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KERNEL-BASED ESTIMATORS FOR THE KIRKWOOD-RIHACZEK TIME-FREQUENCY SPECTRUM

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ABSTRACT

In this paper, we examine kernel-based estimators for the Kirkwood-Rihaczek time-frequency spectrum of harmonizable, nonstationary processes. Based on an inner product consideration, we propose and implement an estimator for the Kirkwood-Rihaczek spectrum. The estimator is constructed from a combination of the complex demodulate with a short-time Fourier transform. Our proposed estimator is less computationally intensive than a theoretically equivalent, known estimator. We compare and test the results of the proposed estimator with an existing estimator from a Matlab time-frequency toolbox on simulated and real-world data. We demonstrate that the proposed estimator is less sensitive to unwanted cross-terms, and less affected by edge effects than the Matlab time-frequency toolbox estimator.

1. INTRODUCTION

The Kirkwood-Rihaczek time-frequency distribution was introduced by Kirkwood [8] in a quantum mechanics context, and later by Rihaczek [12] in deterministic signal theory. The Kirkwood-Rihaczek distribution is a bilinear time-frequency distribution that is covariant to shifts in time and frequency. Hence, it is a member of Cohen's class [3]. It is related to the Wigner-Ville distribution [18, 17] through a general time-frequency distribution [2]. Unlike the Wigner-Ville distribution, the Kirkwood-Rihaczek distribution is a complex-valued quantity.

Time-frequency analysis is an important tool for analyzing nonstationary random processes. In contrast to stationary processes, the frequency content of a nonstationary process may change with time. Therefore, it would be advantageous to represent the process as a function of time and frequency simultaneously. When dealing with random processes, we talk about time-frequency spectra, not time-frequency distributions. The Kirkwood-Rihaczek time-frequency spectrum of a harmonizable process is not a distribution of power/energy, but rather a distribution of correlation, or a complex Hilbert space inner product between the process and its infinitesimal stochastic Fourier generator [13].

In this paper, we examine kernel-based estimators for the Kirkwood-Rihaczek time-frequency spectrum. Based on the inner product consideration, we propose and implement a kernel-based estimator for the Kirkwood-Rihaczek time-frequency spectrum for harmonizable nonstationary processes. This estimator is shown to be theoretically equivalent to the time-domain based estimator with a factored kernel that was proposed, but not implemented, in [13]. Our proposed estimator requires much less computation time than a direct implementation of the estimator in [13]. The performance of the proposed estimator is compared with a different estimator found in the Matlab time-frequency toolbox (TF toolbox) [1], which is implemented from the general class of spectral estimators in [10]. It is well known that Wigner-Ville estimators suffer from interference problems, and this also applies for the estimator for the Kirkwood-Rihaczek spectrum in [1].

2. THE KIRKWOOD-RIHACZEK TIME-FREQUENCY SPECTRUM

Throughout this paper, $X[n]$ will denote a zero-mean, discrete-time, and real-valued process. The process is furthermore assumed to be harmonizable, such that it has a spectral representation [4]

$$X[n] = \int_{-1/2}^{1/2} e^{j2\pi fn} dZ(f), \tag{1}$$

where $dZ(f)$ is the complex-valued increment process of $X[n]$. The increment process has the spectral correlation

$$E\{dZ(f)dZ^*(f-\nu)\} = S_{XX^*}(\nu, f)d\nu df. \tag{2}$$

Here, $E\{\cdot\}$ denotes the expectation operator, f is a global frequency and ν is a local frequency offset. We denote $S_{XX^*}(\nu, f)$ the dual-frequency spectrum (also known as the Loève spectrum [9]) of $X[n]$.The temporal correlation function of $X[n]$ can be written as

$$M_{XX^*}[n, \eta] = E\{X[n]X^*[n-\eta]\} \tag{3}$$

where n is a global time variable, and η is a local time shift. By inserting the spectral representation from (1) in (3), we obtain the following relationship between the correlation function and the dual-frequency spectrum

$$\begin{aligned} M_{XX^*}[n, \eta] &= \iint_{-1/2}^{1/2} e^{j2\pi\nu n} e^{j2\pi f\eta} E\{dZ(\nu+f)dZ^*(f)\} \\ &= \iint_{-1/2}^{1/2} e^{j2\pi\nu n} e^{j2\pi f\eta} S_{XX^*}(\nu, f)d\nu df. \end{aligned} \tag{4}$$

Thus, the correlation function and the dual-frequency spectrum is a two dimensional Fourier transform pair.

The inverse Fourier transform of $S_{XX^*}(\nu, f)$ with respect to ν, or equivalently the Fourier transform of $R_{XX^*}[n,\eta]$ with respect to η, is the Kirkwood-Rihaczek time-frequency spectrum [8, 12]

$$V_{XX^*}[n,f]df = E\left\{X[n]\left(dZ(f)e^{j2\pi fn}\right)^*\right\} \tag{5}$$

of the process $X[n]$. This is a function of global time n and global frequency f. The Kirkwood-Rihaczek time-frequency spectrum is generally complex-valued, thus it cannot be interpreted as a distribution of energy/power in time and frequency. If we define the Hilbert space inner product between Z and W as $\langle Z,W\rangle = E\{ZW^*\}$, we see that

$$V_{XX^*}[n,f]df = \left\langle X[n], dZ(f)e^{j2\pi fn}\right\rangle. \tag{6}$$

The Kirkwood-Rihaczek spectrum can therefore be interpreted as a distribution of correlation over time and frequency [13].

A complex-valued process can also have complementary quantities [11, 14], originating from the fact that the process itself may be correlated at different times with the complex conjugated of the process in addition to the conventional correlation in (3). We will not consider the complementary functions in the following discussion, but all theoretical and practical work in this paper can easily be generalized to the complementary quantities.

3. ESTIMATION OF THE KIRKWOOD-RIHACZEK TIME-FREQUENCY SPECTRUM

The Kirkwood-Rihaczek spectrum will have to be estimated for any practical investigation of the time-frequency properties of $X[n]$. This can be done by estimating the dual-time autocorrelation function or the dual-frequency spectrum, followed by a discrete Fourier or inverse Fourier transform. Based on the inner product in (5), we will instead propose a direct and intuitive estimator which also provides additional insight into the geometry of the Kirkwood-Rihaczek spectrum.

Let $x[n]$, $n = 0,1,\ldots,N-1$, be N samples of a realization of the random process $X[n]$. Since n is a global time variable, we introduce a local data segment $n+\eta$, $\eta = -N_F,\ldots,0,\ldots,N_F$. We calculate the tapered Fourier transform of this local segment as

$$Y[n,k] = \sum_{\eta=-N_F}^{N_F} x[n+\eta]v_F[\eta]e^{-j2\pi f_k\eta}, \tag{7}$$

where $f_k = k/(2N_F+1)$, $k = -N_F,\ldots,N_F$, and $v_F[\eta]$ is a standard data taper (e.g., a Hanning taper).

Inserting the spectral representation from (1) in (7), we see that the relationship between $Y[n,k]$ and the increment process $dZ(f)$ is

$$\begin{aligned} Y[n,k] &= \int_{-1/2}^{1/2}\sum_{\eta=-N_F}^{N_F} v_F[\eta]e^{-j2\pi(f_k-f)\eta}e^{j2\pi fn}dZ(f) \\ &= \int_{-1/2}^{1/2} e^{j2\pi fn}V_F(f_k-f)dZ(f) \\ &\simeq e^{j2\pi f_k n}\int_{-1/2}^{1/2} V_F(f_k-f)dZ(f), \end{aligned} \tag{8}$$

where $V_F(f)$ is the Fourier transform of $v_F[\eta]$. The exponential is moved outside the integral under the assumption that $V_F(f)$ has a narrow main lobe.

We define the complex demodulate [6, 16] as

$$z[n,\eta,k] = \int_{-f_B}^{f_B} e^{j2\pi(f_k+\nu)\eta}V_B(\nu)dZ(f_k+\nu), \tag{9}$$

where $V_B(\nu)$ is the filter response of a low-pass filter. Following [16], we employ the estimate $dZ[k] = Y[n,k]d\nu$ of the increment process. From the convolution in (8) we see that the data taper $v_F[\eta]$ should have low sidelobes to avoid spectral leakage in the estimate. We obtain an estimate of the complex demodulate as

$$\widehat{z}[n,\eta,k] = e^{j2\pi f_k\eta}\int_{-f_B}^{f_B} e^{j2\pi\nu\eta}V_B(\nu)Y[n,k']d\nu, \tag{10}$$

where $k' = k/(2N_F+1)+\nu$. Note that $\widehat{z}[n,\eta,k]$ is a time-domain quantity that can be interpreted as an estimate of the complex modulated increment process, or the infinitesimal stochastic generator [13], $dZ(f)e^{j2\pi fn}$, for each frequency f_k, valid for arbitrary time steps $\eta = -N_T,\ldots,0,\ldots,N_T$.

Finally, we apply time averaging as an approximation of the expectation operator in the definition in (5) to obtain an estimate of the Kirkwood-Rihaczek spectrum

$$\widehat{V}_{XX^*}[n,k] = \sum_{\eta=-N_T}^{N_T} x[n+\eta]v_T[\eta]\left(\widehat{z}[n,\eta,k]\right)^*, \tag{11}$$

where $v_T[\eta]$ is a time domain window controlling the time resolution of the estimate. The main parameters in this estimator are the time smoothing window $v_T[\eta]$ which controls the time resolution of the estimate, and the tapers $v_F[\eta]$ and $v_B[\eta]$ which collectively control the frequency resolution of the estimate. Based on (10) and (11), we find it natural to choose the length of $v_B[\eta]$ to $2N_T+1$.

By inserting (7) in (10), and inserting the result in (11), we can express the estimator of the Kirkwood-Rihaczek spectrum as

$$\begin{aligned} \widehat{V}_{XX^*}[n,k] = &\sum_{\eta=-N_T}^{N_t}\sum_{\mu=-N_F}^{N_F} x[n+\eta]x^*[n+\mu] \\ &\times v_T[\eta]v_F[\mu]v_B[\eta-\mu]e^{-j2\pi f_k(\eta-\mu)}. \end{aligned} \tag{12}$$

We have assumed that all three windows are real-valued. By letting $N_F\to\infty$, $N_T\to\infty$, and performing a change of the variables, we can write (12) as

$$\begin{aligned} \widehat{V}_{XX^*}[n,k] = &\sum_{l=-\infty}^{\infty}\sum_{\lambda=-\infty}^{\infty} x[n+l]\Phi[l,\lambda] \\ &\times x^*[n+l-\lambda]e^{-j2\pi f_k\lambda}, \end{aligned} \tag{13}$$

where

$$\Phi[l,\lambda] = v_T[l]v_B[\lambda]v_F[l-\lambda]. \tag{14}$$

Thus, the estimator based on inner product considerations is identical to the time-domain based estimator proposed, but not implemented, in [13]. The estimator in [13] used the kernel

$$\Phi_{td}[l,\lambda] = w_1[l]w_2[\lambda]w_3[l-\lambda], \tag{15}$$

such that w_1, w_2 and w_3 from their estimator will have similar interpretations as v_T, v_B and v_F, respectively. Since our proposed estimator is formally identical to the estimator proposed in [13], the statistical properties found in [13] apply to our estimator as well. For spectral analysis in general, it is well known that direct estimation method using a Fourier transform, and especially the Fast Fourier Transform (FFT) algorithm, is faster than the corresponding time-domain method. Our time-frequency spectrum estimator is based on a short-time Fourier transform (evaluated by an FFT), which significantly decreases the computational time compared to the dual time-domain kernel proposed in [13]. Hence, the low computational cost of our estimator makes it the estimator of choice.

4. NUMERICAL EXAMPLES

We will now estimate the Kirkwood-Rihaczek time-frequency spectrum for two simulated data sets and one real-world data set with the estimator in (11), and by using an estimator in the Matlab TF toolbox [1]. For the estimator in (11), we have to choose the windows $v_F[\eta]$, $v_T[\eta]$ and $v_B[\eta]$ as well as the window lengths N_F and N_T. These parameters will control the time- and frequency resolution of the estimate. This gives us flexibility in the estimator. The windows and window lengths can be chosen to best match any given data set.

The TF toolbox [1] includes a function that estimates the pseudo Margenau-Hill spectrum of a signal. This estimator is an implementation of a special case of the general class of spectral estimators in [10]. The TF toolbox estimator first estimates the Kirkwood-Rihaczek spectrum, and then takes the real value of this estimate to obtain an estimate of the Margenau-Hill spectrum. Thus, by omitting the real value operator, we get an estimate of the Kirkwood-Rihaczek spectrum. This estimator has one parameter, a smoothing window we denote $w_s[n]$. We chose $w_s[n]$ to be the Discrete Prolate Spheroidal Sequence (DPSS) [15] of order zero. The length of the window N_s and the time-bandwidth product N_sW will control the smoothing of the estimate, and thus, the resolution of the estimate. The values of N_s and N_sW will have to be decided for each data set.

Time-frequency analysis favors the use of analytic signals since it reduces the number of cross-terms that cause interference in the spectrum [5]. Therefore, we carry out the analysis on the analytic signal corresponding to the real-valued signal. Since the underlying signal is real-valued, it will only be necessary to consider the spectrum for positive frequencies. An analytic process is complex-valued, and, in general, an analytic process corresponding to a real-valued nonstationary process will have complementary correlations [14]. However, as stated earlier, the complementary quantities will not be considered here.

4.1 Chirp

In this example, we apply the estimators on simulated data. We generate $N = 256$ samples $x[n]$, $n = 0,\ldots,N-1$, from a real-valued chirp

$$x[n] = \cos\left(\pi b n^2 + 2\pi a n\right), \qquad (16)$$

with normalized starting frequency $a = 0.1$ and chirp rate $b = 0.0012$. We use the DPSS of order zero for all three windows $v_T[\eta]$, $v_B[\eta]$ and $v_F[\eta]$, with time-bandwidth products 6, 10, and 20, respectively. The window lengths are $N_F = N_T = 110$ samples. For the estimator from the TF toolbox [1], we chose a time-bandwidth product $N_sW = 3$ and a window length $N_s = 65$. Figure 1 shows $\left|\widehat{V}_{XX^*}[n,k]\right|$ using the estimator proposed in this paper (upper panel) and the estimator from the TF toolbox [1] (lower panel). We see that both estimators work well for this signal, there is not much difference between the two estimates.

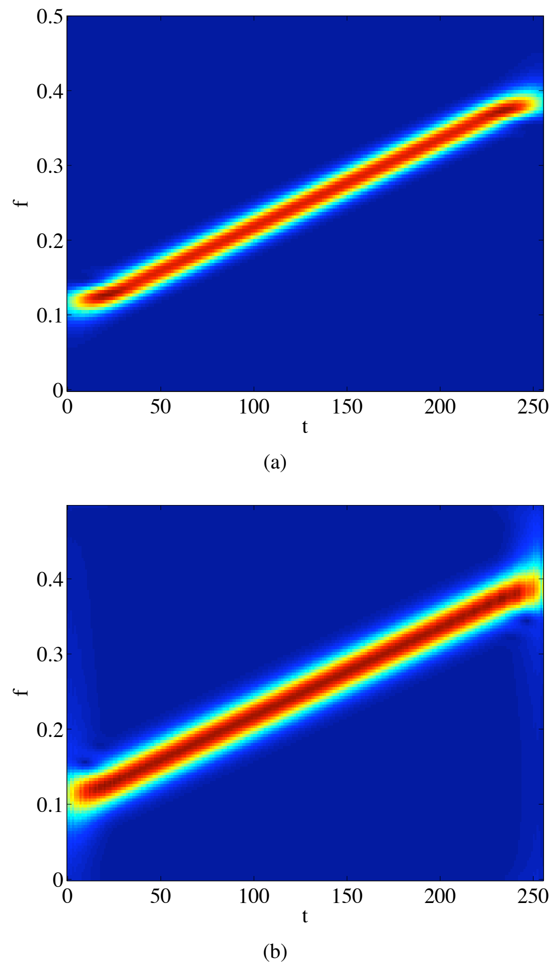

Figure 1: $\left|\widehat{V}_{XX^*}[n,k]\right|$ for the chirp using (a) our proposed estimator and (b) the estimator from the TF toolbox [1].

4.2 Two Pure Tones

We will now consider a simulated example containing two periodic components. We generate $N = 256$ samples $x[n]$, $n = 0,\ldots,N-1$, of the sum of two pure tones

$$x[n] = \cos(2\pi f_1 n) + \cos(2\pi f_2 n), \qquad (17)$$

with normalized frequencies $f_1 = 0.2$ and $f_2 = 0.4$. The windows and window lengths are the same as for the chirp in Section 4.1. The estimate obtained with our proposed estimator (upper panel) and the estimator from the TF toolbox [1] (lower panel) is shown in Figure 2. Both estimators give the

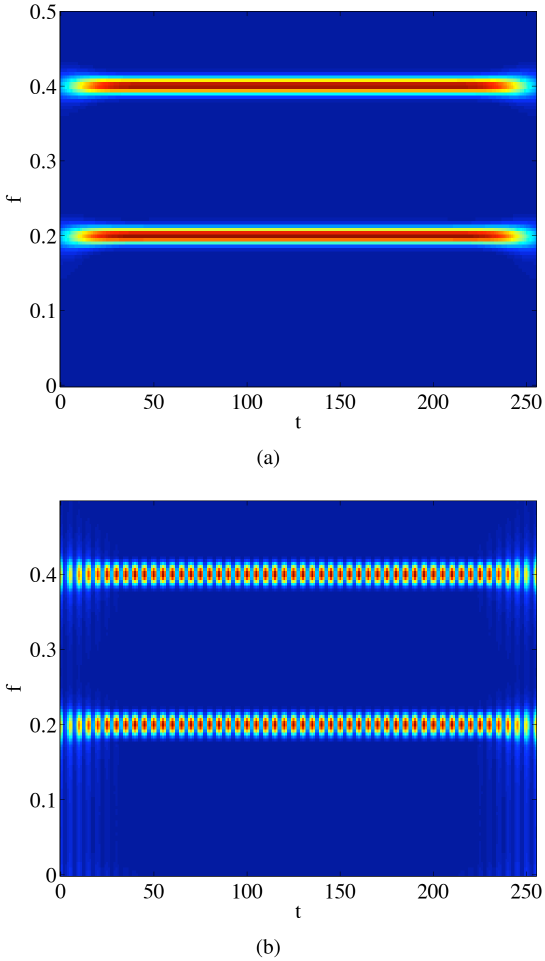

Figure 2: $\left|\widehat{V}_{XX^*}[n,k]\right|$ for the two pure tones using (a) our proposed estimator and (b) the estimator from the TF toolbox [1].

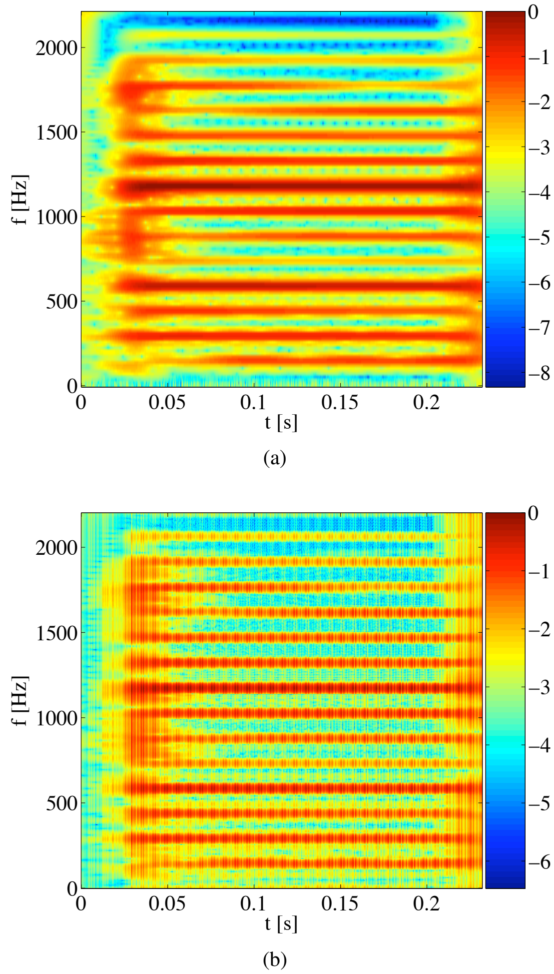

Figure 3: $\log_{10}\left|\widehat{V}_{XX^*}[n,k]\right|$ for the guitar data using (a) our proposed estimator and (b) the estimator from the TF toolbox [1].

correct localization in frequency for the two tones. But here we clearly see that the estimator from the TF toolbox [1] is sensitive to cross-terms between components in the signal. There is a pulsation of the lines which is periodic with frequency $f_2 - f_1 = 0.2$. This pulsation is due to interference between the two components of the process, a phenomenon commonly referred to as a "beat". The estimator we propose in this paper, however, does not suffer from any such unwanted cross-terms.

4.3 Guitar Data

We will now examine a real-world data set. The data are recordings of electric guitar sound emitted by a tube amplifier [7]. The data we consider here is a recording to disk via a microphone of a plectrum plucked D-string of a high-end electric guitar, referenced to standard A=440 Hz. The fundamental frequency, or pitch, occurs at 147 Hz. The guitar was an Infeld Andreas Shark guitar, equipped with Harry Häussel humbucker pickups. We used a Custom Audio Amplifier OD-100SH in a "clean" sound mode. The recording was made in an anechoic room, using an Earthworks M50 omnidirectional calibrated wideband measurement microphone. The M50 has a flat frequency response within $+1/-3$ dB in the frequency range 3 Hz – 50 kHz. The output from the M50 was sent trough a high-end professional "Kiwi" microphone cable manufactured by BLUE, to a Roland VS-2480 hard disk recorder. The signal was recorded with 24 bit resolution and a 44.1 kHz sampling rate.

The data have been downsampled with a factor 10 prior to our analysis, thus, the effective sampling frequency is 4410 Hz. The data set consists of $N = 1024$ points. We used the DPSS of order zero for $v_T[\eta]$ and $v_B[\eta]$ with time-bandwidth product 4 and 8, respectively. We used a Hanning window for $v_F[\eta]$, since the frequency resolution is more critical for these data than for our other examples, and the Hanning window is known for good frequency resolution. The window lengths are $N_T = 128 = N_F = 128$ samples. For the estimator from the TF toolbox [1], we chose a time-bandwidth product $N_s W = 2$ and a window length of $N_s = 251$ samples.

The normalized estimates are shown on a log-scale in Figure 3. When comparing the estimates in Figure 3, we notice the following. First, the pitch component at $f_0 = 147$ Hz, and all its harmonics at integer multiples of f_0, contain interference induced beats for the TF toolbox [1] estimate (lower panel), but not for our proposed estimator (upper panel). Second, the TF toolbox [1] estimate seems to be severely affected by edge effects, while our estimator is much less affected. In particular, the detailed time-frequency behavior of the onset of the guitar pluck is masked for the TF toolbox [1] estimator. Important details concerning the time-frequency evolution of the onset of the guitar tone is evident from the proposed estimator. E.g., we see that the tone starts below pitch, but swings into place subsequently. We also see that the 2nd, 5th, 6th and 7th harmonic make their presence before any of the other harmonics (including the pitch). These interesting details are hard to discern from the TF toolbox [1] estimate.

5. CONCLUSION

We proposed and implemented a kernel-based estimator of the Kirkwood-Rihaczek time-frequency spectrum, based on a Hilbert space inner product interpretation of the spectrum. The estimator was shown to be theoretically equivalent to the estimator proposed in [13]. However, the implementation of our estimator has a far lower computational complexity than the estimator from [13]. The time- and frequency resolution of the estimate are controlled by the choice of the three windows $v_T[\eta]$, $v_B[\eta]$ and $v_F[\eta]$, and the lengths of the windows N_T and N_F. This gives us many parameters to choose for any given data set, which provides the estimator with great flexibility.

We compared the performance of our proposed estimator with that of an estimator from the TF toolbox [1], which is implemented from a general class of spectral estimators proposed in [10]. We observed that for the chirp, which is a single-component signal, the two estimators gave similar results. However, in the example of the two-component signal, the estimator from the TF toolbox [1] suffered from unwanted interference between the two components. Our proposed estimator had no such interference problems. Finally, we considered a real-world recording of a guitar pluck. In this example, the estimate obtained from [1] was severely affected by edge effects in addition to the interference between components. These edge effects masked important details in the time-frequency spectrum, details that were evident in the estimate from our proposed estimator.

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