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PHASE ESTIMATION MEAN SQUARE ERROR AND THRESHOLD EFFECTS FOR WINDOWED AND AMPLITUDE MODULATED SINUSOIDAL SIGNALS

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ABSTRACT

The problem of estimating the phase of (possibly amplitude modulated) sinusoidal signals arises in a variety of signal processing applications [1]. *A closed form expression for the maximum likelihood (ML) estimator exists which achieves the best possible performance given by the Cramer-Rao lower bound (CRLB) asymptotically. However, due to the nonlinear nature of the problem, below a certain level of signal-to-noise ratio (SNR), the so called threshold effect occurs, and the performance of the estimator decreases quickly. In this paper, we investigate this effect, together with the influence of windowing and amplitude modulation on the threshold using the unmodified estimator.*

1. INTRODUCTION

Nonlinear estimators (often derived through the ML principle [2]) exhibit a threshold effect. That is, below a certain level of SNR the performance of the estimator departs from the CRLB due to the occurrence of so called outliers. It is of practical interest to determine this level in order to assess the performance of an estimator. For example, receivers in communication systems are said to operate above or below threshold.

This paper deals with the threshold effect in sinusoidal phase estimation, and the influence of windowing and amplitude modulation on the threshold as well as on the estimation performance below threshold. Furthermore, the inherent bias of the ML phase estimator, as derived in [3], is discussed. It is shown that this bias stems from the circular nature of the phase estimate and that the phase estimator can be considered unbiased depending on the application. The following signal will be investigated:

$$s[n] = A\cos(2\pi\psi_0 n + \phi_0), \qquad (1)$$

with $x[n]$ denoting the sampled data in three different cases: The case of an unmodified signal model (2) in additive white Gaussian noise (AWGN) $v[n]$ with variance σ^2 and zero mean is given by

$$x[n] = s[n] + v[n]. \qquad (2)$$

The windowed case corresponds to

$$x[n] = w[n]s[n] + w[n]v[n], \qquad (3)$$

and

$$x[n] = a[n]s[n] + v[n] \qquad (4)$$

corresponds to the amplitude modulated case. Hereby, N samples are obtained with constant sample rate T_s, A denotes the amplitude ($A > 0$), $\psi_0 = f_0 T_s$ denotes the normalized frequency, and ϕ_0 the phase ($-\pi \le \phi_0 < \pi$). In this work, we investigate the case of unknown A, $a[n]$ and ϕ_0, but known ψ_0. In the case of unknown ψ_0, the threshold effect for phase estimation is influenced by the frequency estimation threshold and will be subject of further work.

Data windows $w[n]$ are typically used if $s[n]$ consists of $p > 1$ sinusoids, i.e.

$$s[n] = \sum_{i=1}^{p} A_i \cos(2\pi\psi_i n + \phi_i) \qquad (5)$$

to suppress interference, but also to suppress interference in the single sinusoid case due to the real valued signal model. See [4] for some general rules a function must fulfill to be called a windowing function. Amplitude modulated sinusoids typically occur e.g. in communication or measurement systems, with the amplitude modulation modelled through the real valued, unknown function $a[n]$, $0 \le a[n] \le 1$. For the signal models (2), (3) and (4), the estimator

$$\hat{\phi}_0 = \arctan\left(\frac{-\sum_{n=0}^{N-1} x[n]\sin(2\pi\psi_0 n)}{\sum_{n=0}^{N-1} x[n]\cos(2\pi\psi_0 n)} \right) \qquad (6)$$

will be considered [2]. Note that only for the signal model (2) above estimator is an (approximate) ML estimator. For the signal model (3), a closed-form expression of the resultant mean-square error (MSE) of the phase estimate above threshold applying (6) has been derived in [5]. Signal model (4) will be considered here since although an optimum parameter estimator for the case of amplitude modulated sinusoidal signals does exist [6], (6) is often used in practice since it is also a consistent estimator [6]. For the signal models (2), (3), and (4) in the case of multiple sinusoids (5), the estimator (6) is only consistent if the frequencies are spaced far apart, i.e.

$$|\psi_i - \psi_k| >> \frac{1}{N} \quad \text{for } i \ne k. \qquad (7)$$

In the following derivations, we generally assume large enough N for (7) to be fulfilled. Furthermore, in the case of amplitude modulated multiple sinusoids, (6) is only a consistent estimator if the modulating function does not disturb the orthogonality of the individual cosines.

2. DERIVATION OF THE ESTIMATION MSE

2.1 CRLB, THRESHOLD EFFECT

ML estimators are known to result in asymptotically unbiased, normal distributed estimates [2], i.e.

$$p\left(\hat{\phi}_0\right)=\frac{1}{\sqrt{2\pi\sigma^2_{\hat{\phi}_0}}}\exp\left(-\frac{1}{2\sigma^2_{\hat{\phi}_0}}\left(\hat{\phi}_0-\phi_0\right)^2\right) \tag{8}$$

with

$$\sigma^2_{\hat{\phi}_0}=\mathrm{var}\left\{\hat{\phi}_0\right\}\geq\frac{1}{\eta N} \tag{9a}$$

denoting the asymptotic estimation variance in case of signal model (2). The SNR is defined as $\eta=A^2/(2\sigma^2)$. Eq. (9a) is valid for signal model (2) only, since for signal models (3) and (4) (6) is a least squares estimator only. Also in these cases the phase estimates are asymptotically distributed according to (8), but with increased estimation variances.
It has been shown in [5] that for signal model (3) the asymptotic estimation variance is given by

$$\sigma^2_{\hat{\phi}_0}=\mathrm{var}\left\{\hat{\phi}_0\right\}\geq\frac{1}{\eta}\frac{\sum_{n=0}^{N-1}w[n]^2}{\left(\sum_{n=0}^{N-1}w[n]\right)^2} \tag{9b}$$

and for signal model (4)

$$\sigma^2_{\hat{\phi}_0}=\mathrm{var}\left\{\hat{\phi}_0\right\}\geq\frac{1}{\eta}\frac{N}{\left(\sum_{n=0}^{N-1}a[n]\right)^2}. \tag{9c}$$

To determine the minimum level of SNR where the asymptotic distribution (8) is valid with a high degree of accuracy (i.e. the threshold level), it is necessary to derive the probability density functions (PDFs) of the phase estimates resulting from applying the estimator (6) under the signal models (2), (3), and (4).

2.2 PDF of PHASE ESTIMATE

The PDFs of the considered signal models can be derived all at once by using the generalized model

$$x[n]=b[n]s[n]+c[n]v[n], \tag{10}$$

with $b[n]=c[n]\equiv 1$ for signal model (1), $b[n]=c[n]=w[n]$ for signal model (3), and $b[n]=a[n]$, $c[n]\equiv 1$ for signal model (4).
First, the PDF of the numerator

$$U=-\sum_{n=0}^{N-1}x[n]\sin(2\pi\psi_0 n) \tag{11}$$

and the denominator

$$V=\sum_{n=0}^{N-1}x[n]\cos(2\pi\psi_0 n) \tag{12}$$

of (6) has to be derived. Since both are linear transformations of the data $x[n]$, the results are normal distributed [2]. Hence, the PDF is completely specified by its mean and variance. The mean of the numerator can be calculated assuming large N and using the fact that $\mathrm{E}\{v[n]\}=0$ and the trigonometric identity $\cos(\beta)\sin(\alpha)=(\sin(\alpha-\beta)+\sin(\alpha+\beta))/2$:

$$\begin{aligned}
&\mathrm{E}\left\{-\sum_{n=0}^{N-1}x[n]\sin(2\pi\psi_0 n)\right\}\\
&=\mathrm{E}\left\{-\sum_{n=0}^{N-1}\left(b[n]A\cos(2\pi\psi_0 n+\phi_0)+c[n]v[n]\right)\sin(2\pi\psi_0 n)\right\}\\
&=\mathrm{E}\left\{-\sum_{n=0}^{N-1}b[n]A\cos(2\pi\psi_0 n+\phi_0)\sin(2\pi\psi_0 n)-c[n]v[n]\sin(2\pi\psi_0 n)\right\}\\
&=-\sum_{n=0}^{N-1}\frac{A}{2}\left[b[n]\sin(-\phi_0)+\sin(4\pi\psi_0 n+\phi_0)\right]-\sum_{n=0}^{N-1}c[n]\mathrm{E}\{v[n]\}\sin(2\pi\psi_0 n)\\
&\approx\frac{A}{2}\sin(\phi_0)\sum_{n=0}^{N-1}b[n].
\end{aligned} \tag{13}$$

Note that for signal model (2), (13) is valid exactly if the sampling frequency is an integer multiple of the signal's frequency.
Analogously, the mean of the denominator of (6) can be shown to be

$$\mathrm{E}\left\{\sum_{n=0}^{N-1}x[n]\cos(2\pi\psi_0 n)\right\}\approx\frac{A}{2}\cos(\phi_0)\sum_{n=0}^{N-1}b[n]. \tag{14}$$

Next, the variance of (11) is defined as

$$\mathrm{var}\{U\}=\mathrm{E}\left\{(U-E\{U\})^2\right\}. \tag{15}$$

But obviously

$$\begin{aligned}
U-\mathrm{E}\{U\}&=-\sum_{n=0}^{N-1}x[n]\sin(2\pi\psi_0 n)+\mathrm{E}\left\{\sum_{n=0}^{N-1}x[n]\sin(2\pi\psi_0 n)\right\}\\
&=-\sum_{n=0}^{N-1}c[n]v[n]\sin(2\pi\psi_0 n)
\end{aligned} \tag{16}$$

and henceforth

$$\begin{aligned}
\mathrm{var}\{U\}&=\mathrm{E}\left\{\left(-\sum_{n=0}^{N-1}c[n]v[n]\sin(2\pi\psi_0 n)\right)^2\right\}\\
&=\mathrm{E}\left\{\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}c[n]c[m]v[n]v[m]\sin(2\pi\psi_0 n)\sin(2\pi\psi_0 m)\right\}\\
&=\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}c[n]c[m]\mathrm{E}\{v[n]v[m]\}\sin(2\pi\psi_0 n)\sin(2\pi\psi_0 m)\\
&=\frac{\sigma^2}{2}\sum_{n=0}^{N-1}c[n]^2.
\end{aligned} \tag{17}$$

Analogously,

$$\mathrm{var}\{V\}=\frac{\sigma^2}{2}\sum_{n=0}^{N-1}c[n]^2. \tag{18}$$

Therefore, U and V are normal distributed according to

$$U \sim N\left(\frac{A}{2}\sin(\phi_0)\sum_{n=0}^{N-1} b[n], \frac{\sigma^2}{2}\sum_{n=0}^{N-1} c[n]^2\right), \qquad V \sim N\left(\frac{A}{2}\cos(\phi_0)\sum_{n=0}^{N-1} b[n], \frac{\sigma^2}{2}\sum_{n=0}^{N-1} c[n]^2\right) \tag{19}$$

and can be furthermore shown to be independent.

To determine the PDF of the phase estimate, first the joint PDF of U and V is given by (after some straightforward manipulations)

$$p(U,V) = p(U)p(V) = \frac{1}{\pi\sigma^2\sum_{n=0}^{N-1} c[n]^2} \times \exp\left(-\frac{1}{\sigma^2\sum_{n=0}^{N-1} c[n]^2}\left(U^2+V^2 - A\left(\sum_{n=0}^{N-1} b[n]\right)[U\sin(\phi_0)+V\cos(\phi_0)] + \frac{A^2}{4}\left(\sum_{n=0}^{N-1} b[n]\right)^2\right)\right). \tag{20}$$

The transformation

$$\xi = \sqrt{U^2+V^2}, \quad \tan\left(\hat{\phi}_0\right) = \frac{\sin\left(\hat{\phi}_0\right)}{\cos\left(\hat{\phi}_0\right)} = \frac{U}{V} \tag{21}$$

leads to

$$U = \xi\sin\left(\hat{\phi}_0\right), \quad V = \xi\cos\left(\hat{\phi}_0\right). \tag{22}$$

The absolute value of the Jacobian, that must be taken into account for transformations [7], can be calculated to

$$\left|\det\begin{bmatrix} \partial U/\partial\xi & \partial U/\partial\hat{\phi}_0 \\ \partial V/\partial\xi & \partial V/\partial\hat{\phi}_0 \end{bmatrix}\right| = \xi. \tag{23}$$

Using (23) and the trigonometric identity

$$\sin\left(\hat{\phi}_0\right)\sin(\phi_0) + \cos\left(\hat{\phi}_0\right)\cos(\phi_0) = \cos\left(\hat{\phi}_0 - \phi_0\right) \tag{24}$$

the joint PDF in the new random variables can be calculated:

$$p\left(\xi,\hat{\phi}_0\right) = \frac{\xi}{\pi\sigma^2\left(\sum_{n=0}^{N-1} c[n]^2\right)}\exp\left(-1\Big/\sigma^2\left(\sum_{n=0}^{N-1} c[n]^2\right)\right)\exp\left(-\xi^2\Big/\sigma^2\left(\sum_{n=0}^{N-1} c[n]^2\right)\right) \times \exp\left(-\left(\xi A\left(\sum_{n=0}^{N-1} b[n]\right)\Big/\sigma^2\left(\sum_{n=0}^{N-1} c[n]^2\right)\right)\cos\left(\hat{\phi}_0-\phi_0\right)\right) \times \exp\left(-A^2\left(\sum_{n=0}^{N-1} b[n]\right)^2\Big/4\sigma^2\left(\sum_{n=0}^{N-1} c[n]^2\right)\right). \tag{25}$$

To obtain the marginal PDF of the phase estimate, the random variable ξ can be removed by integration over ξ, i.e.

$$p\left(\hat{\phi}_0\right) = \int_0^\infty p\left(\xi,\hat{\phi}_0\right)\mathrm{d}\xi. \tag{26}$$

This integral can be solved in closed form, leading to the sought PDF of the phase estimate. After some manipulations, the PDF belonging to signal model (2), indicated by the subscript (1) with $b[n] = c[n] \equiv 1$, is given by

$$p\left(\hat{\phi}_0\right)_{(1)} = \frac{1}{2\pi}\exp\left(-\frac{\eta N}{2}\right) + \frac{1}{2}\sqrt{\frac{\eta N}{2\pi}}\exp\left(-\frac{\eta N}{2}\right)\cos\left(\phi_0-\hat{\phi}_0\right) \times \exp\left(\frac{\eta N}{2}\cos\left(\phi_0-\hat{\phi}_0\right)^2\right)\left(1+\mathrm{erf}\left(\sqrt{\frac{\eta N}{2}}\cos\left(\phi_0-\hat{\phi}_0\right)\right)\right) \tag{27}$$

with erf(·) denoting the error-function. The PDF of the phase estimate for signal model (3) ($b[n] = c[n] = w[n]$), defining the effective noise bandwidth (ENBW) as

$$\mathrm{ENBW} = \frac{\sum_{n=0}^{N-1} w[n]^2}{\left(\sum_{n=0}^{N-1} w[n]\right)^2}, \tag{28}$$

which is $1/N$ for the rectangular window $w[n] \equiv 1$ (and $> 1/N$ for windows other than the rectangular window), is given by

$$p\left(\hat{\phi}_0\right)_{(2)} = \frac{1}{2\pi}\exp\left(-\frac{\eta}{2\mathrm{ENBW}}\right) + \frac{1}{2}\sqrt{\frac{\eta}{2\pi\mathrm{ENWB}}}\exp\left(-\frac{\eta}{2\mathrm{ENBW}}\right)\cos\left(\phi_0-\hat{\phi}_0\right) \times \exp\left(\frac{\eta}{2\mathrm{ENBW}}\cos\left(\phi_0-\hat{\phi}_0\right)^2\right)\left(1+\mathrm{erf}\left(\sqrt{\frac{\eta}{2\mathrm{ENBW}}}\cos\left(\phi_0-\hat{\phi}_0\right)\right)\right). \tag{29}$$

The ENBW has been introduced in [4]. It is commonly used in the literature as a measure of width of data windows. In section 2.4, this point will be discussed more in detail.

Finally, the PDF of the phase estimate for signal model (4), using the abbreviation

$$\gamma = \frac{1}{\left(\sum_{n=0}^{N-1} a[n]\right)^2} \tag{30}$$

is given by

$$p\left(\hat{\phi}_0\right)_{(3)} = \frac{1}{2\pi}\exp\left(-\frac{\eta}{2N\gamma}\right) + \frac{1}{2}\sqrt{\frac{\eta}{2\pi N\gamma}}\exp\left(-\frac{\eta}{2N\gamma}\right)\cos\left(\phi_0-\hat{\phi}_0\right) \times \exp\left(\frac{\eta}{2N\gamma}\cos\left(\phi_0-\hat{\phi}_0\right)^2\right)\left(1+\mathrm{erf}\left(\sqrt{\frac{\eta}{2N\gamma}}\cos\left(\phi_0-\hat{\phi}_0\right)\right)\right). \tag{31}$$

2.3 Asymptotic PDF

To prove the validity of above derivations, the above PDFs and the asymptotic PDFs described in section 2.1 must merge for high SNR (and equivalently for N>>). Exemplarily, this will be checked for the PDF (27). Using the approximation

$$\cos\left(\phi_0-\hat{\phi}_0\right) \approx 1, \tag{32}$$

in (27), since the phase estimate will be near the true value for high SNR, as well as the identity $\cos^2(\alpha) = 1-\sin^2(\alpha)$ yields

$$p\left(\hat{\phi}_0\right)_{(1)} \approx \frac{1}{2\pi}\exp\left(-\frac{\eta N}{2}\right) + \frac{1}{2}\sqrt{\frac{\eta N}{2\pi}} \times \exp\left(-\frac{\eta N}{2}\sin\left(\phi_0-\hat{\phi}_0\right)^2\right)\left(1+\mathrm{erf}\left(\sqrt{\frac{\eta N}{2}}\right)\right). \tag{33}$$

The first term in (33) will vanish for high SNR, and the error-function in the second term of (33) will be approximately 1. Furthermore,

$$\sin\left(\hat{\phi}_0 - \phi_0\right) \approx \hat{\phi}_0 - \phi_0, \qquad (34)$$

and hence (33) can be rewritten for high SNR as

$$p\left(\hat{\phi}_0\right)_{(1)} \approx \sqrt{\frac{\eta N}{2\pi}} \exp\left(-\frac{\eta N}{2}\left(\hat{\phi}_0 - \phi_0\right)^2\right). \qquad (35)$$

This is exactly (8) with inserted (9a). Therefore, the exact PDF (27) merges into the asymptotic PDF (8). Fig. 1 shows a comparison between exact and asymptotic PDF at low SNR of −20 dB. The exact PDF can be seen to have a non-negligible uniformly distributed part, which can be interpreted as outliers. But however, also the shape of the PDF differs from the asymptotic one.

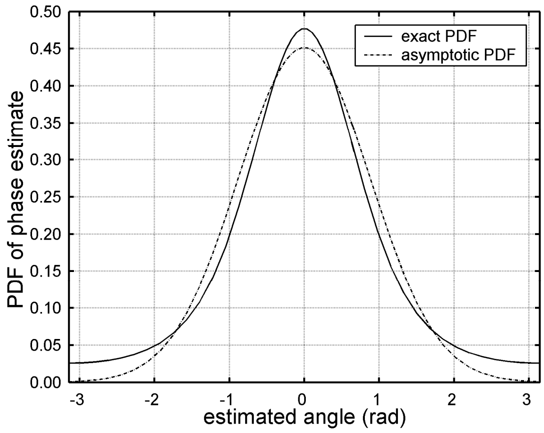

Figure 1 – Comparison of exact (27) and asymptotic PDF (8) of phase estimate, $\eta = -20$ dB, $N = 128$, $\phi_0 = 0$.

2.4 MSE and BIAS, THRESHOLD DEFINITION

From the PDFs described in the preceding section, the mean-square error (MSE) of the phase estimate can be calculated via

$$\mathrm{mse}\left\{\hat{\phi}_0\right\} = \int_{-\pi}^{\pi} \left(\hat{\phi}_0 - \phi_0\right)^2 p\left(\hat{\phi}_0\right) \mathrm{d}\hat{\phi}_0. \qquad (36)$$

However, this approach neglects the circular nature of the phase estimate. As already mentioned in the introduction, in [3] it is shown that the ML phase estimator is in fact biased (for $\phi_0 \neq 0$) when using the classical MSE definition (36) (or equivalently the classical bias definition). This problem not only arises in phase estimation, but in all estimation problems where circular random variables are under investigation. A detailed treatment of this topic can be found e.g. in [8], where generically usable definitions for sample means etc. of circular random variables are given. Note that it is application dependent whether or not definition (36) makes sense. For example, the so called cycle skipping phenomena in phase-locked loops, as mentioned in [3], is caused by the inherent bias of the ML phase estimate. However, in standard measurement applications, it is well known that the estimator (6) yields unbiased and consistent estimates, when defining the MSE as

$$\mathrm{mse}\left\{\hat{\phi}_0\right\} = \begin{cases} \int_{-\pi}^{\pi} \left(\hat{\phi}_0 - \phi_0\right)^2 p\left(\hat{\phi}_0\right) \mathrm{d}\hat{\phi}_0 & \text{for } \left|\hat{\phi}_0 - \phi_0\right| \leq \pi \\ \int_{-\pi}^{\pi} \left(\hat{\phi}_0 - \phi_0 + 2\pi\right)^2 p\left(\hat{\phi}_0\right) \mathrm{d}\hat{\phi}_0 & \text{for } \left|\hat{\phi}_0 - \phi_0\right| > \pi \text{ and } \hat{\phi}_0 - \phi_0 < 0 \\ \int_{-\pi}^{\pi} \left(\hat{\phi}_0 - \phi_0 - 2\pi\right)^2 p\left(\hat{\phi}_0\right) \mathrm{d}\hat{\phi}_0 & \text{for } \left|\hat{\phi}_0 - \phi_0\right| > \pi \text{ and } \hat{\phi}_0 - \phi_0 > 0 \end{cases} \qquad (37)$$

As a simple example, compare the resultant MSE when using definition (36) versus (37) with $\phi_0 = \pi - \pi/180$ and

$$\hat{\phi}_0 = -\pi + \frac{\pi}{180},$$

an estimate which is only 2 degree in error. However, (36) will indicate a large MSE.

In the following discussion, the circular nature of the phase estimate will be taken into account. Unfortunately, the integral (36) cannot be solved in a closed form for the PDFs (27), (29), and (31), and hence must be evaluated numerically.

Hitherto no direct specification in terms of SNR and N of the threshold level has been given. Of course, there is no discontinuity in the MSE curve of an estimator below and above threshold, although the MSE below threshold departs relatively quickly in most cases from the CRLB. To specify a threshold level, it thus makes sense to define a ratio of MSEs were the true MSE deviates more than a certain amount from the asymptotic one, i.e.

$$\frac{\mathrm{mse}\left\{\hat{\phi}_0\right\}_{\mathrm{exact}}}{\mathrm{mse}\left\{\hat{\phi}_0\right\}_{\mathrm{asympt.}}} > \lambda, \qquad (38)$$

for an application dependent constant $\lambda > 1$. Using this definition, it can be easily decided whether or not an estimator operates above or below threshold, assuming knowledge of the SNR. In the following section, a simple method for determining the threshold level will be developed, although it is possible to directly use the derived PDFs (27), (29), or (31), and numerically integrate them according to (36) or (37), respectively.

However, some important conclusions for the case of windowed and amplitude modulated data can be drawn. In [5] the increase of the variance of the phase estimate in the windowed data case has been shown to be exactly the ENBW, see (9b) and (28). Hence, the loss in estimation performance and the increase of the threshold level are affected by the same factor. Note that this is not always the case, e.g. in frequency estimation it is shown in [9] that the threshold level is also affected by the ENBW, whereas the estimator's variance is affected by a more complicated term. Further, note from (27) and (29) that the ENBW appears always pair-wise with η, i.e. the data window acts as decrease of the SNR.

The influence of amplitude modulated data on both the estimation variance and the threshold level are potentially much more stringent. Also in this case, the amplitude modulation can be interpreted as a decrease in SNR, as evident from (30). However, when comparing the influence of e.g. a Han-

ning window and an amplitude modulation in form of a Hanning window onto the SNR, the latter leads to a much higher decrease, as can be seen by comparing (28) and (30).
This findings are of interest in that it is easy to evaluate the possible loss in MSE and threshold level of phase estimation e.g. in measurement systems and therefore to calculate the maximum amount of amplitude modulation allowable to achieve a certain amount of measurement accuracy.

2.5 PRACTICAL IMPLEMENTATION

In Fig. 2 the ratios of MSEs according to (38) are shown for different N for the case of no window and a Hanning window. It is obvious, that at least for a high degree of approximation and for $1 < \lambda \leq 1.4$, the threshold SNR η_{th} for a given λ is inverse proportional to N, that is

$$\eta_{\mathrm{th}} = k\lambda \frac{1}{N} \quad (39)$$

with a constant k yet to be determined. Now, for a given λ, e.g. $\lambda = 1.1$, the constant k can be computed for a given N, e.g. $N = 32$, from rearranging (39) and numerically integrating (27), resulting in (see Fig. 2 for $10\log_{10}(\eta_{\mathrm{th}})$)

$$k = \frac{\eta_{\mathrm{th}} N}{\lambda} = \frac{10^{-3.83/10} \cdot 32}{1.1} = 12.044 \quad (40)$$

and (39) together with the result of (40) can be used subsequently to calculate η_{th} for different values of N. Therefore, the computational burden of numerically computing (36) or store a table of η_{th} for different N in memory can be completely eliminated.

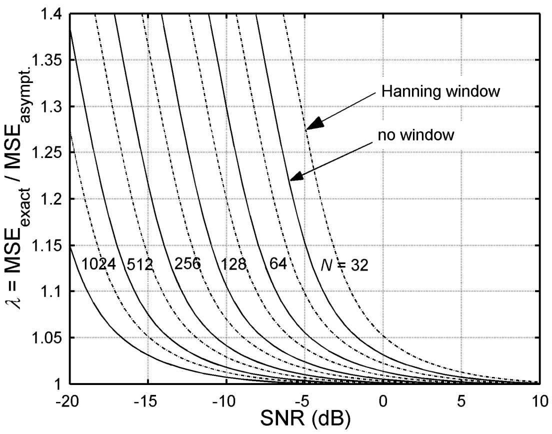

Figure 2 – Ratios of MSEs (38) for different N for the non-windowed and the windowed case.

3. SIMULATION RESULTS

To validate the derived expressions for the PDFs (27), (29), and (31) and the corresponding MSEs, a Monte Carlo simulation, for $\psi_0 = 0.125$, $T_s = 1/4000$, $\phi_0 = \pi/2$, with 500 Monte Carlo trials each has been carried out. The simulation results show a good match to the developed theory. As data window, a Hanning window has been chosen, and the latter has been also used as amplitude modulation function $a[n]$. As can be seen, the case of amplitude modulation leads to a much higher threshold level and MSE, as claimed.

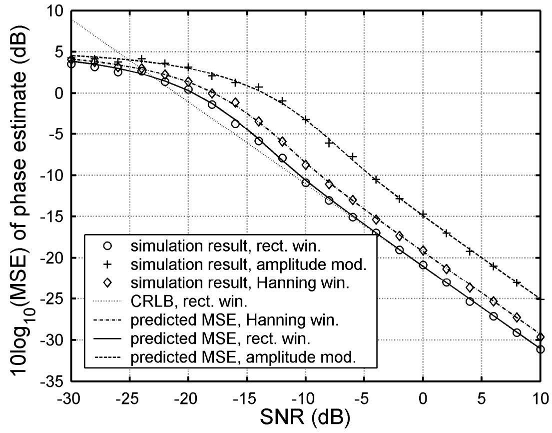

Figure 3 – Monte Carlo simulation of the resultant MSEs of phase estimation for the non-windowed, windowed (Hanning window) and amplitude modulated case with $N = 128$, $\phi_0 = \pi/2$, 500 Monte Carlo trials each.

4. CONCLUSION

In this paper, we presented expressions for the MSE and threshold level for the standard ML phase estimator for windowed and amplitude modulated data. The differences between the non-windowed and windowed respectively amplitude modulated case have been worked out. A simple formula for assessing the threshold level in phase estimation has been given. An investigation of the threshold level for unknown frequency and for using the estimators proposed in [6] is subject of future work.

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