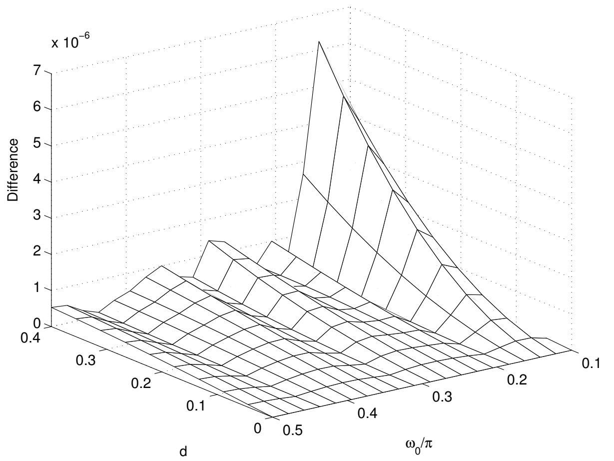

Figure 8: *The difference between the expected estimation error and the simulated error,* $N = 100$. $0.1\pi < \omega_0 < \pi/2$.

When d_{err} was derived we omitted F'_N, assuming that N is infinite or at least large enough. If the batch size N is decreased the expected estimation error will increase, which can be seen in Fig. 7 and 8. As expected, the difference between the estimated error and the simulated error is increased when N becomes smaller so that F'_N no longer can be approximated to be zero. For $N = 100$ samples the estimator still performs quite well, see Fig. 5, but the expected error d_{err} no longer works as a good estimation of the actual error since the expected error is almost as large as the actual difference between the simulated and expected error. This is because for small N noise and the variance of F'_N will dominate, while for large N the filter approximation errors will dominate.

7. CONCLUSIONS

We have presented a novel method to estimate the delay error between two sets of samples using a FD filter. The idea is to use an iterative, Newton-Raphson-based, estimator to minimize the mean-squared-difference between a reference signal and a signal with an unknown delay.

The effects of a nonideal FD filter with magnitude and delay errors have been studied theoretically and in simulations. An expression of the expected estimation error d_{err} was derived, which can be used to optimize the FD filter in the estimator for a minumum estimation error. The expected estimation error d_{err} is a bias, or in other words, the best one we can achieve without noise, i.e when $N \to \infty$.

In the analysis we have assumed a single sinusoid, but the time delay estimator can be used for more general bandlimited signals. However, in this case, the estimation error will differ from the case where a sinusoidal is used. The analysis done in this paper is still useful since it gives an insight into errors caused by the nonideal interpolation performed by the fractional-delay filters. Additionally, in some applications where the training sequence may be chosen freely we can actually choose a sinusoid and achieve the limits computed in this paper.

The time-delay estimation method can easily be extended to more sets of samples, using one set as the reference, which could be useful e.g. in the calibration of time-interleaved analog-to-digital-converters (TIADCs).

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